Some problems on

1st & 2nd Topics

Matrices

Problems on definitions of special types of Matrices

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Now let us use the various definitions of special types of matrices in the following problems:

Q.No.1.: Evaluate $3A-4B$, where $A=\begin{bmatrix}3 & -4 & 6\\ 5 & 1 & 7\end{bmatrix}$ and $B=\begin{bmatrix}1 & 0 & 1\\ 2 & 0 & 3\end{bmatrix}$.

Sol.: Here $A=\begin{bmatrix}3 & -4 & 6\\ 5 & 1 & 7\end{bmatrix}$ and $B=\begin{bmatrix}1 & 0 & 1\\ 2 & 0 & 3\end{bmatrix}$.

Therefore $3A=\begin{bmatrix}9 & -12 & 18\\ 15 & 3 & 21\end{bmatrix}$ and $4B=\begin{bmatrix}4 & 0 & 4\\ 8 & 0 & 12\end{bmatrix}$

Now $3A-4B=\begin{bmatrix}9-4 & -12-0 & 18-4\\ 15-8 & 3-0 & 21-12\end{bmatrix}=\begin{bmatrix}5 & -12 & 14\\ 7 & 3 & 9\end{bmatrix}$. Ans.

Q.No.2.: If $A=\begin{bmatrix}0 & 1 & 2\\ 1 & 2 & 3\\ 2 & 3 & 4\end{bmatrix}$, $B=\begin{bmatrix}1 & -2\\ -1 & 0\\ 2 & -1\end{bmatrix}$, form the product of AB.

Is BA defined ?

Sol.: Since the number of columns of A = the number of rows of B (each being = 3).

$\therefore$ The product AB defined and

$$AB=\begin{bmatrix}0.1+1.(-1)+2.2 & 0.(-2)+1.0+2.(-1)\\ 1.1+2.(-1)+3.2 & 1.(-2)+2.0+3.(-1)\\ 2.1+3.(-1)+4.2 & 2.(-2)+3.0+4.(-1)\end{bmatrix}=\begin{bmatrix}3 & -2\\ 5 & -5\\ 7 & -8\end{bmatrix}.$$

Again, since the number of columns of B $\neq$ the number of rows of A

$\therefore$ The product BA is not possible.

Q.No.3.: If $A=\begin{bmatrix}1 & 3 & 0\\ -1 & 2 & 1\\ 0 & 0 & 2\end{bmatrix}$, $B=\begin{bmatrix}2 & 3 & 4\\ 1 & 2 & 3\\ -1 & 1 & 2\end{bmatrix}$,

compute AB and BA and show that $AB \neq BA$.

Sol.: Here $A=\begin{bmatrix}1 & 3 & 0\\ -1 & 2 & 1\\ 0 & 0 & 2\end{bmatrix}$, $B=\begin{bmatrix}2 & 3 & 4\\ 1 & 2 & 3\\ -1 & 1 & 2\end{bmatrix}$.

$$\text{Now } AB=\begin{bmatrix}1.2+3.1+0.(-1) & 1.3+3.2+0.1 & 1.4+3.3+0.2\\ (-1).2+2.1+1.(-1) & (-1).3+2.1+1.1 & (-1).4+2.3+1.2\\ 0.2+0.1+2.(-1) & 0.3+0.2+2.1 & 0.4+0.3+2.2\end{bmatrix}=\begin{bmatrix}5 & 9 & 13\\ -1 & 2 & 4\\ -2 & 2 & 4\end{bmatrix}.$$

$$BA=\begin{bmatrix}2.1+3.(-1)+4.0 & 2.3+3.2+4.0 & 2.0+3.1+4.2\\ 1.1+2.(-1)+3.0 & 1.3+2.2+3.0 & 1.0+2.1+3.2\\ (-1).1+1.(-1)+2.0 & (-1).3+1.2+2.0 & (-1).0+1.1+2.2\end{bmatrix}=\begin{bmatrix}-1 & 12 & 11\\ -1 & 7 & 8\\ -2 & -1 & 5\end{bmatrix}.$$

Hence $AB \neq BA$.

Q.No.4.: Prove that $A^3-4A^2-3A+(11)I=O$, where $A=\begin{bmatrix}1 & 3 & 2\\ 2 & 0 & -1\\ 1 & 2 & 3\end{bmatrix}$.

Sol.: Here $A=\begin{bmatrix}1 & 3 & 2\\ 2 & 0 & -1\\ 1 & 2 & 3\end{bmatrix}$.

$$\text{Now } A^2=A\times A=\begin{bmatrix}1 & 3 & 2\\ 2 & 0 & -1\\ 1 & 2 & 3\end{bmatrix}\times\begin{bmatrix}1 & 3 & 2\\ 2 & 0 & -1\\ 1 & 2 & 3\end{bmatrix}$$

$$=\begin{bmatrix}1+6+2 & 3+0+4 & 2-3+6\\ 2+0-1 & 6+0-2 & 4+0-3\\ 1+4+3 & 3+0+6 & 2-2+9\end{bmatrix}=\begin{bmatrix}9 & 7 & 5\\ 1 & 4 & 1\\ 8 & 9 & 9\end{bmatrix},$$

and $A^3 = A^2 \times A = \begin{bmatrix} 9 & 7 & 5 \\ 1 & 4 & 1 \\ 8 & 9 & 9 \end{bmatrix} \times \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 9+14+5 & 27+0+10 & 18-7+15 \\ 1+8+1 & 3+0+2 & 2-4+3 \\ 8+18+9 & 24+0+18 & 16-9+27 \end{bmatrix} = \begin{bmatrix} 28 & 37 & 26 \\ 10 & 5 & 1 \\ 35 & 42 & 34 \end{bmatrix}.$$

$$\therefore\ A^3 - 4A^2 - 3A + (11)I = \begin{bmatrix} 28 & 37 & 26 \\ 10 & 5 & 1 \\ 35 & 42 & 34 \end{bmatrix} - 4\begin{bmatrix} 9 & 7 & 5 \\ 1 & 4 & 1 \\ 8 & 9 & 9 \end{bmatrix} - 3\begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix} + 11\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 28-36-3+11, & 37-28-9+0, & 26-20-6+0 \\ 10-4-6-0, & 5-16+0+11, & 1-4+3+0 \\ 35-32-3+0, & 42-36-6+0, & 34-36-9+11 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O.\ \text{Ans.}$$

Q. No.5: Which of the following matrices are singular:

(i) $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 1 & 3 & 4 \end{bmatrix}$, **(ii)** $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 8 \\ 3 & 9 & 25 \end{bmatrix}$, **(iii)** $\begin{bmatrix} 2 & 5 & 19 \\ 1 & -2 & -4 \\ -3 & 2 & 0 \end{bmatrix}$.

Sol.: (i). Here the given matrix is $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 1 & 3 & 4 \end{bmatrix}$.

Since, we know that a matrix A is said to be singular if $|A| = 0$.

$$\therefore |A| = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 1 & 3 & 4 \end{vmatrix} = 1(4-6) - 2(4-2) + 3(3-1) = -2 - 4 + 6 = 0.$$

Hence, the given matrix A is singular.

(ii). Here the given matrix is $B = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 8 \\ 3 & 9 & 25 \end{bmatrix}$.

Now $|B| = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 4 & 8 \\ 3 & 9 & 25 \end{vmatrix} = 1(100-72) - 1(50-24) + 1(18-12) = 28 - 16 + 6 = 18 \neq 0$

Now since $|B| \neq 0$. Hence, the given matrix B is non-singular.

(iii). Here the given matrix is $C = \begin{bmatrix} 2 & 5 & 19 \\ 1 & -2 & -4 \\ -3 & 2 & 0 \end{bmatrix}$.

Now $|C| = \begin{vmatrix} 2 & 5 & 19 \\ 1 & -2 & -4 \\ -3 & 2 & 0 \end{vmatrix} = 2(0+8) - 5(0-12) + 19(2-6) = 16 + 60 - 76 = 0$

Hence, the given matrix C is singular.

Q.No.6.: For what values of x, the matrix $\begin{bmatrix} 3-x & 2 & 2 \\ 2 & 4-x & 1 \\ -2 & -4 & -1-x \end{bmatrix}$ is singular ?

Sol.: Here the given matrix is $A = \begin{bmatrix} 3-x & 2 & 2 \\ 2 & 4-x & 1 \\ -2 & -4 & -1-x \end{bmatrix}$.

Now a matrix is said to be singular is $|A| = 0$.

Here $|A| = \begin{vmatrix} 3-x & 2 & 2 \\ 2 & 4-x & 1 \\ -2 & -4 & -1-x \end{vmatrix}$

$$= (3-x)[(4-x)(-1-x)+4] - 2[2(-1-x)+2] + 2[-8+2(4-x)]$$

$$= (3-x)\left(-4-4x+x+x^2+4\right) - 2(-2-2x+2) + 2(-8+8-2x)$$

$$= -9x + 3x^2 + 3x^2 - x^3 + 4x - 4x = -x^3 + 6x^2 - 9x = -x\left(x^2 + 6x - 9\right)$$

$$= -x\left[(x-3)^2\right].$$

Now $|A| = 0 \Rightarrow -x\left[(x-3)^2\right] = 0 \Rightarrow x = 0$ and $x = 3$. Ans.

Q.No.7.: Find the values of x, y, z and a, which satisfy the matrix equation

$$\begin{bmatrix} x+3 & 2y+x \\ z-1 & 4a-6 \end{bmatrix} = \begin{bmatrix} 0 & -7 \\ 3 & 2a \end{bmatrix}.$$

Sol.: As the given matrices are equal, equating the elements of both the matrices, we get

$x + 3 = 0$; $2y + x = -7$; $z - 1 = 3$; $4a - 6 = 2a$.

$\therefore x = -3, \quad y = -2, \quad z = 4, \quad a = 3$. Ans.

