$$\int_{l=0}^{l=\infty} -\vec{E} \cdot d\vec{l} = (\phi)_{l=0} - (\phi)_{l=\infty}$$

But potential at ∞ is zero, $V_{l=\infty} = 0$

$$\therefore \quad \int_{l=0}^{l=\infty} -\vec{E} \cdot d\vec{l} = (\phi)_{l=0}$$

Potential due to ring at centre *i.e.* $(\phi)_{l=0} = \dfrac{1}{4\pi\varepsilon_0}\dfrac{q}{r}$

$$\therefore \quad \int_{l=0}^{l=\infty} -\vec{E} \cdot d\vec{l} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$$

$$= 9 \times 10^9 \times \frac{1{\cdot}11 \times 10^{-10}}{0{\cdot}5} = +2\text{ V.}$$

2·4. Gauss's Theorem

Karl Friedrich Gauss (1777–1855) gave a theorem, known after his name *Gauss's theorem,* that relates the integral of the normal component of the electric field (*i.e.* net outward electric) flux over any hypothetical closed surface (called a Gaussian surface) to the net charge enclosed by the surface. The theorem states that *the normal component of the electric field (i.e. the net outward electric flux) over any closed surface of any shape drawn in an electric field is equal to* $\dfrac{1}{\varepsilon_0}$ *times the net charge enclosed by the surface, i.e.*

$$\oint_S \mathbf{E} \cdot \mathbf{n}\, da = \frac{1}{\varepsilon_0}(q)$$

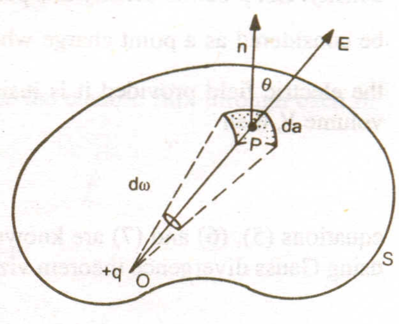

Fig. 2.3

where $\oint_S$ indicates the surface integral over whole of the closed surface and q is net charge enclosed by the surface.

Proof : Case (i) For an internal point. Let a point charge q coulomb be placed at origin O within the closed surface. Let $\mathbf{E}$ be the electric field strength at the point P on the surface due to charge q. Let $\vec{OP} = \mathbf{r}$ and electric field strength vector E make an angle θ with the unit vector $\mathbf{n}$ drawn normal to surface element da surrounding point P.

The surface integral of the normal component of electric field $\mathbf{E}$ over the closed surface S is given by

$$\oint_S \mathbf{E} \cdot \mathbf{n}\, da. \qquad \text{...(1)}$$

But electric field strength at P,

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^3}\mathbf{r} \qquad \text{...(2)}$$

$$\therefore \quad \oint_S \mathbf{E} \cdot \mathbf{n}\, da = \frac{1}{4\pi\varepsilon_0} q \oint_S \frac{\mathbf{r} \cdot \mathbf{n}}{r^3} da \qquad \text{...(3)}$$

$$= \frac{1}{4\pi\varepsilon_0} \frac{q}{r} \qquad ...(19)$$

The derivation may be extended to obtain equations (14) and (15).

As curl **E** = 0 everywhere therefore Stoke's theorem for electric field **E** viz,

$$\oint_S \text{curl } \mathbf{E} \cdot d\,\mathbf{a} = \oint \mathbf{E} \cdot \overrightarrow{dl}$$

implies that

$$\oint \mathbf{E} \cdot \overrightarrow{dl} = 0 \qquad ...(20)$$

The shows that *the electrostatic field is a conservative field, therefore no work is done on a test charge if it is moved around a closed path in the field.*

Another interesting and useful aspect of the electrostatic potential is its relationship with the potential energy associated with the conservative electrostatic force. The potential energy associated with an arbitrary conservative force is

$$V(\mathbf{r}) = -\int_{ref}^{r} \mathbf{F}(\mathbf{r}') \cdot d\,\mathbf{r}' \qquad ...(21)$$

where $V(\mathbf{r})$ is the potential energy at position **r** relative to the reference point at which potential energy is assumed to be zero. Since in electrostatic case $\mathbf{F} = q\,\mathbf{E}$, therefore

$$\mathbf{V}(\mathbf{r}) = -\int_{ref}^{r} q\,\mathbf{E}(\mathbf{r}') \cdot d\,\mathbf{r}'$$

or

$$\phi(\mathbf{r}) = \frac{V(\mathbf{r})}{q} = -\int_{ref}^{r} \mathbf{E}(\mathbf{r}') \cdot dr'.$$

Therefore if the same reference point is chosen for the electrostatic potential and potential energy, then the *electrostatic potential is just the potential energy per unit charge.* This idea may be used in several cases to introduce the electrostatic potential which has its importance in determining the electrostatic field [refer to equation (9)].

Thus we have two definitions of electric potential

1. The electrostatic potential at any point is defined as the work done by some external agency against the direction of field to bring per unit infinitesimal positive test charge from infinity to given point under consideration *i.e.*

Electrostatic potential $\phi(r) = \lim_{q_0 \to 0} \frac{W}{q_0}$ where W is the work done in bringing positive test charge q_0 from infinity to present position $(\vec{\mathbf{r}})$.

2. The electrpsatic potential at any point is defined as the negative of the line integral of electric field from infinity to present position $(\vec{\mathbf{r}})$ *i.e.*

$$\phi(\vec{\mathbf{r}}) = -\int_{\infty}^{r} \vec{\mathbf{E}} \cdot d\,\vec{\mathbf{r}}$$

Ex. 2. *A non-conducting ring of radius* 0·5 *m carries a total charge of* $1{\cdot}11 \times 10^{-10}$ C *distributed non-uniformly as its circumference producing an electric field* $\vec{\mathbf{E}}$ *everywhere in space. Calculate the value of the line integral*

$$\int_{l=\infty}^{l=0} -\vec{\mathbf{E}} \cdot d\,\vec{l} \quad (l = 0 \text{ being centre of the ring})$$

Solution. Potential difference is equal to the negative of line integral

Now it can be seen that the quantity $\left(\frac{\mathbf{r}}{r}\right)\cdot\mathbf{n}\,da$ gives the projection of area da on a plane perpendicular to $\mathbf{r}$. This projected area divided by $r^2\left(i.e.\ \frac{\mathbf{r}\cdot\mathbf{n}\,da}{r^3}\right)$ is the solid angle subtended by da at q, which is written as $d\omega$. Thus

$$\frac{\mathbf{r}\cdot\mathbf{n}}{r^3}\,da = d\omega$$

Hence equation (3) may be written as

$$\oint_S \mathbf{E}\cdot\mathbf{n}\,da = \frac{q}{4\pi\varepsilon_0}\oint_S d\omega \qquad \text{...(4)}$$

But $\int d\omega$ = soild angle subtended by entire closed surface at an internal point = 4π

Hence equation (4) gives

$$\oint_S \mathbf{E}\cdot\mathbf{n}\,da = \frac{q}{4\pi\varepsilon_0}4\pi = \frac{q}{\varepsilon_0} \qquad \text{...(5)}$$

This result is known as Gauss law for a single point charge enclosed by the surface.

If several point charge q_1, q_2, q_n are enclosed by the surface S, then total electric field is given by first term of equation (9) of section 2.2. Each charge subtends a full solid angle 4π, hence equation (5) becomes

$$\oint_S \mathbf{E}\cdot\mathbf{n}\,da = \frac{1}{\varepsilon_0}\sum_{i=1}^{n} q_i \qquad \text{...(6)}$$

This result can be generalised in the case of a continuous distribution of charge characterised by a charge density. Let ρ be the density at a point within an infinitesimal volume element dV; then the charge ρdV may be considered as a point charge which contributes $\frac{\rho dV}{\varepsilon_0}$ to the surface integral fo the normal component of the electric field provided it is inside the surface over which we integrate. Thus if surface S encloses the volume V, then

$$\oint_S \mathbf{E}\cdot\mathbf{n}\,da = \frac{1}{\varepsilon_0}\int_V \rho\,dV \qquad \text{...(7)}$$

equations (5), (6) and (7) are known as Gauss's law, Gauss's law may be expressed in yet another form by using Gauss divergence theorem viz.

$$\int_S \mathbf{E}\cdot\mathbf{n}\,da = \int_V \operatorname{div}\mathbf{E}\,dV \qquad \text{...(8)}$$

using (8), equation (7) becomes

$$\int_V \operatorname{div}\mathbf{E}\,dV = \frac{1}{\varepsilon_0}\int_V \rho\,dV \quad \text{or} \quad \int_V\left(\operatorname{div}\mathbf{E} - \frac{\rho}{\varepsilon_0}\right)dV = 0$$

As volume is arbitrary therefore the integrand must be equal to zero *i.e.*

$$\operatorname{div}\mathbf{E} - \frac{\rho}{\varepsilon_0} = 0 \quad \text{or} \quad \operatorname{div}\mathbf{E} = \frac{\rho}{\varepsilon_0}. \qquad \text{...(9)}$$

This result is known as the *differential form of Gauss's law.*

Case (ii) For an external point. If the charge q is outside the surface, then it is clear from fig. 2·4, that the surface S can be divided into areas S_1, S_2, S_3, S_4 each of which subtends the same solid angle at the charge q. But at S_1 and S_2 the directions of the outward drawn normal are away from q while at S_2 and S_4

they are towards q. Therfore the contributions of two pairs (S_1, S_3) and (S_2, S_4) to the surface integral are equal and opposite. Consequently the net surface integral of the normal component of the electric field **E** vanishes *i.e.*

$$\oint \mathbf{E} \cdot \mathbf{n}\, da = 0$$

It is interesting to note that Gauss's law remains valid as such even for charges in motion.

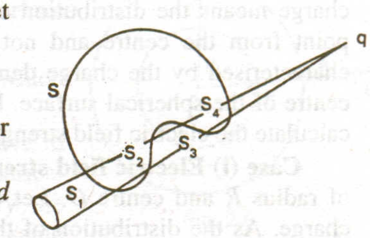

Fig. 2.4

Ex. 3. *Five thousand electric lines of force enter a given volume and three thousand leave it. Find the total charge contained in it.*

(Kanpur 1997)

Solution. According to Gauss' theorem

$$\text{Net electric flux } (\phi) = \frac{1}{\varepsilon_0} \times \text{charge enclosed } (Q)$$

Net flux, ϕ = net number of electric lines of force diverging from given surface

$$= -5000 + 3000 = -2000$$

Charge enclosed $\quad Q = \varepsilon_0 \phi = 8{\cdot}86 \times 10^{-12} \times (-2000)$

$$= -17{\cdot}72 \times 10^{-9} \text{ coulomb} = -\mathbf{1{\cdot}77 \times 10^{-8} \text{ coulomb}}$$

Ex. 4. *A charge* 1 μC *is placed at the centre of a hollow cube. Calculate the electric flux diverging (i) through the centre (ii) through each face.*

Solution. Given $\quad Q = 1\ \mu C = 10^{-6}$ C

(i) Net electric flux diverging through the cube

$$\phi = \frac{1}{\varepsilon_0} Q = \frac{1}{8{\cdot}86 \times 10^{-12}} \times 10^{-6} \text{ volt–m}$$

$$= \frac{1}{8{\cdot}86} \times 10^6 = 1{\cdot}12 \times 10^5 \text{ volt–m}$$

(ii) As charge is placed symmetrically to all six faces of cube; hence the electric flux through each of six faces is divided equally.

$\therefore \quad$ Electric flux through each face, $\phi_1 = \dfrac{1}{6}\left(\dfrac{Q}{\varepsilon_0}\right)$

$$= \frac{1}{6} \times 1{\cdot}12 \times 10^5 = 5{\cdot}6 \times 10^4 \text{ volt-m}$$

Ex. 5. *An electric field in a region is given by* $\vec{E} = 3\hat{i} + 4\hat{j} - 5\hat{k}$.

Calculate the electric flux through the surface $\vec{S} = 2{\cdot}0 \times 10^{-5}\, \hat{k}\ \text{m}^2$

Solution. $\quad \phi = \vec{E} \cdot \vec{S} = (3\hat{i} + 4\hat{j} - 5\hat{k}) \cdot (2{\cdot}0 \times 10^{-5}\, \hat{k})$

$$= 0 + 0 - 10 \times 10^{-5} = \mathbf{1{\cdot}0 \times 10^{-4} \text{ V-m}}$$

2.5. Applications of Gauss's Law

Gauss's law is useful in calculating the electric field in the problems in which it is possible to choose a closed surface such that the electric field has a normal component which is either zero or a single fixed value at every point on the surface. The use of Gauss's law makes the calculations easier as compared to the use of Coulomb's law. Here we shall consider some of the important electrostatic problems where Gauss's law is applicable.

8

Maxwell's Equations and Electromagnetic Waves

8.1. Introduction

In the preceding chapters we have dealt with *steady state problems* in electrostatics and magnetostatics treating electric and magnetic phenomenon independent of each other. The only link between them was the fact that electric currents which produced magnetic fields are basically electric in nature, being charges in motion. Now if we wish to consider more general problems in which field quantities may depend upon time, the almost independent nature of electric and magnetic phenomenon disappears. Time-varying magnetic fields give rise to electric field and vice-versa. We then must speak of *electromagnetic fields* rather than electric and magnetic fields. The behaviour of time dependent electromagnetic fields is described by a set of equations known as *Maxwell's equations.* These equations are mathematical abstractions of experimental results.