Q.No.8.: Find x, y, z and w, given that:

$$3\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} x & 6 \\ -1 & 2w \end{bmatrix} + \begin{bmatrix} 4 & x+y \\ z+w & 3 \end{bmatrix}.$$

Sol.: Given

$$3\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} x & 6 \\ -1 & 2w \end{bmatrix} + \begin{bmatrix} 4 & x+y \\ z+w & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 3x & 3y \\ 3z & 3w \end{bmatrix} = \begin{bmatrix} x+4 & x+y+6 \\ -1+z+w & 2w+3 \end{bmatrix}$$

Now, both the matrices are equal, equating the elements of both the matrices, we get

$3x = x + 4 \qquad \Rightarrow x = 2$

$3y = x + y + 6 \qquad \Rightarrow y = 4$

$3w = 2w + 3 \qquad \Rightarrow w = 3$

$3z = -1 + z + w \qquad \Rightarrow z = 1$. Ans.

Q.No.9.: Matrix A has x rows and $x + 5$ columns. Matrix B has y rows and $11 - y$ columns. Both AB and BA exist. Find x and y.

Sol.: Since the order of A is $x \times (x+5)$ and order of B is $y \times (11-y)$.

Since AB exist $\Rightarrow x + 5 = y \Rightarrow x - y = -5$. (i)

Also BA exist $\Rightarrow 11 - y = x \Rightarrow x + y = 11$. (ii)

Solving (i) and (ii), we get

$2x = 6 \Rightarrow x = 3$. Ans.

$\therefore y = 8$. Ans.

Q.No.10.: If $A + B = \begin{bmatrix} 1 & -1 \\ 3 & 0 \end{bmatrix}$ and $A - B = \begin{bmatrix} 3 & 1 \\ 1 & 4 \end{bmatrix}$. Calculate the product AB.

Sol.: Here given $A + B = \begin{bmatrix} 1 & -1 \\ 3 & 0 \end{bmatrix}$. (i)

and $A - B = \begin{bmatrix} 3 & 1 \\ 1 & 4 \end{bmatrix}$. (ii)

Adding (i) and (ii), we get $2A = \begin{bmatrix} 4 & 0 \\ 4 & 4 \end{bmatrix} \Rightarrow A = \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}$.

Subtracting (i) and (ii), we get $2B = \begin{bmatrix} -2 & -2 \\ 2 & -4 \end{bmatrix} \Rightarrow B = \begin{bmatrix} -1 & -1 \\ 1 & -2 \end{bmatrix}$.

$$\therefore AB=\begin{bmatrix}2 & 0\\ 2 & 2\end{bmatrix}\begin{bmatrix}-1 & -1\\ 1 & -2\end{bmatrix}=\begin{bmatrix}-2+0 & -2+0\\ -2+2 & -2-4\end{bmatrix}=\begin{bmatrix}-2 & -2\\ 0 & -6\end{bmatrix}=-2\begin{bmatrix}1 & 1\\ 0 & 3\end{bmatrix}. \text{ Ans.}$$

Q.No.11.: If $A=\begin{bmatrix}1 & 2 & 3 & 4\\ 2 & 0 & 1 & 2\\ 3 & 1 & 0 & 5\end{bmatrix}_{3\times 4}$ and $B=\begin{bmatrix}2 & 1 & 0\\ 3 & 2 & 1\\ 1 & 0 & 1\end{bmatrix}_{3\times 3}$,

find AB or BA, whichever exist.

Sol.: Here AB does not exist because the number of columns in A is not equal to the number of rows in B and BA exist because the number of columns in B is equal to the number of rows in A.

$$\text{Now } BA=\begin{bmatrix}2 & 1 & 0\\ 3 & 2 & 1\\ 1 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 2 & 3 & 4\\ 2 & 0 & 1 & 2\\ 3 & 1 & 0 & 5\end{bmatrix}=\begin{bmatrix}2+2+0 & 4+0+0 & 6+1+0 & 8+2+0\\ 3+4+3 & 6+0+1 & 8+2+0 & 12+4+5\\ 1+0+3 & 2+0+1 & 3+0+0 & 5+0+4\end{bmatrix}$$

$$\Rightarrow BA=\begin{bmatrix}4 & 4 & 7 & 10\\ 10 & 7 & 11 & 21\\ 4 & 3 & 3 & 9\end{bmatrix}. \text{ Ans.}$$

Q.No.12.: If $A=\begin{bmatrix}1 & 2\\ -2 & 3\end{bmatrix}$, $B=\begin{bmatrix}2 & 1\\ 2 & 3\end{bmatrix}$ and $C=\begin{bmatrix}-3 & 1\\ 2 & 0\end{bmatrix}$,

verify that $(AB)C=A(BC)$ and $A(B+C)=AB+AC$.

Sol.: Now $AB=\begin{bmatrix}1 & 2\\ -2 & 3\end{bmatrix}\begin{bmatrix}2 & 1\\ 2 & 3\end{bmatrix}=\begin{bmatrix}2+4 & 1+6\\ -4+6 & -2+9\end{bmatrix}=\begin{bmatrix}6 & 7\\ 2 & 7\end{bmatrix}$.

$$\therefore (AB)C=\begin{bmatrix}6 & 7\\ 2 & 7\end{bmatrix}\begin{bmatrix}-3 & 1\\ 2 & 0\end{bmatrix}=\begin{bmatrix}-18+14 & 6+0\\ -6+14 & 2+0\end{bmatrix}=\begin{bmatrix}-4 & 6\\ 8 & 2\end{bmatrix}. \qquad \text{(i)}$$

$$\text{Now } BC=\begin{bmatrix}2 & 1\\ 2 & 3\end{bmatrix}\begin{bmatrix}-3 & 1\\ 2 & 0\end{bmatrix}=\begin{bmatrix}-6+2 & 2+0\\ -6+6 & 2+0\end{bmatrix}=\begin{bmatrix}-4 & 2\\ 2 & 2\end{bmatrix}.$$

$$\therefore A(BC)=\begin{bmatrix}1 & 2\\ -2 & 3\end{bmatrix}\begin{bmatrix}-4 & 2\\ 2 & 2\end{bmatrix}=\begin{bmatrix}-4+0 & 2+4\\ 8+0 & -4+6\end{bmatrix}=\begin{bmatrix}-4 & 6\\ 8 & 2\end{bmatrix}. \qquad \text{(ii)}$$

From (i) and (ii), we get $(AB)C=A(BC)$.

$$\text{Now } B+C=\begin{bmatrix}2 & 1\\ 2 & 3\end{bmatrix}+\begin{bmatrix}-3 & 1\\ 1 & 0\end{bmatrix}=\begin{bmatrix}-1 & 2\\ 4 & 3\end{bmatrix}$$

$$\therefore A(B+C)=\begin{bmatrix}1 & 2\\ -2 & 3\end{bmatrix}\begin{bmatrix}-1 & 2\\ 4 & 3\end{bmatrix}=\begin{bmatrix}-1+8 & 2+6\\ 2+12 & -4+9\end{bmatrix}=\begin{bmatrix}7 & 8\\ 14 & 5\end{bmatrix}. \qquad \text{(iii)}$$

$$\text{Now } AC=\begin{bmatrix}1 & 2\\ -2 & 3\end{bmatrix}\begin{bmatrix}-3 & 1\\ 2 & 0\end{bmatrix}=\begin{bmatrix}-3+4 & 1+0\\ 6+6 & -2+0\end{bmatrix}=\begin{bmatrix}1 & 1\\ 12 & -2\end{bmatrix}$$

$$\therefore AB+AC=\begin{bmatrix}6 & 7\\ 2 & 7\end{bmatrix}+\begin{bmatrix}1 & 1\\ 12 & -2\end{bmatrix}=\begin{bmatrix}7 & 8\\ 14 & 5\end{bmatrix}. \qquad \text{(iv)}$$

From(iii) and (iv), we get $A(B+C)=AB+AC$.

Hence verified.

Q.No.13.: Evaluate **(i)** $\begin{bmatrix}x & y & z\end{bmatrix}\begin{bmatrix}a & h & g\\ h & b & f\\ g & f & c\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}$,

(ii) $\begin{bmatrix}2 & 1 & -1\\ 4 & -5 & 6\\ -3 & 7 & 3\end{bmatrix}\times\begin{bmatrix}3 & 1\\ -6 & 4\\ -2 & 5\end{bmatrix}\times\begin{bmatrix}5 & 3\\ -2 & 1\end{bmatrix}$,

(iii) $\begin{bmatrix}1\\ -2\\ 3\end{bmatrix}\times\begin{bmatrix}4 & 5 & 2\end{bmatrix}\times\begin{bmatrix}2\\ -3\\ 5\end{bmatrix}\times\begin{bmatrix}3 & 2\end{bmatrix}$.