In this chapter we shall steady the formulation of *Maxwell's equations* along with their general properties and the basic conservation law of charge and energy.

8.2. Equation of Continuity

Conservation of charge : According to principle of conservation of charge *the net amount of charge in an isolated system remains constant.* For generality let us assume that the charge density is a function of time. Then the principle of conservation of charge may be stated as follows :

If the net charge crossing a surface bounding a closed volume is not zero, then the charge density within the volume must change with time in such a manner that the time rate of increase of charge within the volume equals the net rate of flow charge into the volume. This statement of conservation of charge in a medium may be expressed by the equation of continuity which may be derived as follows :

Let S be the surface enclosing a volume V and let $d\mathbf{S}$ be a small element of this surface. The direction of $d\mathbf{S}$ is taken to be that of the outward normal. If $\mathbf{J}$ is the current density (*i.e.* current per unit area placed normal of direction of current flow) at a point on surface element $d\mathbf{S}$, then $\mathbf{J} \cdot d\mathbf{S}$ represents the charge per unit time leaving volume V across $d\mathbf{S}$. Therefore the *time rate at which charge leaves the volume V* bounded by entire surface S is given by

$$\int_S\int \mathbf{J} \cdot d\mathbf{S}.$$

If q is charge contained is V, then according to charge conservation law, the above integral must be equal to $-dq/dt$, where dq/dt represents the time rate of flow of charge into V, thus

$$\iint \mathbf{J} \cdot d\mathbf{S} = -\frac{dq}{dt}. \qquad \ldots(1)$$

But $\quad q = \iiint_V \rho \, dv,$

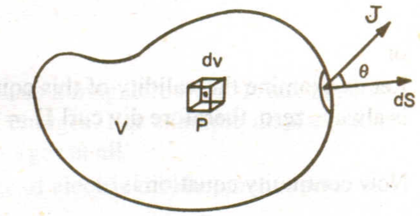

Fig. 8.1

where ρ is the charge density and dv is an element of volume.

Therefore equation (1) takes the form

$$\iint_S \mathbf{J} \cdot d\mathbf{S} = -\frac{d}{dt} \iiint_V \rho \, dv.$$

Since the order of differentiation and integration is interchangeable, therefore

$$\iint_S \mathbf{J} \cdot d\mathbf{S} = -\iiint \frac{\partial \rho}{\partial t} \, dv, \qquad \ldots(2)$$

But from Gauss divergence

$$\iint_S \mathbf{J} \cdot d\mathbf{S} = \iiint_V \text{div } \mathbf{J} \, dv \qquad \ldots(3)$$

Comparing (2) and (3), we get

$$\iiint_V \text{div } \mathbf{J} \, dv = -\iiint \frac{\partial \rho}{\partial t} \, dv.$$

or

$$\iiint_V \left(\text{div } \mathbf{J} + \frac{\partial \rho}{\partial t} \right) dv = 0.$$

Since volume is arbitrary, therefore integrand must be zero

i.e. $\qquad \text{div } \mathbf{J} + \dfrac{\partial \rho}{\partial t} = 0.$

This is the required *equation of continutity and expresses the conservation of charge.*

The current is called stationary if there is no accumulation of charge at any point *i.e.* for stationary current $\partial \rho / \partial t = 0$ at all points. Therefore the criterion for stationary flow is

$$\text{div } \mathbf{J} = \nabla \cdot \mathbf{J} = 0. \qquad \ldots(5)$$

8.3. Maxwell's Postulate ; Displacement Current.

From Amperes circuital law, we have

$$\oint \mathbf{H} \cdot d\mathbf{l} = I \qquad \ldots(1)$$

(refer equation (3) of section 5.8)

$$I = \int_S \mathbf{J} \cdot d\mathbf{S}. \qquad \ldots(3)$$

$$\therefore \qquad \oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot d\mathbf{S}. \qquad \ldots(4)$$

But from Stoke's theorem

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \text{curl } \mathbf{H} \cdot d\mathbf{S}$$

Comparing (3) and (4), we get

$$\int_S \text{curl } \mathbf{H} \cdot d\mathbf{S} = \int_S \mathbf{J} \cdot d\mathbf{S}$$

i.e. $$\int_S (\text{curl } \mathbf{H} - \mathbf{J}) \cdot d\mathbf{S} = 0. \qquad \ldots(5)$$

As the surface is arbitrary, therefore integrand must vanish *i.e.*

$$\text{curl } \mathbf{H} - \mathbf{J} = 0$$

or

$$\text{curl } \mathbf{H} = \mathbf{J}. \quad ...(6)$$

Let us examine the validity of this equation for time-varying fields. Since div of curl of any vector quantity is always zero. therefore div curl $\mathbf{H} = 0$. Then equation (6) implies

$$\text{div } \mathbf{J} = 0. \quad ...(7)$$

Now continuity equation is

$$\text{div } \mathbf{J} + \frac{\partial \rho}{\partial t} = 0. \quad ...(8a)$$

i.e.

$$\text{div } \mathbf{J} = -\frac{\partial \rho}{\partial t}. \quad ...(8b)$$

According to this equation div $\mathbf{J} = 0$ only if $\partial\rho/\partial t = 0$ *i.e.*, charge density is static. Thus we conclude that Ampere's equation (1) is valid only for steady state conditions and is insufficient for the cases of time-varying fields. Hence to include time-varying fields Ampere's law must be modified. Maxwell investigated mathematically how one could alter Ampere's equaion (1) so as the make it consistent with the equation of continuity. Maxwell assumed that the definition for current densiy $\mathbf{J}$ is incomplete and hence some thing say, $\mathbf{J}_d$ must be added to it. Then total current density which must be solenoidal, becomes $C = \mathbf{J} + \mathbf{J}_d$.

Using this postulate, equation (6) becomes

$$\text{curl } \mathbf{H} = \mathbf{C} = \mathbf{J} + \mathbf{J}_d \quad ..(9)$$

In order to identify $\mathbf{J}_d$ let us take the divergence of equation (9) :

$$\text{div curl } \mathbf{H} = \text{div } (\mathbf{J} + \mathbf{J}_d)$$

But div curl $\mathbf{H} = 0$ since div of curl of any vector is always zero ; therefore we get

$$\text{div } (\mathbf{J} + \mathbf{J}_d) = 0$$

or

$$\text{div } \mathbf{J} + \text{div } \mathbf{J}_d = 0$$

or

$$\text{div } \mathbf{J}_d = -\text{div } \mathbf{J}. \quad ...(10)$$

But div $\mathbf{J} = -\partial\rho/\partial t$ from equation of continuity, hence equaion (10) becomes

$$\text{div } \mathbf{J}_d = \frac{\partial \rho}{\partial t} \quad ...(11)$$

But Gauss theorem in differential form gives

$$\text{div } \mathbf{D} = \rho \quad ...(12)$$

Using this equation (11) may be written as

$$\text{div } \mathbf{J}_d = \frac{\partial}{\partial t}(\text{div } \mathbf{D})$$

$$= \text{div}\left(\frac{\partial \mathbf{D}}{\partial t}\right)$$

This gives

$$\mathbf{J}_d = \frac{\partial \mathbf{D}}{\partial t} \quad ...(13)$$

Therefore the modified from the Ampere's law is

$$\boxed{\text{curl } \mathbf{H} = \mathbf{J} + \mathbf{J}_d = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}} \quad ...(14)$$

The term which Maxwell added to Ampere's law to include time varying fields is known as *displacement current* because it arises when electric displacement vector $\mathbf{D}$ changes with time. By addition this term

Maxwell assumed that this term (displacement current) is as effective as the *conduction current* **J** for producing magnetic field.

Characteristics of displacement current

(i) Displacement current is a current only in the sense that it produces a magnetic field. It has none of the other properties of current since it is not linked with the motion of charges. For example displacement current has a finite value even in a perfect vaccum where there are no charges at all.

(ii) The magnitude of displacement current is equal to rate of change of electric displacement vector *i.e.* $\mathbf{J}_d = \partial \mathbf{D}/\partial t$.

(iii) Displacement current serves the purpose to make the total current continuous across the discontinuity in a conduction current. As an example, a battery charging a capacitor produces a closed current loop in terms of total current $\mathbf{J}_{total} = \mathbf{J} + \mathbf{J}_d$.

(iv) Displacement current in a good conductor is negligible as compared to the conduction current at any frequency less than optical frequencies ($\approx 10^{15}$ Hertz).

With the postulate of displacement current Maxwell was able to derive his theory of electromagnetic waves. We may consider the experimental observation of such waves, with the properties predicted, as the experimental basis for *Maxwell's* postulate. Furthermore we shall show that this postulate has a reasonable physical interpretation.

8.4. Physical Interpretation of Maxwell's Postulate

Maxwell's original explanation of displacement current was puzzling, therefore for convenience we shall consider another point of view.

The modified form of Ampere's law may be expressed as

$$\oint_C \mathbf{H}\cdot d\mathbf{l} = \int_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right)\cdot d\mathbf{S}. \qquad ...(1)$$

Just as $\oint \mathbf{E}\cdot d\mathbf{l}$ represents electromotive force in electrostatics, the magnetomotive force (m.m.f.) around the path C is

$$\text{m.m.f.} = \oint_C \mathbf{H}\cdot d\mathbf{l} \qquad \text{(refer section 5.29).} \quad ...(2)$$

Substituting this into (1), we get

$$\text{m.m.f.} = \oint_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right)\cdot d\mathbf{S}. \qquad ...(3)$$

Equations (1) and (2) indicate that there are two ways of producing a magnetic intensity, one with a ordinary conduction current density **J**, as observed by Oersted and postulated by Ampere, and the other by means of time varying electric displacement, as postulated by Maxwell. Since $\mathbf{D} \propto \mathbf{E}$ for air or vacuum, we may say that *a changing electric field gives rise to a magnetic field,* This is the converse of Faraday's discovery that a changing magnetic field gives rise to an electric field.

If we take the case where $\mathbf{J} = 0$ everywhere, then equation (1) becomes

$$\oint_C \mathbf{H}\cdot d\mathbf{l} = \oint_S \frac{\partial \mathbf{D}}{\partial t}\cdot d\mathbf{S}, \quad (\mathbf{J} = 0). \qquad ..(4)$$

This equaion indicates that time varying electric displacement produces a magnetic field : thus varifying Maxwell's postulate.

Direct observation of a magenetic field produced by a changing electric field is difficult. We can not look for an *induced magnetic current* because there are neither free poles nor conductors for magnetic currents. Also we cannot maintain a constant value of $\partial \mathbf{D}/\partial t$ long enough to measure the resulting magnetic field, as we would that due to a steady current. Thus Maxwell's postulate is not as susceptible to direct experimental verification as is Faraday's law. That is why it was the last fundamental law of classical electromagnetism to discover.

If Maxwell's postulate is converse of Faraday's law, then the question arises why is there not converse of Ampere's law ? If we put this quesiton in slight different manner we may say, why does equation (3) contains two terms, while the corresponding equation for e.m.f.

$$e = -\frac{d\phi}{dt} = -\iint \mathbf{B}\cdot d\mathbf{S} \qquad ...(5)$$

has but one term ?

The answer is that the term missing in equation (5) involves *a current density of magnetic current* or a flow of magnetic poles of one sign and since isolated poles of one sign and magnetic currents due to them have no physical significance, therefore the term analogous to **J** in equation (3) and the converse of Ampere's law do not exist. Therefore we must realise the fact that the fundamental role of electric charges leads to certain lack of symmetry in our equations.

8.5. Maxwell's Equations and Their Empirical Basis.

There are four fundamental equations of electromagnetism known as *Maxwell's equations* which may be written in *differential form* as

1. $\nabla\cdot\mathbf{D} = \rho$ (Differential form of Gauss law in electrostatics)
2. $\nabla\cdot\mathbf{B} = 0$ (Differential form of Gauss law in magnetostatics)
3. $\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$ Differential form of Faraday's law of electromagnetic induction
4. $\nabla\times\mathbf{H} = \mathbf{J} + \frac{\partial\mathbf{D}}{\partial t}$ Maxwell's modification of Ampere's law.

In above equations the notation have the following meanings :

D = electric displacement vector in coulomb/m^2.

ρ = charge density of coul/m^3.

B = magnetic induction in weber/m^2.

E = electric field intensity in volt/m or n/coul.

H = magnetic field intensity in amp/m-turn.

Each of Maxwell's equations represents a generalisation of certain experimental observations : Equations (1) represents the differential form of Gauss's law in electrostatics which in turn derives from Coulomb's law. Equation (2) represents Gauss's law in magnetostatics which is usually said to represent the fact that isolated magnetic poles do not exist in our physical world. Equation (3) represents differential form of Faraday's law of electromagnetic induction and finally equation (4) represents Maxwell's modification of Ampere's law to include time varying fields.

It is clear that the Maxwell's equations represent mathematical expression of certain experimental results. As already pointed out these equations can not be verified directly, however their application to any situation can be verified. As a result of extensive experimental work, Maxwell's equations are now known to apply to almost all macroscopic situations and they are usually used, must like conservation of momentum, as guiding principles.

8.6. Derivation of Maxwell's Equation

1. Derivation of first Equation Div $\mathbf{D} = \nabla \cdot \mathbf{D} = \rho$

Let us consider a surface S bounding a volume V in a dielectric medium. In a dielectric medium total charge consists of free charge plus polarisation charge. If ρ and ρ_p are the charge densities of free charge and polarisation charge at a point in a small volume element dV, then *Gauss' law* can be expressed as

$$\int_S \mathbf{E} \cdot d\mathbf{S} = \frac{1}{\varepsilon_0} \int_V (\rho + \rho_p)\, dV$$

But polarisation charge density $\rho_p = -\,\text{div}\,\mathbf{P}$, therefore above equaions takes the form

$$\int_S \mathbf{E} \cdot d\mathbf{S} = \frac{1}{\varepsilon_0} \int_V (\rho - \text{div}\,\mathbf{P})\, dV$$

i.e.
$$\int_S \varepsilon_0 \mathbf{E} \cdot d\mathbf{S} = \int_V \rho\, dV - \int_V \text{div}\,\mathbf{P}\, dV$$

Using Gauss divergence theorem to change surface integral into volume integral, we get

$$\int_V \text{div}\,(\varepsilon_0 \mathbf{E})\, dV = \int_V \rho\, dV - \int_V \text{div}\,P\, dV$$

i.e.
$$\int_V \text{div}\,(\varepsilon_0 \mathbf{E} + \mathbf{P})\, dV = \int_V \rho\, dV \qquad ...(1)$$

But $\varepsilon_0 \mathbf{E} + \mathbf{P} = \mathbf{D}$ = electric displacement vector.

Therefore equation (1) becomes

$$\int_V \text{div}\,\mathbf{D}\, dV = \int_V \rho\, dV$$

$$\int_S \text{div}\,(\mathbf{D} - \rho)\, dV = 0$$

Since this equation is true for all volumes, therefore the integrand in this equation must vanish *i.e.*

$$\text{div}\,\mathbf{D} - \rho = 0$$

or
$$\text{div}\,\mathbf{D} = \rho \quad i.e. \quad \nabla \cdot \mathbf{D} = \rho$$

2. Derivation of Second Equation div $\mathbf{B} = \nabla \cdot \mathbf{B} = 0$

Since isolated magnetic poles and magnetic currents due to them have no physical significance : therefore magnetic lines of force in general are either closed curves or go off to infinity. Consequently the number of magnetic lines of force entering any arbitrary closed surface is exactly the same as leaving it. It means that the flux of magnetic induction **B** across any closed surface is always zero, *i.e.*

$$\int_S \mathbf{B} \cdot d\mathbf{S} = 0$$

Using Gauss divergence theorem to change surface integral into volume integral, we get

$$\int_V \text{div}\,\mathbf{B}\, dV = 0$$

As the surface bounding the volume is arbitrary, therefore this equation holds only if the integrand vanishes *i.e.*

$$\text{div}\,\mathbf{B} = 0 \text{ or } \nabla \cdot \mathbf{B} = 0.$$

Note. For an alternative derivation of div $\mathbf{B} = 0$ refer section 5.10 of chapter 5.

3. Derivation of third equation

$$\text{curl}\,\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

According to Faraday's law of electromagnetic induction it is known that e.m.f. induced in a closed loop is defnied as negative rate of change of magnetic flux *i.e.*

$$e = -\frac{d\phi}{dt}$$

But magnetic flux $\phi = \int_S \mathbf{B} \cdot d\mathbf{S}$ where S is any surface having loop as boundary

$$e = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{S}$$

$$= -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}. \qquad \text{...(2)}$$

(Since surface is fixed in space, hence only **B** changes with time).

But e.m.f. '*e*' can also be computed by calculating the work done in carrying a unit charge round the closed loop *C*. Thus if **E** is the electric field intensity at a small element $d\mathbf{l}$ of loop, we have

$$e = \int_C \mathbf{E} \cdot d\mathbf{l} \qquad \text{...(3)}$$

Comparing equations (2) and (3), we get

$$\int_C \mathbf{E} \cdot d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}. \qquad \text{...(4)}$$

Using Stoke's theorem to change line integral into surface integral, we get

$$\int_S \text{curl } \mathbf{E} \cdot d\mathbf{S} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{E}$$

or

$$\int_S \left(\text{curl } \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t}\right) \cdot d\mathbf{S} = 0. \qquad \text{...(5)}$$

Since surface is arbitrary, therefore equation (5) holds only if the intergrand vanishes *i.e.*

$$\text{curl } \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

or

$$\text{curl } \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad i.e. \quad \nabla \times \mathbf{E} = \frac{\partial \mathbf{B}}{\partial t}.$$

4. Derivation of fourth equation

$$\text{curl } \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}.$$

For its derivation refer section 8.3.

8.7. Maxwell's Equations in Integral Form

Physical significance of Maxwell's Equations.

By means of Gauss' and Stoke's theorems we can put the field equations integral form and hence obtain their physical significance.

i. Maxwell's first equation is $\nabla \cdot \mathbf{D} = \rho$.

Integrating this over an arbitrary volume V, we get

$$\int_V \nabla \cdot \mathbf{D}\, dV = \int_V \rho\, dV.$$

Changing volume integral in L.H.S. of above equation into surface integral by Gauss divergence theorem. We get

$$\int_S \mathbf{D} \cdot d\mathbf{S} = \int_V \rho\, dV. \qquad \text{...(1)}$$

where S is the surface which bounds volume V. Equation (1) represents Maxwell's first equation $\nabla \cdot \mathbf{D} = \rho$ in integral form. Since $\int_V \rho\, dV = q$, the net charge contained in volume V, therefore Maxwell's first equation signifies that :

The net outward flux of electric displacement vector through the surface enclosing a volume is equal to the net charge contained within that volume.

2. Maxwell's second equation is $\nabla \cdot \mathbf{B} = 0$.

Integrating this over an arbitrary volume V, we get

$$\int_V \nabla \cdot \mathbf{B} = 0.$$

Using Gauss divergence theorem to change volume integral into surface integral, we get

$$\int_S \mathbf{B} \cdot d\mathbf{S} = 0. \qquad ...(2)$$

where S is the surface which bounds volume V. Equation (2) represents Maxwell's second equation in integral form and signifies that :

The net outward flux of magnetic induction **B** *through any closed surface is equal to zero.*

3. Maxwell's third equation is $\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$.

Integrating above equation over a surface S bounded by a curve C, we get

$$\int_S (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}.$$

Using Stoke's theorem to convert surface integral on L.H.S. of above equation into line integral along the boundary of C, we get

$$\int_C \mathbf{E} \cdot d\mathbf{l} = -\frac{\partial}{\partial t}\int_S \mathbf{B} \cdot d\mathbf{S}. \qquad ...(3)$$

Equation (3) represents Maxwell's third equation in integral form and signifies that

The electromotive force $\left(e.m.f.\ e = \int_C \mathbf{E} \cdot d\mathbf{l}\right)$ *around a closed path is equal to negative rate of change of magnetic flux linked with the path* $\left(\text{since magnetic flux } \phi = \int_S \mathbf{B} \cdot d\mathbf{S}\right)$.

4. Maxwell's fourth equation is

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}.$$

Taking surface integral over surface S bounded by curve C, we obtain

$$\int_S (\nabla \times \mathbf{H}) \cdot d\mathbf{S} = \int_S \left(\mathbf{J} + \frac{\partial \mathbf{H}}{\partial t}\right) \cdot d\mathbf{S}$$

Using Stoke's theorem to convert surface integral on L.H.S. of above equation into line integral, we get

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right) \cdot d\mathbf{S} \qquad ...(4)$$

This equation represents Maxwell's fourth equation in integral form and signifies that

The magnetomotive force $\left(m.m.f. = \oint_C \mathbf{H} \cdot d\mathbf{l}\right)$ *around a closed path is equal to the conduction current plus displacement current through any surface bounded by the path.*

8.8. Maxwell's Equations in Some Particular Cases.

Maxwell's equation in differential form are

$$\left.\begin{aligned}\nabla\cdot\mathbf{D} &= \rho\\ \nabla\cdot\mathbf{B} &= 0\\ \nabla\times\mathbf{E} &= -\frac{\partial\mathbf{B}}{\partial t}\\ \text{and}\quad \nabla\times\mathbf{H} &= \mathbf{J}+\frac{\partial\mathbf{D}}{\partial t}\end{aligned}\right\}\quad ...(A)$$

Case (i) Maxwell's Equations in Free Space. In free space the volume charge density ρ and current density **J** are zero, hence Maxwell's equations (A) take the form

$$\nabla\cdot\mathbf{D} = 0 \quad ...(1)$$

$$\nabla\cdot\mathbf{B} = 0 \quad ...(2)$$

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \quad ...(3)$$

and

$$\nabla\times\mathbf{H} = \frac{\partial\mathbf{D}}{\partial t} \quad ...(4)$$

with

$$\mathbf{D} = \varepsilon_0\,\mathbf{E} \text{ and } \mathbf{B} = \mu_0\,\mathbf{H} \quad ...(5)$$

where ε_0 and μ_0 are absolute permittivity and permeability of free space respectively.

Case (ii) Maxwell's equations in linear isotropic medium. In a linear isotropic medium, we have

$$\mathbf{D} = \varepsilon\mathbf{E} \text{ and } \mathbf{B} = \mu\mathbf{H} \quad ...(6)$$

where ε and μ are absolute permitivity and permeability of medium respectively.

Using equations (6), *Maxwell's equation (A) for linear isotropic medium take the form*

$$\nabla\cdot\mathbf{E} = \rho/\varepsilon \quad ...(7)$$

$$\nabla\cdot\mathbf{B} = 0 \quad ...(8)$$

$$\nabla\times\mathbf{E} = \mu\,\frac{\partial\mathbf{H}}{\partial t} \quad ...(9)$$

and

$$\nabla\times\mathbf{H} = \varepsilon\,\frac{\partial\mathbf{E}}{\partial t} + \mathbf{J}. \quad ...(10)$$

Case (iii) Maxwell's equations for harmonically varying fields. If electromagnetic fields vary harmonically with time that we may write

$$\mathbf{D} = \mathbf{D}_0\,e^{i\omega t} \text{ and } \mathbf{B} = \mathbf{B}_0\,e^{i\omega t} \quad ...(11)$$

where $\mathbf{D}_0$ and $\mathbf{B}_0$ are peak values of **D** and **B** respectively. Equations (11) yield

$$\left.\begin{aligned}\frac{\partial\mathbf{D}}{\partial t} &= \mathbf{D}_0\,i\omega\,e^{i\omega t} = i\omega\mathbf{D}\\ \frac{\partial\mathbf{B}}{\partial t} &= \mathbf{B}_0\,i\omega\,e^{i\omega t} = i\omega\mathbf{B}\end{aligned}\right\}\quad ...(12)$$

Using (12) Maxwell's equations (A) take the form

$$\nabla\cdot\mathbf{D} = \rho \quad ...(13)$$

$$\nabla\cdot\mathbf{B} = 0 \quad ...(14)$$

$$\nabla\times\mathbf{E} + i\omega\mathbf{B} = 0 \quad ...(15)$$

$$\nabla\times\mathbf{H} - i\omega\mathbf{D} = \mathbf{J}. \quad ...(16)$$

Ex. 1. *Starting from Maxwell's equations, establish Coulomb's law.* **(Meerut 1997)**

Solution. From Maxwell's equation

$$\text{div } D = \rho \quad ...(1)$$

Taking its volume integral over a sphere of radius r, we get

$$\int_V \operatorname{div} \mathbf{D}\, dV = \int_V \rho\, dv$$

As $\int_V \rho\, dv =$ Net charge enclosed by sphere

$$= q \text{ (say)}$$

$$\therefore \int_V \operatorname{div} \mathbf{D}\; dV = q$$

changing vaolume integral into surafce integral, we get

$$\int_S \mathbf{D} \,.\, d\mathbf{S} = q$$

But $\mathbf{D} = \varepsilon \mathbf{E}$

$$\therefore \int_S \varepsilon \mathbf{E} \cdot d\mathbf{S} = q \Rightarrow \varepsilon \int \mathbf{E} \cdot d\mathbf{S} = q$$

$$\Rightarrow \qquad \varepsilon E\, 4\pi r^2 = q \Rightarrow E = \frac{1}{4\pi\varepsilon} \frac{q}{r^2}$$

In vector form

$$\mathbf{E} = \frac{1}{4\pi\varepsilon} \frac{q\,\mathbf{r}}{r^3}$$

Force on charge q_0 will be $\mathbf{F} = q\mathbf{E}$

$$= \frac{1}{4\pi\varepsilon} \frac{q q_0}{r^3}\, r$$

This is Coulomb's law.

Ex. 2. *Show that equation of continuity div* $\mathbf{J} + \partial\rho/\partial t = 0$ *is contained in Maxwell's equations.*

or

Starting from Maxwell's equations, establish the equation of continuity. **(Meerut 1997)**

Solution. From Maxwell's fourth equation, we get

$$\operatorname{curl} \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \qquad \ldots(1)$$

Taking divergence of either sides of (1), we get

$$\operatorname{div} \operatorname{curl} \mathbf{H} = \operatorname{div} \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) \qquad \ldots(2)$$

But div curl $\mathbf{H} = 0$ since divergence of curl of any vector always vanishes, therefore equation (2) gives

$$\operatorname{div} \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) = 0$$

i.e.

$$\operatorname{div} \mathbf{J} + \operatorname{div} \frac{\partial \mathbf{D}}{\partial t} = 0$$

i.e.

$$\operatorname{div} \mathbf{J} + \frac{\partial}{\partial t} (\operatorname{div} \mathbf{D}) = 0,$$

(since space and time operations are interchangeable). Also from Maxwell's first equation div $\mathbf{D} = \rho$, therefore equation (8) gives

$$\operatorname{div} \mathbf{J} + \frac{\partial \rho}{\partial t} = 0$$

This is the required result.