Sol.: (i). $\begin{bmatrix}x & y & z\end{bmatrix}\begin{bmatrix}a & h & g\\ h & b & f\\ g & f & c\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}ax+hy+gz & hx+by+fz & gx+fy+zc\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}$

$$=\begin{bmatrix}ax^2+hxy+gxz+hxy+by^2+fzy+gzx+fyz+z^2c\end{bmatrix}$$

$$=\begin{bmatrix}ax^2+by^2+cz^2+2hxy+2fyz+2gzx\end{bmatrix}. \text{ Ans.}$$

(ii). Now $\begin{bmatrix}2 & 1 & -1\\ 4 & -5 & 6\\ -3 & 7 & 3\end{bmatrix}_{3\times 3}\times\begin{bmatrix}3 & 1\\ -6 & 4\\ -2 & 5\end{bmatrix}_{3\times 2}\times\begin{bmatrix}5 & 3\\ -2 & 1\end{bmatrix}_{2\times 2}$

$$=\begin{bmatrix}6-6+2 & 2+4-5\\ 12+30-12 & 4-20+30\\ -9-42-6 & -3+28+15\end{bmatrix}\times\begin{bmatrix}5 & 3\\ -2 & 1\end{bmatrix}=\begin{bmatrix}2 & 1\\ 30 & 14\\ -57 & 40\end{bmatrix}_{3\times 2}\times\begin{bmatrix}5 & 3\\ -2 & 1\end{bmatrix}_{2\times 2}$$

$$=\begin{bmatrix} 10-2 & 6+1 \\ 150-28 & 90+14 \\ -285-80 & -171+40 \end{bmatrix}=\begin{bmatrix} 8 & 7 \\ 122 & 104 \\ -365 & -131 \end{bmatrix}. \text{ Ans.}$$

(iii). Now $\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}_{3\times1} \times \begin{bmatrix} 4 & 5 & 2 \end{bmatrix}_{1\times3} \times \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}_{3\times1} \times \begin{bmatrix} 3 & 2 \end{bmatrix}_{1\times2}$

$$=\begin{bmatrix} 4 & 5 & 2 \\ -8 & -10 & -4 \\ 12 & 15 & 6 \end{bmatrix}_{3\times3} \times \begin{bmatrix} 6 & 4 \\ -9 & -6 \\ 15 & 10 \end{bmatrix}_{3\times2}$$

$$=\begin{bmatrix} 24-45+30 & 16-30+20 \\ -48+90-60 & -32+60-40 \\ 72-135+90 & 48-90+60 \end{bmatrix}=\begin{bmatrix} 9 & 6 \\ -18 & -12 \\ 27 & 18 \end{bmatrix}. \text{ Ans.}$$

Q.No.14.: Prove that the product of two matrices $\begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$ and $\begin{bmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\phi \end{bmatrix}$ is a null matrix when θ and ϕ differ by an odd multiple of $\frac{\pi}{2}$.

Sol.: Here product of two matrices $=\begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}\begin{bmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\phi \end{bmatrix}$

$$=\begin{bmatrix} \cos^2\theta\cos^2\phi+\cos\theta\cos\phi\sin\theta\sin\phi & \cos^2\theta\cos\phi\sin\phi+\cos\theta\sin\theta\sin^2\phi \\ \cos\theta\sin\phi\cos^2\phi+\sin^2\theta\cos\phi\sin\phi & \cos\theta\cos\phi\sin\theta\sin\phi+\sin^2\theta\sin^2\phi \end{bmatrix}$$

$$=\begin{bmatrix} \cos\theta\cos\phi[\cos(\theta-\phi)] & \cos\theta\sin\phi[\cos(\theta-\phi)] \\ \cos\phi\sin\theta[\cos(\theta-\phi)] & \sin\theta\sin\phi[\cos(\theta-\phi)] \end{bmatrix}.$$

Now if above matrix is a null matrix, then

$$\cos(\theta-\phi)=0 \Rightarrow \theta-\phi=(2n+1)\frac{\pi}{2} \Rightarrow \theta=\phi+(2n+1)\frac{\pi}{2}.$$

Hence, θ and ϕ differ by an odd multiple of $\frac{\pi}{2}$.

This is the required result.

Q.No.15.: If $A = \begin{bmatrix} 0 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 0 \end{bmatrix}$, show that $I + A = (I - A)\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$.

Sol.: Now $I + A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 0 \end{bmatrix} = \begin{bmatrix} 1 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 1 \end{bmatrix}$. (i)

and $I - A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 0 \end{bmatrix} = \begin{bmatrix} 1 & \tan\frac{\alpha}{2} \\ -\tan\frac{\alpha}{2} & 1 \end{bmatrix}$

$$\therefore (I - A)\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} = \begin{bmatrix} 1 & \tan\frac{\alpha}{2} \\ -\tan\frac{\alpha}{2} & 1 \end{bmatrix} \begin{bmatrix} \dfrac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} & \dfrac{-2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} \\ \dfrac{2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} & \dfrac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} + \dfrac{2\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} & \dfrac{-2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} + \dfrac{\tan\frac{\alpha}{2} - \tan^3\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} \\ \dfrac{-\tan\frac{\alpha}{2} + \tan^3\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} + \dfrac{2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} & \dfrac{2\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} + \dfrac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} \end{bmatrix} = \begin{bmatrix} \dfrac{1+\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} & \dfrac{-\tan\frac{\alpha}{2}\left(1+\tan^2\frac{\alpha}{2}\right)}{1+\tan^2\frac{\alpha}{2}} \\ \dfrac{\tan\frac{\alpha}{2}\left(1+\tan^2\frac{\alpha}{2}\right)}{1+\tan^2\frac{\alpha}{2}} & \dfrac{1+\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 1 \end{bmatrix}. \quad \text{(ii)}$$

From (i) and (ii), we get $I + A = (I - A)\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$.

This completes the proof.

Q.No.16.: If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, show that $A^2 - 5A + 7I = O$, where I is a unit matrix of second order.

Sol.: Given $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ $\therefore A^2 = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 9-1 & 3+2 \\ -3-2 & -1+4 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$,

$5A = \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix}$ and $7\,I = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$.

$$\therefore A^2 - 5A + 7I = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} -7 & 0 \\ 0 & -7 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence $A^2 - 5A + 7I = O$. This completes the proof.

Q.No.17.: If $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}$ and I is the unit matrix of order 3,

evaluate $A^2 - 3A + 9\,I$.

Sol.: Given $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}$.

$$\therefore A^2 = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1-4-9 & -2-6+3 & 3+2+6 \\ 2+6+3 & -4+9-1 & 6-3-2 \\ -3+2-6 & 6+3+2 & -9-1+4 \end{bmatrix}$$

$$= \begin{bmatrix} -12 & -5 & 11 \\ 11 & 4 & 1 \\ -7 & 11 & -6 \end{bmatrix},$$

$3A = \begin{bmatrix} 3 & -6 & 9 \\ 6 & 9 & -3 \\ -9 & 3 & 6 \end{bmatrix}$ and $9\,I = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$

$$\therefore A^2 - 3A + 9\,I = \begin{bmatrix} -12 & -5 & 11 \\ 11 & 4 & 1 \\ -7 & 11 & -6 \end{bmatrix} - \begin{bmatrix} 3 & -6 & 9 \\ 6 & 9 & -3 \\ -9 & 3 & 6 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} -15 & 1 & 2 \\ 5 & -5 & 4 \\ 2 & 8 & -12 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} -6 & 1 & 2 \\ 5 & 4 & 4 \\ 2 & 8 & -3 \end{bmatrix}$$

Hence $A^2 - 3A + 9\,I = \begin{bmatrix} -6 & 1 & 2 \\ 5 & 4 & 4 \\ 2 & 8 & -3 \end{bmatrix}$. Ans.

Q.No.18.: If $A=\begin{bmatrix}1 & 2 & -1\\ 2 & 0 & 3\\ 0 & 1 & 2\end{bmatrix}$, $B=\begin{bmatrix}3 & -1 & 1\\ 0 & 0 & 2\\ 4 & -3 & 2\end{bmatrix}$,

verify the result $(A+B)^2 = A^2+BA+AB+B^2$.

Sol.: Now $A+B=\begin{bmatrix}1 & 2 & -1\\ 2 & 0 & 3\\ 0 & 1 & 2\end{bmatrix}+\begin{bmatrix}3 & -1 & 1\\ 0 & 0 & 2\\ 4 & -3 & 2\end{bmatrix}=\begin{bmatrix}4 & 1 & 0\\ 2 & 0 & 5\\ 4 & -2 & 4\end{bmatrix}$

$$\therefore (A+B)^2=\begin{bmatrix}4 & 1 & 0\\ 2 & 0 & 5\\ 4 & -2 & 4\end{bmatrix}\begin{bmatrix}4 & 1 & 0\\ 2 & 0 & 5\\ 4 & -2 & 4\end{bmatrix}=\begin{bmatrix}16+2+0 & 4+0+0 & 0+5+0\\ 4+0+20 & 2+0-10 & 0+0+20\\ 16-4+16 & 4+0-8 & 0-10+16\end{bmatrix}$$

$$=\begin{bmatrix}18 & 4 & 5\\ 28 & -8 & 20\\ 28 & -4 & 6\end{bmatrix}. \qquad \text{(i)}$$

Also $$A^2=\begin{bmatrix}1 & 2 & -1\\ 2 & 0 & 3\\ 0 & 1 & 2\end{bmatrix}\begin{bmatrix}1 & 2 & -1\\ 2 & 0 & 3\\ 0 & 1 & 2\end{bmatrix}=\begin{bmatrix}1+4+0 & 2+0-1 & -1+6-2\\ 2+0+0 & 4+0+3 & -2+0+6\\ 0+2+0 & 0+0+2 & 0+3+4\end{bmatrix}=\begin{bmatrix}5 & 1 & 3\\ 2 & 7 & 4\\ 2 & 2 & 7\end{bmatrix},$$

$$B^2=\begin{bmatrix}3 & -1 & 1\\ 0 & 0 & 2\\ 4 & -3 & 2\end{bmatrix}\begin{bmatrix}3 & -1 & 1\\ 0 & 0 & 2\\ 4 & -3 & 2\end{bmatrix}=\begin{bmatrix}9+0+4 & -3+0-3 & 3-2+2\\ 0+0+8 & 0+0-6 & 0+0+4\\ 12+0+8 & -4+0-6 & 4-6+4\end{bmatrix}=\begin{bmatrix}13 & -6 & 3\\ 8 & -6 & 4\\ 20 & -10 & 2\end{bmatrix},$$