Ex. 3. *Starting from Maxwell's equations.*

Curl $\mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$ and curl $\mathbf{H} = \mathbf{J} + \dfrac{\partial \mathbf{D}}{\partial t}$ *respectively, show that div* $\mathbf{B} = 0$ *and* div $\mathbf{E} = \rho$.

Solution. Given curl $\mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$

Taking divergence of both sides

$$\text{div curl } \mathbf{E} = -\text{div}\left(\frac{\partial \mathbf{B}}{\partial t}\right)$$

As div curl of any vector is zero and space and time operations are interchangeable,

$$\text{div}\frac{\partial \mathbf{B}}{\partial t} = 0 \Rightarrow \frac{\partial}{\partial t}(\text{div } \mathbf{B}) = 0$$

$$\Rightarrow \text{div } \mathbf{B} = \text{constant}$$

As isolated magnetic poles do not exist in nature, therefore for $\boxed{\text{div } \mathbf{B} = 0}$

Given $$\text{curl } \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

Taking div of both sides

$$\text{div curl } \mathbf{H} = \text{div}\left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right)$$

As $$\text{div curl } \mathbf{H} = 0$$

$\Rightarrow$ $$\text{div } \mathbf{J} + \text{div}\left(\frac{\partial \mathbf{D}}{\partial t}\right) = 0$$

$$\text{div } \mathbf{J} = -\text{div}\left(\frac{\partial \mathbf{D}}{\partial t}\right)$$

But div $\mathbf{J} + \dfrac{\partial q}{\partial t} = 0$ (equation of continuity)

$\therefore$ $$-\text{div}\left(\frac{\partial \mathbf{D}}{\partial t}\right) + \frac{\partial \rho}{\partial t} = \Rightarrow \frac{\partial}{\partial t}(\text{div } \mathbf{D}) = \frac{\partial \rho}{\partial t}$$

on integrating $$\boxed{\text{div } \mathbf{D} = \rho}$$

8.9. Electromagnetic Energy, Poynting Theorem.

In preceding chapters we have seen that

electromagnetic potential energy, $U_c = \frac{1}{2}\int_V \mathbf{E} \cdot \mathbf{B} \, dv$ ...(1)

and energy stored in a magnetic field $U_m = \frac{1}{2}\int_V \mathbf{H} \cdot \mathbf{B} \, dv$ ...(2)

Now let us see whether these expressions apply to non-static situations.

Maxwell's equations in differential form are

$$\nabla \cdot \mathbf{D} = \rho \text{ or } \text{div } \mathbf{D} = \rho \quad ...(3)$$

$$\nabla \cdot \mathbf{B} = 0 \text{ or } \text{div } \mathbf{B} = 0 \quad ...(4)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{D}}{\partial t} \text{ or } \text{curl } \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad ...(5)$$

and $$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \text{ or } \text{curl } \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \quad ...(6)$$

Taking scalar product of equation (5) with **H** and equation (6) with **E**, we get

$$\mathbf{H} \cdot \text{curl } \mathbf{E} = -\mathbf{H}\frac{\partial \mathbf{B}}{\partial t} \quad ...(7)$$

and
$$\mathbf{E}\cdot\operatorname{curl}\mathbf{H} = \mathbf{E}\cdot\mathbf{J} + \mathbf{E}\cdot\frac{\partial \mathbf{D}}{\partial t} \qquad ...(8)$$

Now subtracting equation (8) from equation (7), we get

$$\mathbf{H}\cdot\operatorname{curl}\mathbf{E} - \mathbf{E}\cdot\operatorname{curl}\mathbf{H} = -\,\mathbf{H}\cdot\frac{\partial \mathbf{B}}{\partial t} - \mathbf{E}\cdot\frac{\partial \mathbf{D}}{\partial t} - \mathbf{E}\cdot\mathbf{J}$$
$$= -\left(\mathbf{H}\cdot\frac{\partial \mathbf{B}}{\partial t} + \mathbf{E}\cdot\frac{\partial \mathbf{D}}{\partial t}\right) - \mathbf{E}\cdot\mathbf{J} \qquad ...(9)$$

Using vector identity.
$$\operatorname{div}(\mathbf{E}\times\mathbf{H}) = \mathbf{H}\cdot\operatorname{curl}\mathbf{E} - \mathbf{E}\cdot\operatorname{curl}\mathbf{H} \qquad ...(10)$$

equation (9) may be expressed as
$$\operatorname{div}(\mathbf{E}\times\mathbf{H}) = -\left(\mathbf{H}\cdot\frac{\partial \mathbf{B}}{\partial t} + \mathbf{E}\cdot\frac{\partial \mathbf{D}}{\partial t}\right) - \mathbf{E}\cdot\mathbf{J} \qquad ...(11)$$

Now if the medium is *linear* so that the relations
$$\mathbf{B} = \mu\mathbf{H} \quad\text{and}\quad \mathbf{D} = \varepsilon\mathbf{E} \qquad ...(12)$$

apply, then we may write
$$\mathbf{E}\cdot\frac{\partial \mathbf{D}}{\partial t} = \mathbf{E}\cdot\frac{\partial}{\partial t}(\varepsilon\mathbf{E}) = \frac{1}{2}\,\varepsilon\,\frac{\partial}{\partial t}(\mathbf{E})^3 = \frac{\partial}{\partial t}\left(\tfrac{1}{2}\,\mathbf{E}\cdot\mathbf{D}\right)$$

and
$$\mathbf{H}\cdot\frac{\partial \mathbf{B}}{\partial t} = \mathbf{H}\cdot\frac{\partial}{\partial t}(\mu\mathbf{H}) = \frac{1}{2}\,\mu\,\frac{\partial}{\partial t}(\mathbf{H})^2 = \frac{\partial}{\partial t}\left(\tfrac{1}{2}\,\mathbf{H}\cdot\mathbf{B}\right)$$

Using the relationships, equation (11) takes the form
$$\operatorname{div}(\mathbf{E}\times\mathbf{H}) = -\frac{\partial}{\partial t}\left[\frac{1}{2}(\mathbf{E}\cdot\mathbf{D} + \mathbf{H}\cdot\mathbf{B})\right] - \mathbf{J}\cdot\mathbf{E} \qquad ..(13)$$

Each term in above equation has certain physical significance which may be seen by integrating equation (13) over a volume V bounded by surface S. Thus
$$\int_V \operatorname{div}(\mathbf{E}\times\mathbf{H})\,dv = -\int_V \left\{\frac{\partial}{\partial t}\frac{1}{2}(\mathbf{E}\cdot\mathbf{D} + \mathbf{H}\cdot\mathbf{B})\right\} dv - \int_V \mathbf{J}\cdot\mathbf{E}\,dv$$

Using Gauss divergence theorem to change volume integral L.H.S. of above equation into surface integral, we get
$$\oint_S (\mathbf{E}\times\mathbf{H})\cdot d\mathbf{S} = -\frac{d}{dt}\int_V \tfrac{1}{2}(\mathbf{E}\cdot\mathbf{D} + \mathbf{H}\cdot\mathbf{B})\,dv - \int_V \mathbf{J}\cdot\mathbf{E}\,dv$$

Rearranging this equation, we get
$$-\int_V \mathbf{J}\cdot\mathbf{E}\,dv = \frac{d}{dt}\int_V \tfrac{1}{2}(\mathbf{E}\cdot\mathbf{D} + \mathbf{H}\cdot\mathbf{B})\,dv + \oint_S (\mathbf{E}\times\mathbf{H})\cdot d\mathbf{S} \qquad ...(14)$$

To understand the physical significance of above equation, let us now interpret each term in it.

Interpretation of $\int_V \mathbf{J}\cdot\mathbf{E}\,dv$. To understand the meaning of this term let us consider a charged particle q moving with velocity $\mathbf{v}$ under the combined effect of mechanical, electric and magnetic forces.

The electromagnetic force due to field vectors $\mathbf{E}$ and $\mathbf{B}$ acting on the charged particle is
$$\mathbf{F} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B})$$

As the magnetic force $q(\mathbf{v}\times\mathbf{B})$ is always perpendicular to velocity, hence the magnetic field does no work. Therefore for a single charge q the rate of doing work by electromagnetic field $\mathbf{E}$ and $\mathbf{B}$ is
$$\frac{\partial W}{\partial t} = \mathbf{F}\cdot\mathbf{v} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B})\cdot\mathbf{v} = q\,\mathbf{E}\cdot\mathbf{v}$$

If $\mathbf{F}_m$ is the mechanical force, then work done by *mechanical force against electromagnetic field vectors* per unit time *i.e.* the rate at which mechanical work is done on the particle is

$$\frac{\partial W_m}{\partial t} = \mathbf{F}_m \cdot \mathbf{v} = -\mathbf{F} \cdot \mathbf{v} = -q\,\mathbf{E} \cdot \mathbf{v} \qquad \text{...(15)}$$

If the electromagnetic field consists of a group of charges moving with different velocities *e.g.* n_i charge carriers each of charge q_i moving with velocity $\mathbf{v}_i$ $(i = 1, 2, 3, ...)$; the equation (15) may be written as

$$\frac{\partial W_m}{\partial t} = -\sum_i n_i q_i \mathbf{v}_i \cdot \mathbf{E}_i \qquad \text{...(16)}$$

In this case current density $\qquad \mathbf{J} = \sum_i \mathbf{J}_i = \sum_i n_i q_i \mathbf{v}_i$

using this substitution, equation (16) becomes

$$\frac{\partial W_m}{\partial t} = -\sum \mathbf{J}_i \cdot \mathbf{E}_i = -\mathbf{J} \cdot \mathbf{E} \qquad \text{...(17)}$$

This equation represents the *power density* that it transferred into electromagnetic field.

Therefore the expression $-\int \mathbf{E} \cdot \mathbf{J}\, dv$ represents ***rate of energy transferred into the electromagnetic field through the motion of free charge in volume V.***

If there are no sources of e.m.f. in volume V, then the term

$$-\int \mathbf{E} \cdot \mathbf{J}\, dv = -\int \frac{\mathbf{J}^3}{\sigma}\, dv \qquad (\text{Since } \mathbf{J} = \sigma\mathbf{E})$$

is negative and represents negative of rate of heat energy produced

Interpretation of $\frac{d}{dt}\int \frac{1}{2}(\mathbf{E} \cdot \mathbf{D} + \mathbf{H} \cdot \mathbf{B})\, dv$; We know

$$\int_V \frac{1}{2}\mathbf{E} \cdot \mathbf{D}\, dv = U_e, \qquad \text{electrostatic potential energy in volume } V$$

$$\int_V \frac{1}{2}\mathbf{H} \cdot \mathbf{B}\, dv = U_m, \qquad \text{magnetic energy in volume } V$$

$\therefore$ Obviously

$$U = \int_V \frac{1}{2}(\mathbf{E} \cdot \mathbf{D} + \mathbf{H} \cdot \mathbf{B})\, dv \qquad \text{...(18)}$$

represents some sort of potential energy of electromagnetic field. One need not ascrible this potential energy to the charged particles but consider this term as a *field energy.* This is known as *electromagnetic field energy* in volume *V. A concept such as energy stored in the field itself rather than residing with the particles is a basic concept of electromagnetic theory.* Obviously $\frac{1}{2}(\mathbf{E} \cdot \mathbf{D} + \mathbf{H} \cdot \mathbf{B})$ represents energy density of electromagnetic field *i.e.*

$$u = \frac{1}{2}(\mathbf{E} \cdot \mathbf{D} + \mathbf{D} \cdot \mathbf{B}) \qquad \text{...(19)}$$

Consequently the term $\frac{d}{dt}\int \frac{1}{2}(\mathbf{E} \cdot \mathbf{D} + \mathbf{H} \cdot \mathbf{B})\, dv$ *represents the* rate *of electromagnetic energy stored in volume V.*

Interpretation of $\oint_S (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{S}$ **:** Since surface integral in this term involves only electric and magnetic fields, it is feasible to interpret this term as the rate of energy flow across the surface. It means that $(\mathbf{E} \times \mathbf{H})$ itself represents the energy flow per unit time per unit area. The latter interpretation, however, leads to certain difficulties : the only interpretation which survives is that the *surface integral of* $(\mathbf{E} \times \mathbf{H})$

over a closed surface represents the amount of electromagnetic energy crossing the closed surface per second. The vector (**E** × **H**) is known as the *Poynting vector* and usually represented by the symbol **S** *i.e.*

Poynting vector $$\mathbf{S} = \mathbf{E} \times \mathbf{H} \qquad ...(20)$$

Interpretation of Energy Equation (13) or (14)

In view of above interpretations, equation (13) may be expressed as

$$-\mathbf{J} \cdot \mathbf{E} = \frac{\partial u}{\partial t} + \nabla \cdot \mathbf{S} \qquad ...(21)$$

The physical meaning of this equation is that the *time rate of change of electromagnetic energy with a certain volume plus time rate of the energy flowing out through the boundary surface is equal to the power transferred into the electromagnetic field.*

This is the statement of conservation of energy in electromagnetism and is known as **Poynting theorem.**

8.10. Poynting Vector

In preceding section we have seen that

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} \qquad ...(1)$$

is known as Poynting vector and is interpreted as the power flux *i.e.* amount of energy crossing unit area placed perpendicular to the vector, per unit time. The conception of energy of the electromagnetic field as residing in the medium is very fundamental one and has great advantage in the development of the theory. Maxwell thought of the medium as resembling an elastic solid, the electrical energy representing the potential energy of strain of the medium, the magnetic energy the kinetic energy of motion. Though such a mechanical view no longer exists, still the energy is regarded as being localised in space and as travelling in the manner indicated by Poynting vector. In a light wave there is certain energy per unit volume, proportional to the square of the amplitude (E or H). This energy travels along and Poynting vector is the vector that measures the rate of flow or the intensity of the wave. In a plane electromagnetic wave **E** and **H** are at right angles to each other and at right angles to the direction of flow ; thus **E** × **H** must be along the direction of flow. In more complicated waves as well, Poynting vector points along the direction of flow of radiation. For example if we have a source of light and we wish to find at what rate it is emitting energy, we surround it by a closed surface and integrate the normal component of Poynting vector over the surface. The whole conception of energy being transported in the medium is fundamental to the electromagnetic theory of light.

In case of time varying field **S** = **E** × **H** *gives the instantaneous value of the Poynting vector.* Let us find the form of Poyting vector for such cases. Let *the fields* **E** *and H be given by real parts of complex exponentials of the form*

$$\mathbf{E} = \mathbf{E}_0(\mathbf{r})\, e^{i\omega t}$$

At a given point of space let us assume that **E** is given by the real part of $\mathbf{E}_0 e^{i\omega t}$ and **H** by real part of $\mathbf{H}_b e^{i\omega t}$ where $\mathbf{E}_0$ and $\mathbf{H}_0$ are complex vector functions of position. Let the real and imaginary parts of $\mathbf{E}_0$ be denoted by $\mathbf{E}_r$ and $\mathbf{H}_{im}$ respectively. Similarly real and imaginary parts of $\mathbf{H}_0$ are $\mathbf{H}_r$ and $\mathbf{E}_{im}$. Then

$$\mathbf{E} = \text{Real part of } (\mathbf{E}_0 e^{i\omega t}) = \text{Re}\,(\mathbf{E}_0 e^{i\omega t})$$

where Re denotes '*real part of*'

$$\therefore \quad \mathbf{E} = \text{Re}\,(\mathbf{E}_r + i\,\mathbf{E}_{im})(\cos \omega t + i \sin \omega t) = \mathbf{E}_r \cos \omega t - \mathbf{E}_{im} \sin \omega t$$

Similarly, $\mathbf{H} = \mathbf{H}_r \cos \omega t - \mathbf{H}_{im} \sin \omega t$

Then *Poynting vector is*

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} = (\mathbf{E}_r \times \mathbf{H}_r)\cos^2 \omega t + (\mathbf{E}_{im} \times \mathbf{H}_{im})\sin^2 \omega t$$
$$- [(\mathbf{E}_r \times \mathbf{H}_{im}) + (\mathbf{E}_{im} \times \mathbf{H}_r)] \sin \omega t \cos \omega t$$

We notice that there are two types of terms in the above expression ; the first two whose time average is different from zero, since $\cos^2 \omega t$ and $\sin^2 \omega t$ average to $\frac{1}{2}$; and the last term whose time average is zero since $\sin \omega t \cos \omega t$ averages to zero. Thus the *time average* of Poynting vector (average being) taken over a complete cycle) is

$$< \mathbf{S} > = < \mathbf{E} \times \mathbf{H} > = \tfrac{1}{2}[(\mathbf{E}_r \times \mathbf{H}_r) + (\mathbf{E}_{im} \times \mathbf{H}_{im})] \qquad ...(2)$$

This equation can be written in a convenient way by using the notation of complex conjugates, where the complex conjugate of a complex number is the number obtained from the original one by changing the sign of i, wherever it appears is indicated by a (*) over the number. In terms of this notation

$$(\mathbf{E} \times \mathbf{H}^*) = (\mathbf{E}_0 e^{i\omega t}) \times (\mathbf{H}_0^* e^{-i\omega t})$$
$$= \mathbf{F}_0 \times \mathbf{H}_0^* = (\mathbf{E}_r + i\,\mathbf{E}_{im}) \times (\mathbf{H}_r - i\,\mathbf{H}_{im})$$
$$= (\mathbf{E}_r \times \mathbf{H}_r + \mathbf{E}_{im} \times \mathbf{H}_{im}) + i\,(\mathbf{E}_{im} \times \mathbf{H}_r - \mathbf{E}_r \times \mathbf{H}_{im}) \qquad ...(3)$$

Comparing (2) and (3), we note that, except for the factor $\frac{1}{2}$, the real part of equation (3) is just the same as the quantity appearing in equation (2). That is, we have

$$< \mathbf{S} > = < \mathbf{E} \times \mathbf{H} > = \tfrac{1}{2}\, Re\,(\mathbf{E} \times \mathbf{H}^*) \qquad ...(4)$$

where **E** and **H** appearing on the R.H.S. of above equation are the complex quantities whose real parts give the real **E** and **H** appearing on the L.H.S. of the equation.

Similarly we may show that

Average electrostatic energy density.

$$< u_e > = < \tfrac{1}{2}\varepsilon E^2 > = \tfrac{1}{4}\varepsilon \mathbf{E} \cdot \mathbf{E}^* \qquad ...(5)$$

and average magnetic energy density,

$$< u_m > = < \tfrac{1}{2}\mu H^2 > = \tfrac{1}{4}\mu \mathbf{H} \cdot \mathbf{H}^* \qquad ...(6)$$

Ex. 4. *Calculate the magnitude of Poynting vector at the surface of the sun. Given that power radiated by sun* = $3{\cdot}8 \times 10^{26}$ *watts and radius of sun* = 7×10^8 *m.*

Solution. From definition of Poynting vector S, it is the power radiated per unit area and surface area of sun is $4\pi R^2$, R being radius of sun.

$\therefore$ If P is the total power radiated by sun.

$$P = S 4\pi R^2$$

i.e.
$$S = \frac{P}{4\pi R^2} = \frac{3{\cdot}8 \times 10^{26}}{4 \times 3{\cdot}14 \times (7 \times 10^8)^2} = 6{\cdot}175 \times 10^7 \text{ watt/m}^2.$$

Ex. 5. *If the average distance between the sun and earth is* $1{\cdot}5 \times 10^{11}$ *m, thow that the average solar energy incident on the earth is* ≈ 2 *cal/cm²-min (called the solar constant).* **(Meerut 1969)**

Solution. If r is the distance between sun and earth, and Poynting vector at the surface of earth, then

$$S_e . 4\pi r^2 = P$$

Energy stored in capacitor $U = \frac{1}{2}\varepsilon_0 E^2 \pi a^2 l$

Rate of increase of electrical energy

$$R = \frac{dV}{dt}$$

$$= \frac{1}{2}\varepsilon_0 2E\frac{\partial E}{dt}(\pi a^2 l)$$

$$= \varepsilon_0 \pi a^2 l E \frac{\partial E}{\partial t} \qquad ...(3)$$

Clearly (2) and (3) are identical.

8.11. The Wave Equation

We shall now derive the equations for electromagnetic waves be the use of Maxwell's equations. This is one of the most important applications of Maxwell's equations.

Let us consider a uniform linear medium having permitivity ε, permeability μ and conductivity σ; but not any charge or any current other than that determined by Ohm's law. Then

$$\mathbf{D} = \varepsilon\mathbf{E}\,;\ \mathbf{B} = \mu\mathbf{H}\,;\ \mathbf{J} = \sigma\mathbf{E} \text{ and } \rho = 0. \qquad ...(1)$$

So the Maxwell's equations

$$\left.\begin{aligned} \operatorname{div}\mathbf{D} &= \rho \\ \operatorname{div}\mathbf{B} &= 0 \\ \operatorname{curl}\mathbf{E} &= -\frac{\partial\mathbf{B}}{\partial t} \\ \text{and} \qquad \operatorname{curl}\mathbf{H} &= \mathbf{J} + \frac{\partial\mathbf{D}}{\partial t} \end{aligned}\right\} \qquad ...(2)$$

in this case take the form

$$\operatorname{div}\mathbf{E} = 0 \qquad ...(3)$$

$$\operatorname{div}\mathbf{H} = 0 \qquad ...(4)$$

$$\operatorname{curl}\mathbf{E} = -\mu\frac{\partial\mathbf{H}}{\partial t} \qquad ...(5)$$

and

$$\operatorname{curl}\mathbf{H} = \sigma E + \varepsilon\frac{\partial\mathbf{E}}{\partial t} \qquad ...(6)$$

Taking curl of equation (5), we get

$$\operatorname{curl}\operatorname{curl}\mathbf{E} = -\mu\frac{\partial}{\partial t}(\operatorname{curl}\mathbf{H})$$

Substituting curl **H** from equation (6), we get

$$\operatorname{curl}\operatorname{curl}\mathbf{E} = -\mu\frac{\partial}{\partial t}\left(\sigma\mathbf{E} + \varepsilon\frac{\partial\mathbf{E}}{\partial t}\right)$$

i.e.

$$\operatorname{curl}\operatorname{curl}\mathbf{E} = -\sigma\mu\frac{\partial\mathbf{E}}{\partial t} - \varepsilon\mu\frac{\partial^2\mathbf{E}}{\partial t^2} \qquad ...(7)$$

Similarly, if we take the curl of equation (6) and substitute **E** from equation (5), we obtain

$$\operatorname{curl}\operatorname{curl}\mathbf{H} = -\sigma\mu\frac{\partial\mathbf{H}}{\partial t} - \varepsilon\mu\frac{\partial^2\mathbf{H}}{\partial t^2} \qquad ...(8)$$

Now using vector identity

$$\text{curl curl } \mathbf{A} = \text{grad div } \mathbf{A} - \nabla^2 \mathbf{A}$$

and keeping in view equations (3) and (4) (*i.e.*, div $\mathbf{E} = 0$ and div $\mathbf{H} = 0$) ; equation (7) and (8) take the form

$$\nabla^2 \mathbf{E} - \sigma\mu \frac{\partial \mathbf{E}}{\partial t} - \varepsilon\mu \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

and $$\nabla^2 \mathbf{H} - \sigma\mu \frac{\partial \mathbf{H}}{\partial t} - \varepsilon\mu \frac{\partial^2 \mathbf{H}}{\partial t^2} = 0. \qquad ...(10)$$

Equations (9) and (10) represent wave equations which govern the electromagnetic field in a homogeneous, linear medium in which the charge density is zero; whether this medium is conducting or non-conducting. However, it is not enough that these equations be satisfied; Maxwell's equations must also be satisfied. It is clear that equations (9) and (10) are consequence of Maxwell's equations; but the converse is not true. Now the problem is to solve wave equations (9) and (10) in such a manner the Maxwell's equations are also satisfied. One method that works very well for *monochromatic wave (i.e.* waves characterised by a single frequency) is to obtain a solution for **E.** Then curl **E** will give time derivative of **B** $\left(\text{since curl } E = -\frac{\partial \mathbf{B}}{\partial t}\right)$ so that **B** can be computed.

It is more convenient to use the method of complex variable analysis for the solution of wave equations. The time dependence of the field (for certainty we take vector **E**) is taken to be $e^{-i\omega t}$, so that

$$\mathbf{E}(\mathbf{r}, t) = E_S(\mathbf{r})\, e^{-i\omega t} \qquad ...(11)$$

It may be noted that the *physical electric field is obtained by taking the real part of* (11) : furthermore $\mathbf{E}_S(\mathbf{r})$ is in general complex so that the actual electric field is proportional to cos $(\omega t + \phi)$, where ϕ is phase of $\mathbf{E}_S(\mathbf{r})$. Using equation (11), equation (9) (droping common factor $e^{-i\omega t}$) gives

that $$\nabla^2 \mathbf{E}_S + \omega^2 \varepsilon\mu\, \mathbf{E}_S + i\omega\sigma\mu\, \mathbf{E}_S = 0 \qquad ...(12)$$

Here the spatial electric field $\mathbf{E}_S$ depends on the space co-ordinates *i.e.*

$$\mathbf{E}_S = \mathbf{E}_S(\mathbf{r}).$$

For plane electromagnetic waves it is convenient to put

$$\mathbf{E}_S = \mathbf{E}_0\, e^{i\,\mathbf{k}\cdot\mathbf{r}}$$

where **k** is the propagation wave vector defined as

$$\mathbf{k} = \frac{2\pi}{\lambda}\,\mathbf{n} = \frac{\omega}{v}\,\mathbf{n}, \ \mathbf{n} \text{ being unit vector along } \mathbf{k}$$

and **r** is position vector from origin, **v** is the phase velocity of the wave.