$$AB=\begin{bmatrix}1 & 2 & -1\\ 2 & 0 & 3\\ 0 & 1 & 2\end{bmatrix}\begin{bmatrix}3 & -1 & 1\\ 0 & 0 & 2\\ 4 & -3 & 2\end{bmatrix}=\begin{bmatrix}3+0-4 & -1+0+3 & 1+4-2\\ 6+0+12 & -2+0-9 & 2+0+6\\ 0+0+8 & 0+0-6 & 0+2+4\end{bmatrix}=\begin{bmatrix}-1 & 2 & 3\\ 18 & -11 & 8\\ 8 & -6 & 6\end{bmatrix},$$

$$BA=\begin{bmatrix}3 & -1 & 1\\ 0 & 0 & 2\\ 4 & -3 & 2\end{bmatrix}\begin{bmatrix}1 & 2 & -1\\ 2 & 0 & 3\\ 0 & 1 & 2\end{bmatrix}=\begin{bmatrix}3-2+0 & 6+0+1 & -3-3+2\\ 0+0+0 & 0+0+2 & 0+0+4\\ 4-6+0 & 8+0+2 & -4-9+4\end{bmatrix}=\begin{bmatrix}1 & 7 & -4\\ 0 & 2 & 4\\ -2 & 10 & -9\end{bmatrix},$$

$$\therefore A^2+BA+AB+B^2=\begin{bmatrix}5 & 1 & 3\\ 2 & 7 & 4\\ 2 & 2 & 7\end{bmatrix}+\begin{bmatrix}1 & 7 & -4\\ 0 & 2 & 4\\ -2 & 10 & -9\end{bmatrix}+\begin{bmatrix}-1 & 2 & 3\\ 18 & -11 & 8\\ 8 & -6 & 6\end{bmatrix}+\begin{bmatrix}13 & -6 & 3\\ 8 & -6 & 4\\ 20 & -10 & 2\end{bmatrix}$$

$$= \begin{bmatrix} 5+1-1+13 & 1+7+2-6 & 3-4+3+3 \\ 2+0+18+8 & 7+2-11-6 & 4+4+8+4 \\ 2-2+8+20 & 2+10-6-10 & 7-9+6+2 \end{bmatrix}$$

$$= \begin{bmatrix} 18 & 4 & 5 \\ 28 & -8 & 20 \\ 28 & -4 & 6 \end{bmatrix}. \quad \text{(i)}$$

From (i) and (ii), we get $(A+B)^2 = A^2 + BA + AB + B^2$.

Hence, the result is verified.

Q.No.19.: If $E = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ and $F = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$,

calculate the products EF and FE and show that $E^2F + F^2E \neq E$.

Sol.: Now $EF = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0+1+0 & 0+0+0 & 0+0+0 \\ 0+0+0 & 0+0+1 & 0+0+0 \\ 0+0+0 & 0+0+0 & 0+0+0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

and $FE = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Ans.

Now $E^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $E^2F = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$,

$F^2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, $F^2E = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

$\therefore E^2F + F^2E = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

$\Rightarrow E^2F + F^2E \neq E$.

Q.No.20.: By mathematical induction, prove that if $A = \begin{bmatrix} 11 & -25 \\ 4 & -9 \end{bmatrix}$, then

$$A^n = \begin{bmatrix} 1+10n & -25n \\ 4n & 1-10n \end{bmatrix}.$$

Sol.: For n = 1, $A^1 = \begin{bmatrix} 11 & -25 \\ 4 & -9 \end{bmatrix} = \begin{bmatrix} 1+10.1 & -25.1 \\ 4.1 & -1-10.1 \end{bmatrix}$.

Thus, the result is true for n = 1.

Now, let us suppose that the result is true for n = k, then $A^k = \begin{bmatrix} 1+10k & -25k \\ 4k & 1-10k \end{bmatrix}$.

Now, we have to prove that the result is true for n = k + 1.

Now $A^{k+1} = A^k.A = \begin{bmatrix} 1+10k & -25k \\ 4k & 1-10k \end{bmatrix}\begin{bmatrix} 11 & -25 \\ 4 & -9 \end{bmatrix} = \begin{bmatrix} 11+10k & -25-25k \\ 4k+4 & -9-10k \end{bmatrix}$

$$A^{k+1} = \begin{bmatrix} 1+10(k+1) & -15(k+1) \\ 4(k+1) & 1-10(k+1) \end{bmatrix}.$$

Thus the result is also true for n = k + 1.

Hence, this proves the result.

Q.No.21.: If $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, show that $A^n = \begin{bmatrix} \cos n\alpha & \sin n\alpha \\ -\sin n\alpha & \cos n\alpha \end{bmatrix}$,

where n is a positive integer.

Sol.: For n = 1, $A^1 = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} = \begin{bmatrix} \cos 1\alpha & \sin 1\alpha \\ -\sin 1\alpha & \cos 1\alpha \end{bmatrix}$.

Thus, the result is true for n = 1.

Now, let us suppose that the result is true for n = k, then $A^k = \begin{bmatrix} \cos k\alpha & \sin k\alpha \\ -\sin k\alpha & \cos k\alpha \end{bmatrix}$.

Now, we have to prove that the result is true for n = k + 1.

Now $A^{k+1} = A^k.A = \begin{bmatrix} \cos k\alpha & \sin k\alpha \\ -\sin k\alpha & \cos k\alpha \end{bmatrix}\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$

$$= \begin{bmatrix} \cos k\alpha\cos\alpha - \sin k\alpha\sin\alpha & \cos k\alpha\sin\alpha + \sin k\alpha\cos\alpha \\ -(\cos\alpha\sin k\alpha + \sin\alpha\cos k\alpha) & -\sin k\alpha\sin\alpha + \cos\alpha\cos k\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos(k+1)\alpha & \sin(k+1)\alpha \\ -\sin(k+1)\alpha & \cos(k+1)\alpha \end{bmatrix}.$$

Thus, the result is also true for n = k + 1.

Hence, this proves the result.

Q.No.22.: Factorize the matrix $A = \begin{bmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{bmatrix}$ into LU, where L is lower triangular matrix and U is the upper triangular matrix.

Sol.: Let $L = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ and $U = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ 0 & b_{22} & b_{23} \\ 0 & 0 & b_{33} \end{bmatrix}$ be the lower triangular matrix and upper triangular matrix respectively.

Now $LU = A \Rightarrow \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ 0 & b_{22} & b_{23} \\ 0 & 0 & b_{33} \end{bmatrix} = \begin{bmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{11}b_{13} \\ a_{21}b_{11} & a_{21}b_{12} + a_{22}b_{22} & a_{21}b_{13} + a_{22}b_{23} \\ a_{31}b_{11} & a_{31}b_{12} + a_{32}b_{22} & a_{31}b_{13} + a_{32}b_{23} + a_{33}b_{33} \end{bmatrix} = \begin{bmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{bmatrix}$$

Equating, we get

$a_{11}b_{11} = 5$, $a_{11}b_{12} = -2$, $a_{11}b_{13} = 1$, $a_{21}b_{11} = 7$, $a_{21}b_{12} + a_{22}b_{22} = 1$,

$a_{21}b_{13} + a_{22}b_{23} = -5$, $a_{31}b_{11} = 3$, $a_{31}b_{12} + a_{32}b_{22} = 7$, $a_{31}b_{13} + a_{32}b_{23} + a_{33}b_{33} = 4$.

Since, we have 9 equations and we have to find 12 unknowns, so we can choose 3 unknowns arbitrary.

In other way, we have infinite number of such type of matrices whose product is A.

Now let us suppose $a_{11} = a_{22} = a_{33} = 1$.

$\therefore b_{11} = 5$, $b_{12} = -2$, $b_{13} = 1$, $a_{21} = \frac{7}{5}$, $a_{31} = \frac{3}{5}$,

$\frac{7}{5} \times (-2) + b_{22} = 1 \Rightarrow b_{22} = 1 + \frac{14}{5} = \frac{19}{5}$,

$\frac{7}{5} \times 1 + 1 \times b_{23} = -5 \Rightarrow b_{23} = -5 - \frac{7}{5} = \frac{-32}{5}$,

$\frac{7}{5} \times (-2) + a_{32} \times \frac{19}{5} = 7 \Rightarrow \frac{19}{5} a_{32} = \frac{41}{5} \Rightarrow a_{32} = \frac{41}{19}$,

$$\frac{3}{5}\times 1+\frac{41}{19}\times\frac{-32}{5}+b_{33}=4\Rightarrow\frac{57-3112}{95}+b_{33}=4\Rightarrow\frac{-251}{19}+b_{33}=4,$$

$$\Rightarrow b_{33}=4+\frac{251}{19}=\frac{76+251}{19}\Rightarrow b_{33}=\frac{327}{19}.$$

Thus $\begin{bmatrix}5 & -2 & 1\\ 7 & 1 & -5\\ 3 & 7 & 4\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ \frac{7}{5} & 1 & 0\\ \frac{3}{5} & \frac{41}{19} & 1\end{bmatrix}\begin{bmatrix}5 & -2 & 1\\ 0 & \frac{19}{5} & \frac{-32}{5}\\ 0 & 0 & \frac{327}{19}\end{bmatrix}$.

$\Rightarrow A = LU$.

Thus $L=\begin{bmatrix}1 & 0 & 0\\ \frac{7}{5} & 1 & 0\\ \frac{3}{5} & \frac{41}{19} & 1\end{bmatrix}$ and $U=\begin{bmatrix}5 & -2 & 1\\ 0 & \frac{19}{5} & \frac{-32}{5}\\ 0 & 0 & \frac{327}{19}\end{bmatrix}$ be the lower triangular and upper triangular matrices, respectively.

Q.No.23.: Show that $\begin{bmatrix}3 & 7-4i & -2+5i\\ 7+4i & -2 & 3+i\\ -2-5i & 3-i & 4\end{bmatrix}$ is a Hermitian matrix.

Sol.: A given matrix A is said to be Hermitian if $A = A^{\theta}$ or $A' = \overline{A}$.