With this in mind, equation (11) may be written as

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0\, e^{i\,\mathbf{k}\cdot\mathbf{r} - i\omega t} \qquad ...(13)$$

Here $\mathbf{E}_0$ is complex amplitude and is constant in space and time. *It is important to note that when field vector is in form* (13), *i.e.* operation of grad, div and curl on field vector is equivalent to

$$\left.\begin{aligned} &\text{grad} \rightarrow i\mathbf{k}\ ;\ \text{div} = \nabla\cdot - i\,\mathbf{k}\cdot\ ;\ \text{curl} = \nabla\times \rightarrow i\mathbf{k}\times \\ &\text{Also } \frac{\partial}{\partial t} \rightarrow -\,i\omega. \end{aligned}\right\} \qquad ...(14)$$

Now we shall conside various cases of interest to determine field vectors **E** and **H** in electromagnetic field.

8.12. Plane Electromagnetic Waves in Free Space.

Maxwell's equaitons are

$$\left.\begin{aligned} \text{div}\,\mathbf{D} &= \nabla\cdot\mathbf{D} = \rho \\ \text{div}\,\mathbf{B} &= \nabla\cdot\mathbf{B} = 0 \\ \text{curl}\,\mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\ \text{and} \qquad \text{curl}\,\mathbf{H} &= \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \end{aligned}\right\} \text{ and } \begin{aligned} \mathbf{B} &= \mu\mathbf{H} \\ \mathbf{D} &= \varepsilon\mathbf{E} \\ \mathbf{J} &= \sigma\mathbf{E} \end{aligned} \qquad ...(1)$$

Free space is characterised by

$$\rho = 0,\ \sigma = 0,\ \mu = \mu_0 \text{ and } \varepsilon = \varepsilon_0 \qquad ...(2)$$

Therefore Maxwell's equations reduce to

$$\left.\begin{aligned} \text{div}\,\mathbf{E} &= 0 & ...(a) \\ \text{div}\,\mathbf{H} &= 0 & ...(b) \\ \text{curl}\,\mathbf{E} &= -\mu_0 \frac{\partial \mathbf{H}}{\partial t} & ...(c) \\ \text{and} \qquad \text{curl}\,\mathbf{H} &= \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}. & ...(d) \end{aligned}\right\} \qquad ...(3)$$

Taking curl of equation 3(c), we get

$$\text{curl curl}\,\mathbf{E} = -\mu_0 \frac{\partial}{\partial t}(\text{curl}\,\mathbf{H})$$

Substituting curl **H** from [3(d)], we get

$$\text{curl curl}\,\mathbf{E} = \mu_0 \frac{\partial}{\partial t}\left(\varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right)$$

i.e.

$$\text{curl curl}\,\mathbf{E} = -\mu_0\varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} \qquad ...(4)$$

Now

$$\text{curl curl}\,\mathbf{E} = \text{grad div}\,\mathbf{E} - \nabla^2\mathbf{E}$$

i.e.

$$\text{curl curl}\,\mathbf{E} = -\nabla^2\mathbf{E} \qquad [\text{since div}\,\mathbf{E} = 0 \text{ from 3(a)}]$$

Making this substitution equation (4) becomes

$$\nabla^2\mathbf{E} - \mu_0\varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0, \qquad ...(5)$$

Now taking curl of equation [3(d)], we get

$$\text{curl curl}\,\mathbf{H} = \varepsilon_0 \frac{\partial}{\partial t}(\text{curl}\,\mathbf{E}).$$

Substituting curl from [3(c)], we get

$$\text{curl curl}\,\mathbf{H} = \varepsilon_0 \frac{\partial}{\partial t}\left(-\mu_0 \frac{\partial \mathbf{H}}{\partial t}\right) = -\mu_0\varepsilon_0 \frac{\partial^2 \mathbf{H}}{\partial t^2} \qquad ...(6)$$

Again using identity curl curl **H** = grad div **H** − ∇^2**H** and noting that div **H** = 0 from [3(b)], we obtain

$$\text{curl curl}\,\mathbf{H} = -\nabla^2\mathbf{H}.$$

Making this substitution in equation (6), we get

$$\nabla^2 \mathbf{H} - \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{H}}{\partial t^2} = 0. \qquad ...(7)$$

Equations (5) and (7) represent *wave equations governing electromagnetic fields* **E** *and* **H** *in free space.* It may be noted that these equations may be obtained by using (2) in equations (9) and (10) of preceding section. Equations (5) and (7) are vector equations of identical form which means that each of the six components of **E** and **H** separately satisfies the same scalar wave equation of the form

$$\nabla^2 u - \mu_0 \varepsilon_0 \frac{\partial^2 u}{\partial t^2} = 0 \qquad ...(8)$$

where u is a scalar *and can stand for one of the components of* **E** and **H.** It is obvious that equation (8) resembles with the general wave equation

$$\nabla^2 u = \frac{1}{v^2} \frac{\partial^2 u}{\partial t^2} \qquad ...(9)$$

where v is the velocity of wave.

Comparing (8) and (9), we see that *the field vectors* **E** *and* **H** *are propagated in free space as waves at a speed equal to*

$$v = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \qquad \text{Since } \mu_0 = 4\pi \times 10^{-7} \text{ weber/Amp–m}$$

$$\varepsilon_0 = 8{\cdot}542 \times 10^{-12} \text{ farad/m}$$

$$= \sqrt{\left(\frac{4\pi}{\mu_0 . 4\pi\varepsilon_0}\right)}$$

So that $\frac{1}{4\pi\varepsilon_0} = 9 \times 10^9$ m/farad.

$$= \sqrt{\left(\frac{4\pi}{4\pi \times 10^{-7}} \times 9 \times 10^9\right)}$$

$$= 3 \times 10^8 \text{ m/sec} = c, \text{ *the speed of light.*}$$

Therefore it is reasonable to write c the speed of light in place of $\frac{1}{\sqrt{\mu_0 \varepsilon_0}}$; so equations (5) and (7) take the form

$$\nabla^2 \mathbf{E} - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \qquad ...(10)$$

$$\nabla^2 \mathbf{H} - \frac{1}{c} \frac{\partial^2 \mathbf{H}}{\partial t^2} = 0 \qquad ...(11)$$

and

$$\nabla^2 u - \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} = 0. \qquad ...(12)$$

Now let us find the solution of above equations for plane electromagnetic waves. ***A plane wave is defined as a wave whose amplitude is the same at any point in a plane perpendicular to a specified direction.***

The plane wave solutions of above equations in well known form may be written as

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t} \qquad ...(13)$$

$$\mathbf{H}(\mathbf{r}, t) = \mathbf{H}_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t} \quad ...(14)$$

$$u(\mathbf{r}, t) = u_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t} \quad ...(15)$$

where $\mathbf{E}_0$, $\mathbf{H}_0$ and u_0 are complex amplitudes which are constant in space and time while $\mathbf{k}$ is a wave propagation vector denoted as

$$\mathbf{k} = k\mathbf{n} = \frac{2\pi}{\lambda}\mathbf{n} = \frac{2\pi\nu}{c}\mathbf{n} = \frac{\omega}{c}\mathbf{n} \quad ...(16)$$

Here $\mathbf{n}$ is a unit vector in the direction of wave propagation. Now in order to apply the conditions $\nabla \cdot \mathbf{E} = 0$ and $\nabla \cdot \mathbf{H} = 0$, let us first find $\nabla \cdot \mathbf{E}$ and $\nabla \cdot \mathbf{H}$.

$$\nabla \cdot \mathbf{E} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot [(\hat{i}E_{0x} + \hat{j}E_{0y} + \hat{k}E_{0z})\, e^{i(k_x x + k_y y + k_z z) - i\omega t}]$$

$$[\text{since } \mathbf{k}\cdot\mathbf{r} = (\hat{i}k_x + \hat{j}k_y + \hat{k}k_z)\cdot(\hat{i}x + \hat{j}y + \hat{k}z)$$

$$= [k_x x + k_y y + k_z z]$$

$$\therefore \quad \nabla \cdot \mathbf{E} = (E_{0x}\, ik_x + E_{0y}\, ik_z\, E_0)\, e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}$$

$$= i(k_x E_{0x} + k_y E_{0y} + k_z E_{0z})\, e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}$$

$$= i(\hat{i}k_x + \hat{j}k_y + \hat{k}k_z)\cdot(\hat{i}E_{0x} + \hat{j}E_{0y} + \hat{k}E_{0z})\, e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}$$

$$= i\mathbf{k}\cdot\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t} = i\mathbf{k}\cdot\mathbf{E}$$

Similarly $$\nabla \cdot \mathbf{H} = i\mathbf{k}\cdot\mathbf{H}$$

Thus the requirements $\nabla \cdot \mathbf{E} = 0$ and $\nabla \cdot \mathbf{H} = 0$ demand that

$$\mathbf{k}\cdot\mathbf{E} = 0 \text{ and } \mathbf{k}\cdot\mathbf{H} = 0 \quad ...(17)$$

This means that ***electromagnetic field vectors* E *and* H *are both perpendicular to the direction of propagation vector* k*. This implies that electromagnetic waves are transverse in character.*** Further restrictions are provided by curl equations (3c) and 3(d) viz.

$$\text{curl } \mathbf{E} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t} \quad \text{and} \quad \text{curl } \mathbf{H} = \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

using equations (13) and (14), above equations yield

$$i\mathbf{k} \times \mathbf{E} = -\mu_0\cdot(-i\omega\mathbf{H}) \text{ or } \mathbf{k} \times \mathbf{E} = \mu_0\omega\mathbf{H} \quad ...(18)$$

and $$i\mathbf{k} \times \mathbf{H} = \varepsilon_0\cdot(-i\omega\mathbf{E}) \text{ or } \mathbf{k} \times \mathbf{H} = -\varepsilon_0\omega\mathbf{E}. \quad ...(19)$$

From equation (18) it is obvious that field vector **H** is perpendicular to both **k** and **E** and according to equation (19) **E** perpendicular to both **k** and **H**. This simply means that *field vectors* **E** *and* **H** *are mutually perpendicular and also they are also perpendicular to the direction of propagation of wave.* This all in turn implies that in a plane electromagnetic wave, vectors (**E**, **H**, **k**) form a set of orthogonal vectors which form a right handed co-ordinate system in tha order (fig. 8.2).

Further from equation (18).

$$\mathbf{H} = \frac{1}{\mu_0\omega}(\mathbf{k} \times \mathbf{E}) = \frac{k}{\mu_0\omega}(\mathbf{n} \times \mathbf{E}) \quad (\text{since } \mathbf{k} = k\mathbf{n})$$

$$= \frac{1}{\mu_0 c}(\mathbf{n} \times \mathbf{E}) \quad ...(20)$$

This equation in term of moduli

$$\mathbf{H} = \frac{1}{\mu_0 c}\,\mathbf{E}\left(\text{since } \frac{1}{\sqrt{\mu_0\varepsilon_0}} = c\right)$$

Now the ratio of magnitude of **E** to the magnitude of **H** is symbolised as Z_0 *i.e.*

$$\mathbf{Z}_0 = \left|\frac{\mathbf{E}}{\mathbf{H}}\right| = \left|\frac{\mathbf{E}_0}{\mathbf{H}_0}\right| = \mu_0 c = \sqrt{\left(\frac{\mu_0}{\varepsilon_0}\right)}$$

$$\left(\text{since } c = \frac{1}{\sqrt{\mu_0\varepsilon_0}}\right)$$

$$= \sqrt{\left(\frac{4\pi \times 10^{-7}}{8{\cdot}85 \times 10^{-12}}\right)} = \mathbf{376{\cdot}6\ ohms} \qquad ...(21)$$

where the units of $\mathbf{Z}_0$ are most easily seen from the fact that it measures a ratio of E in volt/m to H in amp-turn/m and therefore must equal volt/amp or Ohms. Because the units of E/H are the same as those of impedance, the value of $\mathbf{Z}_0$ is often referred to as the **wave impedance** of free space. Further since the ratio $\mathbf{Z}_0 = \left|\frac{\mathbf{E}}{\mathbf{H}}\right|$ is real and positive ; this implies *that field vectors* **E** *and* **H** *are in the same phase i.e.* they have the same relative magnitude at all points at all times (fig. 8.2).