Let $A=\begin{bmatrix}3 & 7-4i & -2+5i\\ 7+4i & -2 & 3+i\\ -2-5i & 3-i & 4\end{bmatrix}$.

$\therefore \overline{A}=\begin{bmatrix}3 & 7+4i & -2-5i\\ 7-4i & -2 & 3-i\\ -2+5i & 3+i & 4\end{bmatrix}$.

Also $A'=\begin{bmatrix}3 & 7+4i & -2-5i\\ 7-4i & -2 & 3-i\\ -2+5i & 3+i & 4\end{bmatrix}$.

$\therefore A' = \overline{A}$.

Hence, the given matrix is Hermitian.

Q.No.24.: If $A=\begin{bmatrix} 2 & 3+2i & -4 \\ 3-2i & 5 & 6i \\ -4 & -6i & 3 \end{bmatrix}$.

Then show that A is Hermitian and iA is Skew-Hermitian.

Sol.: Since, here $A=\begin{bmatrix} 2 & 3+2i & -4 \\ 3-2i & 5 & 6i \\ -4 & -6i & 3 \end{bmatrix}$.

Therefore $\overline{A}=\begin{bmatrix} 2 & 3-2i & -4 \\ 3+2i & 5 & -6i \\ -4 & 6i & 3 \end{bmatrix}$ and $\overline{A}'=\begin{bmatrix} 2 & 3+2i & -4 \\ 3-2i & 5 & 6i \\ -4 & -6i & 3 \end{bmatrix}=A$.

Thus A is Hermitian.

Let $B=iA=i\begin{bmatrix} 2 & 3+2i & -4 \\ 3-2i & 5 & +6i \\ -4 & -6i & 3 \end{bmatrix}=\begin{bmatrix} 2i & -2+3i & -4i \\ 2+3i & 5i & -6 \\ -4i & 6 & 3i \end{bmatrix}$.

Therefore $\overline{B}=\begin{bmatrix} -2i & -2-3i & 4i \\ 2-3i & -5i & -6 \\ 4i & 6 & -3i \end{bmatrix}$ and $B^T=\begin{bmatrix} 2i & 2+3i & -4i \\ -2+3i & 5i & 6 \\ -4i & -6 & 3i \end{bmatrix}$.

Thus $\overline{B}=-B^T \Rightarrow B$ is Skew-Hermitian.

Q.No.25.: If $A=\begin{bmatrix} 2+i & 3 & -1+3i \\ -5 & i & 4-2i \end{bmatrix}$, shows that AA* is a Hermitian matrix, where A* is the conjugate transpose of A.

Sol.: We have $A'=\begin{bmatrix} 2+i & -5 \\ 3 & i \\ -1+3i & 4-2i \end{bmatrix}$ and $A^*=\begin{bmatrix} 2-i & -5 \\ 3 & -i \\ -1-3i & 4+2i \end{bmatrix}$.

$$\therefore AA^*=\begin{bmatrix} 2-i & 3 & -1+3i \\ -5 & i & 4-2i \end{bmatrix}\begin{bmatrix} 2-i & -5 \\ 3 & -i \\ -1-3i & 4+2i \end{bmatrix}$$

$$=\begin{bmatrix} 4-i^2+9+1-9i^2, & -10-5-3i-10+10i \\ -10+5i+3i-10-10i, & 25-i^2+16-4i^2 \end{bmatrix}$$

$$= \begin{bmatrix} 24 & -20+2i \\ -20-2i & 46 \end{bmatrix}$$, which is a Hermitian matrix.

Q.No.26.: Prove that $\frac{1}{2}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix}$ is a unitary matrix.

Sol.: A given matrix A is said to be unitary if $AA^{\theta} = I$.

Let $A = \frac{1}{2}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix}$.

$\therefore \overline{A} = \frac{1}{2}\begin{bmatrix} 1-i & -1-i \\ 1-i & 1+i \end{bmatrix}$ and $A^{\theta} = \frac{1}{2}\begin{bmatrix} 1-i & 1-i \\ -1-i & 1+i \end{bmatrix}$.

Now $AA^{\theta} = \frac{1}{2}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix} \cdot \frac{1}{2}\begin{bmatrix} 1-i & 1-i \\ -1-i & 1+i \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 2+2 & 2-2 \\ 2-2 & 2+2 \end{bmatrix}$

$= \frac{1}{4}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$.

$\therefore AA^{\theta} = I$.

Hence, the given matrix is unitary.

Q.No.27.: Given that $A = \begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix}$, show that $(1-A)(1+A)^{-1}$ is a unitary matrix.

or

If $N = \begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix}$, Obtain the matrix $(I-N)(I+N)^{-1}$, and show that it is unitary.

Sol.: $I + A = \begin{bmatrix} 1 & 1+2i \\ -1+2i & 1 \end{bmatrix}$, $|I + A| = 1 - (-1-4) = 6$.

$(I+A)^{-1} = \begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix} \div 6$. Also $I - A = \begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}$

$$\therefore (I-A)(I+A)^{-1} = \begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}\begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix} \div 6 = \frac{1}{6}\begin{bmatrix} -4 & -2-4i \\ 2-4i & -4 \end{bmatrix} \qquad \text{(i)}$$

$$\text{Its conjugate-transpose } = \frac{1}{6}\begin{bmatrix} -4 & 2+4i \\ -2+4i & -4 \end{bmatrix} \qquad \text{(ii)}$$

$\therefore$ Product of (i) and (ii) $\frac{1}{36}\begin{bmatrix}-4 & -2-4i\\ 2-4i & -4\end{bmatrix}\begin{bmatrix}-4 & 2+4i\\ -2+4i & -4\end{bmatrix}=\frac{1}{36}\begin{bmatrix}36 & 0\\ 0 & 36\end{bmatrix}=I$.

Hence the result.

Or

Sol.: Since here $I-N=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}-\begin{bmatrix}0 & 1+2i\\ -1+2i & 0\end{bmatrix}=\begin{bmatrix}1 & -1-2i\\ 1-2i & 1\end{bmatrix}$.

$I+N=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}+\begin{bmatrix}0 & 1+2i\\ -1+2i & 0\end{bmatrix}=\begin{bmatrix}1 & 1+2i\\ -1+2i & 1\end{bmatrix}$.

$|I+N|=\begin{bmatrix}1 & 1+2i\\ -1+2i & 1\end{bmatrix}=1-\left(4i^2-1\right)=6$.

adj (I +N) $=\begin{bmatrix}1 & -1-2i\\ 1-2i & 1\end{bmatrix}$.

$(I+N)^{-1}=\frac{1}{|I+N|}\operatorname{adj}(I+N)=\frac{1}{6}\begin{bmatrix}1 & -1-2i\\ 1-2i & 1\end{bmatrix}$

$\therefore (I-N)(I+N)^{-1}=\begin{bmatrix}1 & -1-2i\\ 1-2i & 1\end{bmatrix}\frac{1}{6}\begin{bmatrix}1 & -1-2i\\ 1-2i & 1\end{bmatrix}=\frac{1}{6}\begin{bmatrix}-4 & -2-4i\\ 2-4i & -4\end{bmatrix}=A$ (say)

$A'=\frac{1}{6}\begin{bmatrix}-4 & 2-4i\\ -2-4i & -4\end{bmatrix}$

$\overline{(A')}=A^*=\frac{1}{6}\begin{bmatrix}-4 & 2+4i\\ -2+4i & -4\end{bmatrix}$

$A^*A=\frac{1}{6}\begin{bmatrix}-2 & 2+4i\\ -2+4i & -4\end{bmatrix}\frac{1}{6}\begin{bmatrix}-4 & -2-4i\\ 2-4i & -4\end{bmatrix}=\frac{1}{36}\begin{bmatrix}36 & 0\\ 0 & 36\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=1$

$\Rightarrow A=(I-N)(I+N)^{-1}$ is unitary.

Q.No.28.: If $S=\begin{bmatrix}1 & 1 & 1\\ 1 & a^2 & a\\ 1 & a & a^2\end{bmatrix}$, where $a=e^{i2\pi/3}$, then show that $S^{-1}=\frac{1}{3}\overline{S}$.

Sol.: Now $a=e^{i2\pi/3}=\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}=-\frac{1}{2}+i\frac{\sqrt{3}}{2}=\omega$ (cube root of unity).

$\therefore a^2=\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}=-\frac{1}{2}-i\frac{\sqrt{3}}{2}=\omega^2$

and $a^3 = e^{6i\pi/3} = e^{2i\pi} = \cos 2\pi + i\sin 2\pi = 1 = \omega^3$.

$$\therefore S = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{bmatrix}.$$

$$\text{Now } \overline{S} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \frac{1}{\omega^2} & \frac{1}{\omega} \\ 1 & \frac{1}{\omega} & \frac{1}{\omega^2} \end{bmatrix} \Rightarrow \overline{S} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}. \qquad [\because \omega^3 = 1]$$

$$\text{Also } |S| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{vmatrix} = (\omega^4 - \omega^2) - (\omega^2 - \omega) + (\omega - \omega^2)$$

$$= (\omega - \omega^2) + (\omega - \omega^2) + (\omega - \omega^2) = 3(\omega - \omega^2)$$

$$\text{And Adj } S = \begin{bmatrix} \omega^4 - \omega^2 & \omega - \omega^2 & \omega - \omega^2 \\ \omega - \omega^2 & \omega^2 - 1 & 1 - \omega \\ \omega - \omega^2 & 1 - \omega & \omega^2 - 1 \end{bmatrix}$$

$$\therefore S^{-1} = \frac{\text{Adj } A}{|S|} = \frac{1}{3(\omega - \omega^2)} \begin{bmatrix} \omega^4 - \omega^2 & \omega - \omega^2 & \omega - \omega^2 \\ \omega - \omega^2 & \omega^2 - 1 & 1 - \omega \\ \omega - \omega^2 & 1 - \omega & \omega^2 - 1 \end{bmatrix}$$

$$= \frac{1}{3(\omega - \omega^2)} \begin{bmatrix} \omega - \omega^2 & \omega - \omega^2 & \omega - \omega^2 \\ \omega - \omega^2 & \omega^2 - 1 & 1 - \omega \\ \omega - \omega^2 & 1 - \omega & \omega^2 - 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -\frac{1+\omega}{\omega} & \frac{1}{\omega} \\ 1 & \frac{1}{\omega} & -\frac{1+\omega}{\omega} \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix} \qquad \begin{bmatrix} \because 1 + \omega + \omega^2 = 0 \Rightarrow 1 + \omega = -\omega^2 \\ \omega^3 = 1 \Rightarrow \frac{1}{\omega} = \omega^2 \end{bmatrix}$$

$$= \frac{1}{3}\overline{S}.$$

Thus $\quad S^{-1} = \frac{1}{3}\overline{S}$.