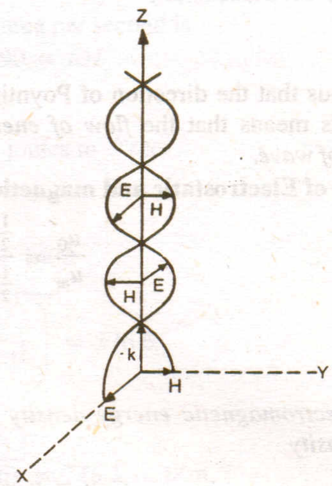

Fig. 8.2

The Poynting vector (i.e. energy flow per unit area per unit time) for a plane electromagnetic wave is given by

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} = \mathbf{E} \times \frac{\mathbf{n} \times \mathbf{E}}{\mu_0 c} \qquad \text{using (20)}$$

$$= \frac{1}{\mu_0 c}\,\mathbf{E} \times (\mathbf{n} \times \mathbf{E}) = \frac{1}{\mu_0 c}\,[(\mathbf{E}\cdot\mathbf{E})\,\mathbf{n} - (\mathbf{E}\cdot\mathbf{n})\,\mathbf{E}]$$

$$= \frac{1}{\mu_0 c}\,E^2\mathbf{n}$$

(since $\mathbf{E}\cdot\mathbf{n} = 0$, **E** being prependicular to **n**)

$$= \frac{E^2}{\mathbf{Z}_0}\,\mathbf{n} \qquad \text{[refer equation (21)]}$$

For a plane electromagnetic wave of angular frequency ω, the average value of **S** over a complete cycle is given by

$$<\mathbf{S}> = \frac{1}{\mathbf{Z}_0} <E^2> \mathbf{n}$$

$$= \frac{1}{Z_0} < \left(E_0\,e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}\right)^2 >_{real} \mathbf{n}$$

$$= \frac{1}{Z_0}\,E_0^2 < \cos^2(\omega t - \mathbf{k}\cdot\mathbf{r}) > \mathbf{n}$$

$$= \frac{1}{Z_0}\,\frac{E_0^2}{2}\,\mathbf{n} \qquad \left[\text{since } (\cos^2 < \omega t - \mathbf{k}\cdot\mathbf{r}) > = \tfrac{1}{2}\right]$$

$$= \frac{1}{Z_0} E_{rms}^2 \mathbf{n} \qquad ...(22)$$

$$\left(\text{since } E_{rms} = \frac{E_0}{\sqrt{2}}\right)$$

It is obvious that the direction of Poynting vector is along the direction of propagation of electromagnetic wave. This means that the *flow of energy in a plane electromagnetic wave in free space is along the direction of wave.*

Ratio of Electrostatic and magnetic energy densities is given by

$$\frac{u_0}{u_m} = \frac{\frac{1}{2}\varepsilon_0 E^2}{\frac{1}{2}\mu_0 H^2} = \frac{\varepsilon_0}{\mu_0}\frac{E^2}{H^2} = \frac{\varepsilon_0}{\mu_0}\cdot\frac{\mu_0}{\varepsilon_0} = 1. \qquad ...(23)$$

$$\left[\text{since } \frac{E}{H} = \sqrt{\left(\frac{\mu_0}{\varepsilon_0}\right)}\right]$$

i.e. the electromagnetic energy density is equal to magnetostatic energy density. Total *electromagnetic energy density*

$$\therefore \qquad u = u_e + u_m = 2u_e = 2 \times \tfrac{1}{2}\varepsilon_0 E^2 = \varepsilon_0 E^2$$

$\therefore$ Time average of *energy density*

$$< u > = < \varepsilon_0 E^2 > = \varepsilon_0 < (E_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t})^2 >_{real}$$

$$= \varepsilon_0 E_0^2 < \cos^2(\omega t - \mathbf{k}\cdot\mathbf{r}) > = \tfrac{1}{2}\varepsilon_0 E_0^2 = \varepsilon_0 E_{rms}^2 \qquad ...(24)$$

Dividing (22) by (24), we obtain

$$\frac{< \mathbf{S} >}{< u >} = \frac{1}{Z_0\varepsilon_0}\mathbf{n} = \frac{1}{\sqrt{\left(\frac{\mu_0}{\varepsilon_0}\right)}\,\varepsilon_0}\mathbf{n} = \frac{n}{\sqrt{\mu_0\varepsilon_0}} = c\,\mathbf{n} \qquad ...(25)$$

Thus we obtain

$$< \mathbf{S} > = < u > c\,\mathbf{n} \qquad ...(26a)$$

i.e. energy flux = energy density $\times$ c. ...(26b)

This equation implies that the *energy density associated with an electromagnetic wave in free space propagates with the speed of light with which the field vectors do.*

Summarising we may say for electromagnetic waves in free space that :

1. In free space the electromagnetic waves travel with the speed of light.

2. The electromagnetic field vectors **E** and **H** are mutually perpendicular and they are also perpendicular to the direction of propagation of electromagnetic waves. Thereby indicating the electromagnetic waves are transverse in nature.

3. The field vectors **E** and **H** are in same phase.

4. The direction of flow of electromagnetic energy is along the direction of wave propagation and the energy flow per unit area per second is represented by

$$< \mathbf{S} > = \frac{E_{rms}^2}{Z_0}\mathbf{n} = < u > c\,\mathbf{n}.$$

5. The electrostatic energy density is equal to the magnetic energy density and the energy density associated with the electromagnetic wave in free space propagates with the speed of light.

Ex. 7. *If the earth receives 2 cal* min^{-1} cm^{-2} *solar energy, what are the amplitudes of electric and magnetic field of radiation.* **(Rohilkhand 1997; U.P.C.S. 1979)**

Solution. From Poynting theorem the energy flux per unit area per second is

$$|\mathbf{S}| = |\mathbf{E} \times \mathbf{H}| = EH \sin 90 = EH \qquad \text{...(1)}$$

The energy flux per unit area per second at earth is (given)

$$2 \text{ cal min}^{-1} \text{cm}^{-2} = \frac{2 \times 4{\cdot}2 \times 10^4}{60} \text{ joules m}^{-2} \text{sec}^{-2} \qquad \text{...(2)}$$

Comparing (1) and (2), we get

$$EH = \frac{2 \times 4{\cdot}2 \times 10^4}{60} = 1400 \qquad \text{...(3)}$$

But

$$\frac{E}{H} = \sqrt{\frac{\mu_0}{\varepsilon_0}} = \sqrt{\left(\frac{4\pi \times 10^{-7}}{8{\cdot}85 \times 10^{-12}}\right)} = 376{\cdot}6. \qquad \text{...(4)}$$

Multiplying equations (3) and (4), we get

$$E^2 = 1400 \times 376{\cdot}6$$

$$\therefore \quad E = \sqrt{(1400 \times 376{\cdot}6)} = \mathbf{726{\cdot}1 \text{ volt/m.}}$$

Substituting this value of E in (3), we get

$$H = \frac{1400}{E} = \frac{1400}{726{\cdot}1} = \mathbf{1{\cdot}928 \text{ amp-turn/m}}.$$

$\therefore$ Amplitudes of electric and magnetic fields of radiation are

$$E_0 = E\sqrt{2} = 726{\cdot}1 \times \sqrt{2} = \mathbf{1027 \text{ volt/m.}}$$

$$H_0 = H\sqrt{2} = 1{\cdot}928\sqrt{2} = \mathbf{2{\cdot}730 \text{ amp-turn/m}}.$$

Ex. 8. *Assuming that all the energy from a 1000 watt lamp is radiated uniformly, calculate the average values of the intensities of electric and magnetic fields of radiation at a distance of 2m from the lamp.*

Solution. Considering the lamp as a point source, the total flux energy over a sphere drawn round the lamp as centre is 1000 watt = 1000 joule/sec.

This energy flux falls on area $4\pi r^2 = 4\pi \times 2^2 = 16\pi^2 m^2$; therefore the energy flux per unit area per second $= \dfrac{1000}{16\pi^3}$ watt/m^2.

Hence from Poynting theorem

$$|\mathbf{S}| = |\mathbf{E} \times \mathbf{H}| = EH = \frac{1000}{16\pi^2} \qquad \text{...(5)}$$

and

$$\frac{E}{H} = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 376{\cdot}6. \qquad \text{...(6)}$$

Multiplying equations (5) and (6), we get

$$E^2 = \frac{1000}{16\pi^2} \times 376{\cdot}6,$$

$$\therefore \quad \mathbf{E} = \sqrt{\left[\frac{1000}{16\pi^2} \times 376{\cdot}6\right]} = \mathbf{48{\cdot}87 \text{ volt/m.}}$$

Substituting the value of E in equation (5), we get

$$\mathbf{H} = \frac{1000}{16\pi^2 E} = \frac{1000}{16\pi^2 \times 48{\cdot}87} = 0{\cdot}1297 \text{ amp-turn/m}.$$

8.14. Plane Electromagnetic Waves in a Non-conducting Isotropic medium. (*i.e.* Isotropic Dielectric)

A non-conducting medium which has same properties in all directions is called an isotropic dielectric.

Maxwell's equations are

$$\left.\begin{aligned} \operatorname{div} \mathbf{D} &= \nabla \cdot \mathbf{D} = \rho \\ \operatorname{div} \mathbf{B} &= \nabla \cdot \mathbf{B} = 0 \\ \operatorname{curl} \mathbf{E} &= \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \\ \text{and} \quad \operatorname{curl} \mathbf{H} &= \nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \end{aligned}\right\} \qquad \ldots(7)$$

In an *isotropic dielectric* (or non-conducting isotropic medium)

$$\mathbf{D} = \varepsilon\, \mathbf{E},\ \mathbf{B} = \mu \mathbf{H},\ \mathbf{J} = \sigma \mathbf{E} = 0 \text{ and } \rho = 0$$

Therefore Maxwell's equations in this case take the form

$$\left.\begin{aligned} \operatorname{div} \mathbf{E} &= \nabla \cdot E = 0 && \ldots(a) \\ \operatorname{div} \mathbf{H} &= \nabla \cdot \mathbf{H} = 0 && \ldots(b) \\ \text{and} \quad \operatorname{curl} \mathbf{E} &= -\mu \frac{\partial \mathbf{H}}{\partial t} && \ldots(c) \\ \operatorname{curl} \mathbf{H} &= \varepsilon \frac{\partial \mathbf{E}}{\partial t} && \ldots(d) \end{aligned}\right\}$$

Taking curl of equation (2c), we get

$$\operatorname{curl} \operatorname{curl} \mathbf{E} = -\mu \frac{\partial}{\partial t} (\operatorname{curl} \mathbf{H})$$

Substituting curl **H** from (2d) in above equation

$$\operatorname{curl} \operatorname{curl} \mathbf{E} = -\mu \frac{\partial}{\partial t}\left(\varepsilon \frac{\partial \mathbf{E}}{\partial t}\right)$$

i.e.

$$\operatorname{curl} \operatorname{curl} \mathbf{E} = -\mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} \qquad \ldots(3)$$

Similarly if we take curl of (2*d*) and substitute curl **E** from (2c), we get

$$\operatorname{curl} \operatorname{curl} \mathbf{H} = -\mu\varepsilon \frac{\partial^2 \mathbf{H}}{\partial t^2} \qquad \ldots(4)$$

using vector identity

$$\operatorname{curl} \operatorname{curl} \mathbf{A} = \operatorname{grad} \operatorname{div} \mathbf{A} - \nabla^2 \mathbf{A}$$

and keeping in mind equations (2a) and (2b) *i.e.*, div **E** = 0 and div **H** = 0 equations (3) and (4) give

$$\nabla^2 \mathbf{E} - \mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \qquad \ldots(5)$$

and

$$\nabla^2 \mathbf{H} - \mu\varepsilon \frac{\partial^2 \mathbf{H}}{\partial t^2} = 0 \qquad \ldots(6)$$

These equations are vector equations of identical form which means that each of the six components of **E** and **H** separately satisfies the same scalar wave equation of the form

$$\nabla^2 u - \mu\varepsilon \frac{\partial^2 u}{\partial t^2} = 0 \qquad ...(7)$$

where u is a scalar and can stand for any one of components of **E** and **H**. It is obvious that equation (6) resembles with the general wave equation

$$\nabla^2 u - \frac{1}{v^2} \frac{\partial^2 u}{\partial t^2} = 0 \qquad ...(8)$$

where v is the speed of wave.

This means that the field vector **E** and **H** are propagated in isotropic dielectric as waves with speed v given by

$$v = \frac{1}{\sqrt{(\mu\varepsilon)}} = \frac{1}{\sqrt{(K_m \mu_0 K_e \varepsilon_0)}}$$

where K_m is relative permeability of medium and K_e is relative permittivity (or dielectric constant) of the medium.

As $\dfrac{1}{\sqrt{\mu_0\varepsilon_0}} = c$, speed of electromagnetic waves in free space.

$$\therefore \qquad v = \frac{c}{\sqrt{K_m K_e}} \qquad ...(10)$$

Since $K_m > 1$ and $K_e > 1$: thereby indicating ***that the speed of electromagnetic waves is an isotropic dielectric is less than the speed of electromagnetic waves in free space.***

As $$n = \frac{c}{v} \quad i.e.\; v = \frac{c}{n} \qquad ...(11)$$

$\therefore$ Comparing (10) and (11) we note that the refractive index n *in this particular case is*

$$n = \sqrt{(K_m K_e)} \qquad ...(12)$$

For a non-magnetic material $K_m = 1$; therefore

$$n = \sqrt{K_e} \quad i.e. \quad n^2 = K_e \qquad ...(13)$$

This relation is known as Maxwell's relation and has been verified by a number of experiments.

Replacing $\mu\varepsilon$ by $\dfrac{1}{v^2}$, wave equations (5) and (6) may be expressed as

$$\nabla^2 \mathbf{E} - \frac{1}{v^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \qquad ...(14)$$

and $$\nabla^2 \mathbf{H} - \frac{1}{v^2} \frac{\partial^2 \mathbf{H}}{\partial t^2} = 0. \qquad ...(15)$$

The *plane-wave solutions* of equations (14) and (15) in well known from may be written as

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t} \qquad ...(16)$$

$$\mathbf{H}(\mathbf{r}, t) = \mathbf{H}_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t} \qquad ...(17)$$

where $\mathbf{E}_0$ and $\mathbf{H}_0$ are complex amplitudes which are constant in space and time : while **k** is wave propagation vector given by

$$\mathbf{k} = k\hat{n} = \frac{2\pi}{\lambda}\hat{n} = \frac{\omega}{v}\hat{n} \qquad ...(18)$$

Here $\hat{n}$ is a unit vector in the direction of wave propagation.

Relative directions of E and H. The requirement $\nabla \cdot \mathbf{E} = 0$ and $\nabla \cdot \mathbf{H} = 0$, demand that

$$\mathbf{k} \cdot \mathbf{E} = 0 \quad \text{and} \quad \mathbf{k} \cdot \mathbf{H} = 0 \qquad ...(19)$$

Comparing (7) and (8), we see $$v = \frac{1}{\sqrt{(\mu\varepsilon)}} \qquad ...(19a)$$

This means that the field vector **E** and **H** are both perpendicular to the direction of propagation vector **k**. This implies that ***electromagnetic waves in isotropic dielectric are transverse in nature*** Further restrictions are provided by curl equations (2c) and (2b) *viz.*

$$\text{curl } \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t} \quad \text{and} \quad \text{curl } \mathbf{H} = \varepsilon \frac{\partial \mathbf{E}}{\partial t}$$

Using (16) and (17), these equations yield

$$\mathbf{k} \times \mathbf{E} = \mu\omega\mathbf{H} \qquad ...(20)$$

and $$\mathbf{k} \times \mathbf{H} = -\varepsilon\omega\mathbf{E} \qquad ...(21)$$

From these equations it is obvious that *field vectors* **E** *and* **H** *are mutually perpendicular and also they are perpendicular to the direction of propagation vector* **k**. *This in turn implies that in a plane electromagnetic wave in isotropic dielectric, vector* **(E, H, k)** *from a set of orthogonal vectors which form a right handed coordinate system in given order (fig. 8.2).*

Phase of E and H and Wave Impedance. From equation (20)

$$\mathbf{H} = \frac{1}{\mu\omega}(\mathbf{k} \times \mathbf{E}) = \frac{k}{\mu\omega}(\hat{n} \times \mathbf{E})$$

$$= \frac{1}{\mu v}(\hat{n} \times \mathbf{E}) = \sqrt{\frac{\varepsilon}{\mu}}\,(\hat{n} \times \mathbf{E}) \qquad ...(22)$$

$$\left(\text{since } k = \frac{\omega}{v} \text{ and } v = \frac{1}{\sqrt{(\mu\varepsilon)}}\right)$$

Now the ratio of magnitude of **E** to the magnitude of **H** is symbolised by Z *i.e.*

$$Z = \left|\frac{\mathbf{E}}{\mathbf{H}}\right| = \frac{\mathbf{E}_0}{\mathbf{H}_0} = \sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\left(\frac{K_m\mu_0}{K_e\varepsilon_0}\right)} = \text{real quantity} \qquad ...(23)$$

This implies ***that the field vectors* E *and* H *are in the same phase, i.e., they have same relative magnitudes at all points at all time. The unit of Z comes out to be ohm, since***

$$\mathbf{Z} = \frac{\mathbf{E}}{\mathbf{H}} = \frac{\text{volt/m}}{\text{amp–turn/m}} = \frac{\text{volt}}{\text{amp}} = \text{ohm};$$

hence the value of Z is referred to as *wave impedance of isotropic dielectric medium.* The *wave impedance of medium is related to that of free space by the relation*

$$Z = \sqrt{\left(\frac{K_m\mu_0}{K_e\varepsilon_0}\right)} = \sqrt{\left(\frac{K_m}{K_e}Z_0\right)} \qquad ...(24)$$

where $Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}$ is called the wave impedance of free space.

Poynting vector for a plane electromagnetic wave in an isotropic dielectric is given by

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} = \mathbf{E} \times \left(\sqrt{\frac{\varepsilon}{\mu}}\,\hat{n} \times \mathbf{E}\right)$$

$$= \frac{\mathbf{E} \times (\hat{n} \times \mathbf{E})}{Z} \left(\text{since } Z = \sqrt{\frac{\mu}{\varepsilon}} \right)$$

$$= \frac{(\mathbf{E} \cdot \mathbf{E})\,\hat{n} - (\mathbf{E} \cdot \hat{n})\,\mathbf{E}}{\mathbf{Z}}$$

$$= \frac{\mathbf{E}^2}{Z}\,\hat{n} \text{ (since } \mathbf{E} \cdot \hat{n} = 0 \text{ because } \mathbf{E} \text{ is perpendicular to } \hat{n})$$

The time average of *Poynting vector* is

$$<\mathbf{S}> = <\mathbf{E} \times \mathbf{H}> = <\frac{\mathbf{E}^2}{Z}\,\hat{n}>$$

$$= \frac{1}{Z} < \left(\mathbf{E}_0\, e^{i\,\mathbf{k} \cdot \mathbf{r} - i\omega t} \right)^2 >_{real}\, \hat{n}$$

Since for finding actual physical fields we often take real parts of complex exponentials.

$$\therefore \quad <\mathbf{S}> = \frac{1}{Z} E_0^2 < \cos^2 (\omega t - \mathbf{k}\ \mathbf{r}) > \hat{n}$$

$$= \frac{1}{Z} E_0^2 \cdot \tfrac{1}{2}\,\hat{n} = \frac{1}{2Z} E_0^2\,\hat{n}$$

$$= \frac{1}{Z} E_{rms}^2\,\hat{n} \left(\text{since } E_{rms} = \frac{E_0}{\sqrt{2}} \right) \quad \text{...(25a)}$$

$$= \sqrt{\left(\frac{K_e}{K_m}\right)}\, \frac{1}{Z_0}\, E_{rms}^2\,\hat{n}$$

$$= \frac{\sqrt{(K_e K_m)}}{K_m}\, \frac{1}{Z_0}\, E_{rms}^2\,\hat{n} = \frac{n}{K_m}\,\frac{1}{Z_0}\, E_{rms}^2\,\hat{n}. \quad \text{...(25b)}$$

[becuase refractive index $n = \sqrt{(K_e K_m)}$]

$$= \frac{n}{K_m} <\mathbf{S}>_{\text{free space}} \quad \text{...(25c)}$$

$$\left[\text{because } <\mathbf{S}>_{\text{free space}} = \frac{1}{Z_0} E_{rms}^2\,\hat{n} \right]$$

Equations (25a) and (25b), show that the *flow of energy is along the direction of propagation of electromagnetic wave.* Equation (25c) shows that the Poynting vector for electromagnetic wave in isotropic dielectric is $\sqrt{\left(\frac{K_e}{K_m}\right)}$ or $\frac{n}{K_m}$ times of the Poynting vector if the same electromagnetic wave were propagated through free space. It may be noted that the average of Poynting vector may also be obtained as

$$<\mathbf{S}> = <\mathbf{E} \times \mathbf{H}> = \tfrac{1}{2}\,.\,\text{Real part of } (\mathbf{E} \times \mathbf{H}^*)$$

Power flow and Energy density. Let us find the ratio of electrostatic and magnetostatic energy densities in an electromagnetic wave field *i.e.*

$$\frac{u_e}{u_m} = \frac{\frac{1}{2}\varepsilon E^2}{\frac{1}{2}\mu H^2} = \frac{\varepsilon}{\mu}\,\frac{E^2}{H^2} = \frac{\varepsilon}{\mu}\,Z^2 = \frac{\varepsilon}{\mu}\,.\,\frac{\mu}{\varepsilon} = 1. \quad \text{...(26)}$$

$$\left[\text{since } Z = \frac{E}{H} = \sqrt{\left(\frac{\mu}{\varepsilon}\right)} \right]$$

This implies that for the case of electromagnetic waves in an isotropic dielectric the electrostatic energy density (u_e) is equal to the magnetostatic energy density (u_m).

Therefore total *electromagnetic energy density*

$$u = u_e + u_m = 2u_e \text{ (since } u_e = u_m)$$
$$= 2 \cdot \tfrac{1}{2}\varepsilon E^2 = \varepsilon E^2$$

Therefore *time average of energy density*

$$< u > = < \varepsilon E^2 > = \varepsilon < E^2 > = \varepsilon < \left(E_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}\right)^2_{real} >$$
$$= \varepsilon E^2 < \cos^2(\omega t - \mathbf{k}\cdot\mathbf{r}) > = \frac{\varepsilon E_0^2}{2}$$
$$= \varepsilon E_{rms}^2 \qquad ...(27)$$

Dividing equaion (23a) with equation (27), we obtain

$$\frac{< \mathbf{S} >}{< \mu >} = \frac{(E_{rms}^2 \hat{n}/Z)}{\varepsilon E_{rms}^2} = \frac{1}{Z\varepsilon}\hat{n} = \frac{2}{\sqrt{\left(\frac{\mu}{\varepsilon}\right)}\,\varepsilon}\hat{n} \text{ since } Z = \sqrt{\frac{\mu}{\varepsilon}}$$
$$= \frac{1}{\sqrt{\mu\varepsilon}}\hat{n} = v\hat{n} \left(\text{since } v = \frac{1}{\sqrt{\mu\varepsilon}}\right)$$

Thus we obtain

$$< \mathbf{S} > = < u > v\hat{n}$$

or

$$< \mathbf{S} >_{av} = u_{av}\, v\hat{n} \qquad ...(28a)$$

or in words **energy flux = v × energy density.** ...(28b)

This equation has a simple meaning. If the energy were flowing with velocity v (= phase velocity of electromagnetic wave with which electromagnetic field vectors propagate), in the direction of propagation of wave, all the energy contained in a cylinder of unit cross-section and height equal to v would cross unit cross-section per second, forming the flux. This in turn implies *that the energy density associated with an electromagnetic wave in a stationary homogeneous nonconducting medium propagates with the same speed with which the field vectors do.*

Summarising we may say for the case of electromagnetic waves in isotropic dielectric that :

1. In isotropic dielectric the electromagnetic waves travel with a speed less than the speed of light.
2. The electromagnetic field vectors **E** and **H** are mutually perpendicular and they are also perpendicular to the direction of propagation of electromagnetic wave. Thereby indicating that electromagnetic waves are transverse in nature.
3. The field vectors **E** and **H** are in the same phase.
4. The direction of flow of electromagnetic energy is along the direction of wave propagation and the energy flow per unit area per second is represented as

$$< S > = \frac{E_{rms}^2 \hat{n}}{Z} = < u > v\hat{n}$$

5. The electrostatic energy density is equal to the magnetostatic energy density and the total energy density is given by

$$< u > = \varepsilon E_{rms}^2 .$$

6. The energy density associated with an electromagnetic wave propagates with the phase velocity of the wave.

Ex. 8. *A plane electromagnetic wave travelling in positive z-direction in an unbounded lossless dielectric medium with relative permeability $\mu_r = 1$ and relative permittivity $\varepsilon_r = 3$ has a peak electric field intensity $E_0 = 6$ V/m. Find*

(i) the speed of the wave,

(ii) the intrinsic impedance of the medium,

(iii) the peak magnetic field intensity (H_0),

and (iv) the peak Poynting vector $\mathbf{S}$ *(z, t)* **(Banaras 1990)**

Solution. $E_0 = \sqrt{(E_{0x}^2 + E_{0y}^2)} = 6$ V/m, $\varepsilon_r = 3$, $\mu_r = 1$

(i) The speed of electromagnetic wave

$$v = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{1}{\sqrt{\mu_r \mu_0 \varepsilon_r \varepsilon_0}}$$

$$= \frac{1}{\sqrt{(\mu_0\varepsilon_0)}} \cdot \frac{1}{\sqrt{(\mu_r\varepsilon_r)}} = \frac{c}{\sqrt{(\mu_r\varepsilon_r)}}$$

$$= \frac{3 \times 10}{\sqrt{(1 \times 3)}} = \mathbf{1{\cdot}73 \times 10^8\ m/s.}$$

(ii) Impedance of medium

$$Z = \sqrt{\left(\frac{\mu}{\varepsilon}\right)} = \sqrt{\left(\frac{\mu_r\mu_0}{\varepsilon_r\varepsilon_0}\right)} = \sqrt{\left(\frac{\mu_0}{\varepsilon_0}\right)} \cdot \sqrt{\left(\frac{\mu_r}{\varepsilon_r}\right)}$$

$$= \sqrt{\left(\frac{4\pi \times 10^{-7}}{8{\cdot}86 \times 10^{-12}}\right)} \cdot \sqrt{\left(\frac{1}{3}\right)} = \frac{376{\cdot}6}{\sqrt{3}}$$

$$= \mathbf{217{\cdot}6\ \Omega.}$$

(iii) Peak value of magnetic field

$$H_0 = \frac{E_0}{Z} = \frac{6}{217{\cdot}6} = \mathbf{2{\cdot}76 \times 10^{-2}\ A/m.}$$

(iv) Poynting vector $\mathbf{S} = \mathbf{E} \times \mathbf{H}$

$$\text{Peak Poynting vector} = E_0 H_0 = \frac{E_0^2}{Z}$$

$$= \frac{6^2}{217{\cdot}6} = \mathbf{0{\cdot}165\ W/m^2}.$$

8.14. Plane Electromagnetic Waves in Anisotropic Non-conducting Medium (*i.e.* Anisotropic Dielectric) :

In Anisotropic medium in one in which electromagnetic field properties depend on direction. Let us consider a non-magnetic non-conducting homogeneous anisotropic medium. In such a medium

$$\mathbf{J} = 0;\ \rho = 0 \ \text{ and } \ \mu = \mu_0 \qquad ...(1)$$

Moreover the permitivity ε is no longer a scalar; but it is a tensor; so that components of electric displacement $\mathbf{D}$ are in general related to components of $\mathbf{E}$ by the equations