Hence, this proves the result.

Home Assignments

Q.No.1.: Express A as the sum of a symmetric and skew-symmetric matrix where

$$A=\begin{bmatrix}4 & 2 & -3\\ 1 & 3 & -6\\ -5 & 0 & -7\end{bmatrix}$$

Ans.: $A+A^T=\frac{1}{2}\begin{bmatrix}8 & 3 & -8\\ 3 & 6 & -6\\ -8 & -6 & -14\end{bmatrix}$ symmetric,

$A-A^T=\frac{1}{2}\begin{bmatrix}0 & 1 & 2\\ -1 & 0 & -6\\ -2 & 6 & 0\end{bmatrix}$ skew-symmetric.

Q.No.2.: Prove that the inverse of a non-singular symmetric matrix A is symmetric.

Q.No.3.: Write $A=\begin{bmatrix}3 & -4 & -1\\ 6 & 0 & -1\\ -3 & 13 & -4\end{bmatrix}$ as the sum of a symmetric R and skew-symmetric S.

Ans.: $R=\frac{1}{2}\left[A+A^T\right]=\begin{bmatrix}3 & 1 & -2\\ 1 & 0 & 6\\ -2 & 6 & -4\end{bmatrix}$, $S=\frac{1}{2}\left[A-A^T\right]=\begin{bmatrix}0 & -5 & 1\\ 5 & 0 & -7\\ -1 & 7 & 0\end{bmatrix}$.

Q.No.4.: Prove that the product AB of two symmetric matrices A and B is symmetric if AB = BA.

Q.No.5.: Determine for what values of numbers a and b, c = aA + bB is Skew-Hermitian given that A and B are Skew-Hermitian.

Ans.: both a and b must be real.

Q.No.6.: If $A=\begin{bmatrix}0 & 1+2i\\ -1+2i & 0\end{bmatrix}$, show that $(I-A)(1+A)^{-1}$ is a unitary matrix.

Q.No.7.: Show that $A=\begin{bmatrix}a+ic & -b+id\\ b+id & a+ic\end{bmatrix}$ is unitary matrix if $a^2+b^2+c^2+d^2=1$.

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2^{nd} Topic

Matrices

Definitions of special types of Matrices-II

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Here, we will discuss some more definitions:

1. **Transpose of a matrix and their properties**
2. **Conjugate of a matrix and their properties**
3. **Transposed conjugate of a matrix and their properties**
4. **Symmetric, skew-symmetric matrices and their properties**

Complex Matrices:

5. **Hermitian, skew-Hermitian matrices and their properties**
6. **Normal matrix, Orthogonal (orthonormal) matrix and Unitary matrix**

Transpose of a matrix:

Definition: Let $A = \left[a_{ij}\right]_{m\times n}$. Then the $n \times m$ matrix obtained from A by changing its rows into columns and its columns into rows is called the transpose of A and is denoted by the symbol A' or A^T.

Symbolically: If $A = \left[a_{ij}\right]_{m\times n}$ then $A' = \left[b_{ji}\right]_{n\times m}$,

where $b_{ji} = a_{ij}$,

i.e., $(j,\ i)^{th}$ element of A' is the $(i,\ j)^{th}$ element of A.

Transposition: **The operation of interchanging rows with columns is called transposition.**

Example: The transpose of 3×4 matrix $A=\begin{bmatrix}1&2&3&4\\2&3&4&1\\3&4&2&1\end{bmatrix}_{3\times4}$ is the 4×3 matrix

$$A'=\begin{bmatrix}1&2&3\\2&3&4\\3&4&2\\4&1&1\end{bmatrix}_{4\times3}.$$

The first row of A is the first column of A'. The second row of A is the second column of A'. The third row of A is the third column of A'.

Properties of the transpose of matrix:

Theorem: If A' and B' be the transposes of A and B respectively, then

(i) $(A')'=A$,

(ii) $(A+B)'=A'+B'$, A and B being of the same size,

(iii) $(kA)'=kA'$, k being any complex number,

(iv) $(AB)'=B'A'$, A and B being comfortable to multiplication.

Proof:

(i). Let A be an $m\times n$ matrix.

Then A' will be an $n\times m$ matrix.

Therefore, $(A')'$ will be an $m\times n$ matrix.

Thus, the matrices A and $(A')'$ are the same type.

Also, the $(i,\ j)^{th}$ element of $(A')'=$ the $(j,\ i)^{th}$ element of $A'=$ the $(i,\ j)^{th}$ element of A.

Hence $A=(A')'$.

(ii). Let $A=\left[a_{ij}\right]_{m\times n}$ and $B=\left[b_{ij}\right]_{m\times n}$.

Then A+B will be a matrix of the type $m\times n$ and consequently $(A+B)'$ will be matrix of the type $n\times m$.

Again, A' and B' are both $n\times m$ matrices.

Therefore, the sum $A'+B'$ exist and will also be a matrix of the type $n\times m$.

Further, $(j,\ i)^{th}$ element of $(A+B)' =$ the $(i,\ j)^{th}$ element of $A+B = a_{ij}+b_{ij}$

= the $(i,\ j)^{th}$ element of A + the $(i,\ j)^{th}$ element of B

= the $(j,\ i)^{th}$ element of A' + the $(j,\ i)^{th}$ element of B'

= the $(j,\ i)^{th}$ element of $A'+B'$.

Thus the matrices $(A+B)'$ and $A'+B'$ are the same type and their $(j,\ i)^{th}$ elements are equal. Hence $(A+B)' = A'+B'$.

(iii). Let $A=\left[a_{ij}\right]_{m\times n}$. If k is any complex number, then kA will also be an $m\times n$ matrix and consequently $(kA)'$ will be an $n\times m$ matrix.

Again A' will be an $n\times m$ matrix and therefore kA' will also be $n\times m$ matrix .

Further, the $(j,\ i)^{th}$ element of $(kA)' =$ the $(i,\ j)^{th}$ element of $kA = k.(i, j)^{th}$ element of A

$= k.(j,\ i)^{th}$ element of $A' =$ the $(j,\ i)^{th}$ element of kA'.

Thus, the matrices $(kA)'$ and kA' are the same size and their $(j,\ i)^{th}$ elements are equal. Hence $(kA)' = kA'$.

(iv). Let $A=\left[a_{ij}\right]_{m\times n}$ and $B=\left[b_{ij}\right]_{m\times n}$,

then $A'=\left[c_{ji}\right]_{n\times m}$, where $c_{ji}=a_{ij}$ and $B'=\left[d_{kj}\right]_{p\times n}$, where $d_{kj}=b_{jk}$.

The matrix AB will be of the type $m\times p$.

Therefore the matrix $(AB)'$ will be of the type $p\times m$.

Again the matrix A' will be of the type $n\times m$ and the matrix B' will be of the type $p\times n$.

Therefore, the product $B'A'$ exists and will be a matrix of the type $p\times m$.

Thus, the matrices (AB) and $(k,\ i)^{th}$ are of the same type.

Now the $(k,\ i)^{th}$ element of $(AB)' =$ the $(i,\ k)^{th}$ element of $AB = \sum_{j=1}^{n} a_{ij}.b_{jk}$

$= \sum_{j=1}^{n} c_{ji}d_{kj} = \sum_{j=1}^{n} d_{kj}c_{ji} =$ the $(k,\ i)^{th}$ element of $B'A'$.

Thus, the matrices $(AB)'$ and $B'A'$ are the same size and their $(k,\ i)^{th}$ element is equal.

Hence $(AB)' = B'A'$.

The above law is called the reversal law for transposes, i.e., the transpose of the product of the transposes taken in reverse order.

Conjugate of a matrix:

Definition: The matrix obtained from any given matrix A on replacing its elements by the corresponding conjugate complex numbers is called the conjugate of A and is denoted by $\overline{A}$.

Symbolically: If $A = \left[a_{ij}\right]_{m\times n}$, then $\overline{A} = \left[\overline{a}_{ij}\right]_{m\times n}$,

where $\overline{a}_{ij}$ denotes the conjugate complex of a_{ij}.

If A be a matrix over the field of real numbers, then obviously $\overline{A}$ coincide with A.

Example: If $A = \begin{bmatrix} 2+3i & 4-7i & 8 \\ -i & 6 & 9+i \end{bmatrix}$, then $\overline{A} = \begin{bmatrix} 2-3i & 4+7i & 8 \\ i & 6 & 9-i \end{bmatrix}$.

Properties of the conjugate of a matrix:

Theorem: If $\overline{A}$ and $\overline{B}$ be the conjugates of A and B respectively, then

(i) $\overline{\left(\overline{A}\right)} = A$,

(ii) $\overline{(A+B)} = \overline{A} + \overline{B}$,

(iii) $\overline{(kA)} = \overline{k}\ \overline{A}$, k being any complex number,

(iv) $\overline{(AB)} = \overline{A}\ \overline{B}$, A and B conformable to multiplication.

Proof:

(i). Let $A = \left[a_{ij}\right]_{m\times n}$.

Then $\overline{A} = \left[\overline{a}_{ij}\right]_{m\times n}$, where $\overline{a}_{ij}$ is the conjugate complex of a_{ij}.

Obviously, both A and $\overline{\left(\overline{A}\right)}$ are matrices of the same type $m\times n$.

The $(i,\ j)^{th}$ element of $\overline{\left(\overline{A}\right)}$ = the conjugate complex of $(i,\ j)^{th}$ element of $\overline{A}$

= the conjugate complex of $\overline{a}_{ij}$ = $\overline{\left(\overline{a}_{ij}\right)}$ = a_{ij} = the $(i,\ j)^{th}$ element of A.

Hence $\overline{\left(\overline{A}\right)} = A$.

(ii). Let $A = \left[a_{ij}\right]_{m\times n}$ and $B = \left[b_{ij}\right]_{m\times n}$.

Then $\overline{A} = \left[\overline{a}_{ij}\right]_{m\times n}$ and $\overline{B} = \left[\overline{b}_{ij}\right]_{m\times n}$.

First we see both $\overline{(A+B)}$ and $\overline{A}+\overline{B}$ are $m\times n$ matrices.

Again the $(i,\ j)^{th}$ element of $\overline{(A+B)}$ = the conjugate complex of $(i,\ j)^{th}$ element of A + B

= the conjugate complex of $a_{ij}+b_{ij}$ = $\overline{\left(a_{ij}+b_{ij}\right)} = \overline{a}_{ij}+\overline{b}_{ij}$

= the $(i,\ j)^{th}$ element of $\overline{A}$ + the $(i,\ j)^{th}$ element of $\overline{B}$

= the $(i,\ j)^{th}$ element of $\overline{A}+\overline{B}$.

Hence $\overline{(A+B)} = \overline{A}+\overline{B}$.

(iii). Let $A = \left[a_{ij}\right]_{m\times n}$.

If k is any complex number, then both $\overline{(kA)}$ and $\overline{k}\ \overline{A}$ will be $m\times n$ matrices.

The $(i,\ j)^{th}$ element of $\overline{(kA)}$

= the conjugate complex of the $(i,\ j)^{th}$ element of kA

= the conjugate complex of $ka_{ij} = \overline{\left(ka_{ij}\right)} = \overline{k}\ \overline{a}_{ij}$

$= \overline{k}$. the $(i,\ j)^{th}$ element of $\overline{A}$ = the $(i,\ j)^{th}$ element of $\overline{k}\ \overline{A}$.

Hence $\overline{(kA)} = \overline{k}\ \overline{A}$.

(iv). Let $A = \left[a_{ij}\right]_{m\times n}$ and $B = \left[b_{jk}\right]_{n\times p}$

Then $\overline{A} = \left[\overline{a}_{ij}\right]_{m\times n}$ and $\overline{B} = \left[\overline{b}_{jk}\right]_{n\times p}$.

First we see that both the matrices $\overline{(AB)}$ and $\overline{A}\ \overline{B}$ are of the type $m\times p$.

Again the $(i,\ k)^{th}$ element of $\overline{(AB)}$

= the conjugate complex of the $(i,\ k)^{th}$ element of AB

= the conjugate complex of $\sum_{j=1}^{n} a_{ij}b_{jk}$

$$=\overline{\left(\sum_{j=1}^{n} a_{ij}b_{jk}\right)}=\sum_{j=1}^{n} \overline{a_{ij}b_{jk}}=\sum_{j=1}^{n} \overline{a}_{ij}\overline{b}_{jk}$$

= the $(i,\ k)^{th}$ element of $\overline{A}\ \overline{B}$.

Hence $\overline{(AB)}=\overline{A}\ \overline{B}$. **Hermitian conjugate**, or **transjugate**

Transposed conjugate of a matrix or Hermitian conjugate or Hermitian transpose or Adjoint matrix or Transjugate :

Definition: The **conjugate transpose**, **Hermitian transpose**, or **adjoint matrix** of an *m*-by-*n* matrix *A* with complex entries is the *n*-by-*m* matrix A^* obtained from *A* by taking the transpose and then taking the complex conjugate of each entry (i.e., negating their imaginary parts but not their real parts).

or

The transpose of the conjugate of a matrix A is called transposed conjugate of A and is denoted by A^{θ} or by $A*$.

Obviously, the conjugate of the transpose of A is the same as the transpose of the conjugate of A, i.e., $\overline{(A')}=\overline{(A)}\,'=A^{\theta}$.

Symbolically: If $A=\left[a_{ij}\right]_{m\times n}$, then $A^{\theta}=\left[b_{ji}\right]_{n\times m}$,

where $b_{ji}=\overline{a}_{ij}$,

i.e., $(j,\ i)^{th}$ element of A^{θ} = the conjugate complex of the $(i,\ j)^{th}$ element of A.

Example: If $A=\begin{bmatrix} 1+2i & 2-3i & 3+4i \\ 4-5i & 5+6i & 6-7i \\ 8 & 7+8i & 7 \end{bmatrix}$

then $A'=\begin{bmatrix} 1+2i & 4-5i & 8 \\ 2-3i & 5+6i & 7+8i \\ 3+4i & 6-7i & 7 \end{bmatrix}$ and $\overline{(A')}=A^{\theta}=\begin{bmatrix} 1-2i & 4+5i & 8 \\ 2+3i & 5-6i & 7-8i \\ 3-4i & 6+7i & 7 \end{bmatrix}$.

Motivation for developing conjugate transpose:

The conjugate transpose can be motivated by noting that complex numbers can be usefully represented by 2×2 skew-symmetric matrices, obeying matrix addition and multiplication:

$$a+ib \equiv \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

An *m*-by-*n* matrix of complex numbers could therefore equally well be represented by a *2m*-by-*2n* matrix of real numbers. It therefore arises very naturally that when transposing such a matrix which is made up of complex numbers, one may in the process also have to take the complex conjugate of each entry.

Properties of transposed conjugate of a matrix:

Theorem: If A^{θ} and B^{θ} be the transposed conjugates of A and B respectively, then

(i) $\left(A^{\theta}\right)^{\theta} = A$ **,**

(ii) $\left(A+B\right)^{\theta} = A^{\theta} + B^{\theta}$, A and B being the same size,

(iii) $\left(kA\right)^{\theta} = \bar{k}\ A^{\theta}$ **,** k being any complex number,

(iv) $\left(AB\right)^{\theta} = B^{\theta}A^{\theta}$ **,** A and B being conformable to multiplication.

Proof: **(i).** $\left(A^{\theta}\right) = \overline{\left[\left\{\left(\bar{A}\right)'\right\}'\right]} = \overline{\left(\bar{A}\right)} = A$,

since $\left\{\left(\bar{A}\right)'\right\}' = A$.

(ii). $\left(A+B\right)^{\theta} = \overline{\left\{\left(A+B\right)'\right\}}\ \overline{\left(A'+B'\right)} = \overline{\left(A'\right)} + \overline{\left(B'\right)} = A^{\theta} + B^{\theta}$ **.**

(iii). $\left(kA\right)^{\theta} = \overline{\left\{\left(kA\right)'\right\}} = \overline{\left(kA'\right)} = \bar{k}\overline{\left(A'\right)} = \bar{k}A^{\theta}$ **.**

(iv). $\left(AB\right)^{\theta} = \overline{\left\{\left(AB\right)'\right\}} = \overline{\left(B'A'\right)} = \overline{\left(B'\right)}\overline{\left(A'\right)} = B^{\theta}A^{\theta}$ **.**

Thus, the reversal law holds for the transposed conjugate also.

Symmetric and skew-symmetric matrices:

Symmetric matrix:

Definition: A **symmetric matrix** is a square matrix that is equal to its transpose.

or

A square matrix $A=\left[a_{ij}\right]$ is said to be symmetric if its $(i,\ j)^{th}$ element is the same as its $(j,\ i)^{th}$ element.

Symbolically: If $a_{ij}=a_{ji}$ for all j, i, then a square matrix $A=\left[a_{ij}\right]$ is said to be symmetric.

Examples: The matrices $\begin{bmatrix} a & h & g & p \\ h & b & f & q \\ g & f & c & r \\ p & q & r & s \end{bmatrix}, \begin{bmatrix} 1 & i & -2i \\ i & -2 & 4 \\ -2i & 4 & 3 \end{bmatrix}, \begin{bmatrix} 2 & 4 \\ 4 & 3 \end{bmatrix}$ are symmetric matrices.

Theorem: A necessary and sufficient condition for a matrix A to be symmetric is that A and A' are equal.

Proof: Necessary condition:

Let $A=\left[a_{ij}\right]$ to be an n-rowed symmetric matrix. Then $a_{ij}=a_{ji}$.

To show $A=A'$.

Now A' will also be an n-rowed square matrix.

Also the $(i,\ j)^{th}$ element of $A'=$ the $(j,\ i)^{th}$ element of $A=a_{ji}$

$$=a_{ij}=\text{the } (i,\ j)^{th} \text{ element of A.}$$

Hence $A'=A$.

Sufficient condition:

Let if $A'=A$, then A must be a square matrix.

To show; A is symmetric.

Also $(i,\ j)^{th}$ element of A = the $(i,\ j)^{th}$ element of A' $\quad [\because A=A']$

$$=\text{the } (j,\ i)^{th} \text{ element of A.}$$

Hence A is a symmetric matrix.

Skew-symmetric matrix or Antisymmetric matrix or antimetric matrix:

Definition: A **skew-symmetric** (or **antisymmetric** or **antimetric**) **matrix** is a square matrix *A* whose transpose is also its negative

A square matrix $A=\left[a_{ij}\right]$ is said to be skew-symmetric if the $(i,\ j)^{th}$ element of A is negative of the $(j,\ i)^{th}$ element of A.

$$\mathrm{Skew}\left(\begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}\right)=\begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}$$

$$\mathrm{Skew}(\bar{a})\bar{x}=\bar{a}\times\bar{x}$$

Symbolically: If $a_{ij}=-a_{ji}$ for all i, j, then a square matrix $A=\left[a_{ij}\right]$ is said to be skew-symmetric.

Result: Show that the diagonal elements of a skew-symmetric matrix are all zero:

Proof: If A is a skew-symmetric matrix, then $a_{ij}=-a_{ji}$. [by definition]

$\therefore a_{ii}=-a_{ii}$ for all values of i .

$\Rightarrow 2a_{ii}=0\Rightarrow a_{ii}=0$.

Thus, the **diagonal elements** of a skew-symmetric matrix are **all zero**.

Examples: The matrices $\begin{bmatrix} 0 & h & g \\ -h & 0 & f \\ -g & -f & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & -3i & -4 \\ 3i & 0 & 8 \\ 4 & -8 & 0 \end{bmatrix}$ are skew-symmetric matrices.

Theorem: A necessary and sufficient condition for a matrix A to be skew-symmetric is that $A=-A'$.

Proof: Necessary condition:

Let A be an n-rowed skew-symmetric matrix. Then $a_{ij}=-a_{ji}$

To show : $A=A'$.

Now $-A$ and A' are both n-rowed square matrices.

Also the $(i,\ j)^{th}$ element of $A' =$ the $(j,\ i)^{th}$ element of A

$$= a_{ji} = -a_{ij} = \text{the } (i,\ j)^{th} \text{ element of } -A.$$

Hence $A' = -A$.

Sufficient condition:

Let $A' = -A$, then A must be a square matrix.

To show: A is skew-symmetric matrix.

Now the $(i,\ j)^{th}$ element of A= the negative of the $(i,\ j)^{th}$ element of A' $[\because A = -A']$

$$= \text{the negative of the } (j,\ i)^{th} \text{ element of A.}$$

Hence, A is a skew-symmetric matrix.

Some important properties of symmetric and skew-symmetric matrices:

(1). If A is a symmetric (skew-symmetric) matrix, then show that kA is also symmetric (skew symmetric).

Proof: (i). Let A be symmetric matrix. Then $A' = A$.

We have $(kA)' = kA'$ $[\because A' = A]$

$= kA.$

Since $(kA)' = kA$, therefore kA is a symmetric matrix.

(ii). Let A be skew symmetric matrix. Then $A' = -A$.

We have $(kA)' = kA' = k(-A)$ $[\because A' = -A]$

$= -(kA)$.

Since $(kA)' = -(kA)$, therefore, kA is a skew-symmetric matrix.

(2). If A, B are symmetric (skew-symmetric), then so is also A + B.

Proof: (i). Let A and B be two symmetric matrices of the same order.

Then $A' = A$ and $B' = B$.

Now $(A+B)' = A' + B' = A + B$.

Since $(A+B)' = A + B$, therefore, A + B is a symmetric matrix.

(ii). Let A and B be two skew-symmetric matrices of the same order.

Then $A' = -A$ and $B' = -B$.

Now $(A+B)'=A'+B'=(-A)+(-B)=-(A+B$.

Since $(A+B)'=-(A+B)$, therefore, A + B is a skew-symmetric matrix.

(3). If A and B are symmetric matrices, then show that AB is symmetric if and only if A and B commute i. e. AB = BA.

Proof: It is given that A and B are two symmetric matrices.

Therefore $A'=A$ and $B'=B$.

Now suppose that AB = BA.

Then to prove that AB is symmetric.

We have $(AB)'=B'A'=BA$ $\quad [\because A'=A, \quad B'=B]$

$=AB$ $\quad [\because AB=BA]$

Since $(AB)'=AB$, therefore AB is symmetric matrix.

Conversely, suppose that AB is a symmetric matrix.

Then to prove that AB + BA.

We have $AB=(AB)'$ $\quad [\because$ AB is a symmetric matrix$]$

$=B'A'=BA$.

(4). If A be any matrix, then prove that AA' and $A'A$ are both symmetric matrices.

Proof: Let A be any matrix.

We have $(AA')'=(A')'A'$ [by reversal law for transposes]

$=AA'$ $\quad [\because (A')'=A]$

Since $(AA')'=AA'$, therefore AA' is a symmetric matrix.

Again $(AA')'=A'A$, therefore $A'A$ is a symmetric matrix.

(5). If A be any square matrix, then show that $A+A'$ is a symmetric and $A-A'$ is skew-symmetric.

Proof: We have $(A+A')'=A'+(A')'=A'+A=A+A'$.

Hence $A+A'$ is symmetric.

Again $(A-A')'=A'-(A')'=A'-A=-(A-A')$.

Hence, $A-A'$ is skew-symmetric.

(6). Show that the matrix $B'AB$ is symmetric or skew-symmetric according as A is symmetric or skew-symmetric.

Proof: Case I. Let A be a symmetric matrix. Then $A' = A$.

Now $(B'Ab)' = B'A'(B')'$, by the reversal law for the transposes

$$= B'A'B \qquad [\text{since } (B')' = B]$$

$$= B'AB.$$

Hence, $B'AB$ is symmetric.

Case II. Let A be a skew symmetric matrix. Then $A' = -A$.

Now $(B'AB)' = B'A'(B)' = B'A'B = B'(-A)B$,

$$= -(B'A)B = -B'AB$$

Hence, $B'AB$ is skew-symmetric.

(7). Show that every square matrix is uniquely expressible as the sum of a symmetric matrix and a skew-symmetric matrix.

Proof: Let A be any square matrix.

We can write $A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A') = P + Q$, say,

where $P = \frac{1}{2}(A + A')$ and $Q = \frac{1}{2}(A - A')$.

We have $P' = \left\{\frac{1}{2}(A + A')\right\}' = \frac{1}{2}(A + A')' \quad [\because (kA)' = kA']$

$$= \frac{1}{2}\{A' + (A')'\} \qquad [\because (A+B)' = A' + B']$$

$$= \frac{1}{2}(A' + A) \qquad [\because (A')' = A]$$

$$= \frac{1}{2}(A' + A) = P.$$

Therefore, P is symmetric matrix.

Again $Q' = \left\{\frac{1}{2}(A - A')\right\}' = \frac{1}{2}(A - A')' = \frac{1}{2}(A - A')' = \frac{1}{2}\{A' - (A')'\}$

$$= \frac{1}{2}(A' - A) = -\frac{1}{2}(A - A') = -Q.$$

Therefore, Q is a skew-symmetric matrix.

Thus we have expressed the square matrix A as the sum of a symmetric and a skew-symmetric matrix.

To prove: The above representation is unique.

Let A= R + S be another such representation of A, where R is symmetric and S skew-symmetric.

Then to prove that R = P and S = Q.

We have $A' = (R+S)' = R' + S' = R - S \qquad [\because R' = R \text{ and } S' = -S]$

$\therefore A + A' = 2R$ and $A' - A = 2S$.

This gives $R = \frac{1}{2}(A + A')$ and $S = \frac{1}{2}(A - A')$.

Thus, R = P and S + Q.

Therefore, the representation is unique.

Thus, every square matrix is uniquely expressible as the sum of a symmetric matrix and a skew-symmetric matrix.

Hermitian and skew-Hermitian matrices:

Hermitian matrix (or Self-adjoint matrix):

 

Charles Hermite

(December 24, 1822 – January 14, 1901) ,

(French mathematician)

Definition: A square matrix $A = [a_{ij}]$ is said to be Hermitian if the $(i,\ j)^{th}$ element of A is equal to the conjugate complex of the $(j,\ i)^{th}$ element of A,

i.e., $a_{ij} = \bar{a}_{ji}$ for all i, j.

or

A Hermitian matrix (or self-adjoint matrix) is a square matrix with complex entries which is equal to its own conjugate transpose

Examples: The matrices $\begin{bmatrix} a & b+ic \\ b-ic & d \end{bmatrix}, \begin{bmatrix} 1 & 2-3i & 3+4i \\ 2+3i & 0 & 4-5i \\ 3-4i & 4+5i & 2 \end{bmatrix}$ are Hermitian matrices.

Result: Show that every diagonal element of a Hermitian matrix must be real.

Proof: If A is a Hermitian matrix, then $a_{ii} = \bar{a}_{ii}$. [by definition]

$\Rightarrow a_{ii}$ is real for all i.

Thus, **every diagonal element** of a Hermitian matrix must be **real**.

Remarks: A Hermitian matrix over the field of real numbers is nothing but a real symmetric matrix.

Thus, a Hermitian matrix is a generalization of a real symmetric matrix as every real symmetric matrix is Hermitian.

Obviously, a necessary and sufficient condition for a matrix A to be Hermitian is that $A = A^{\theta}$.

Skew-Hermitian matrix or Antihermitian matrix:

Definition: A square matrix $A = \left[a_{ij}\right]$ is said to be skew-Hermitian if the $(i,\ j)^{th}$ element of A is equal to the negative of the conjugate complex of the $(j,\ i)^{th}$ element of A, i.e., if $a_{ij} = -\bar{a}_{ji}$ for all i and j.

or

A square matrix with complex entries is said to be **skew-Hermitian** or **antihermitian** if its conjugate transpose is equal to its negative

Result: Show that the diagonal elements of a skew-Hermitian matrix must be pure imaginary numbers or zero.

Proof: If A is a skew-Hermitian matrix, then $a_{ii} = -\bar{a}_{ii}$. [by definition]

$\Rightarrow a_{ii} + \bar{a}_{ii} = 0$

i.e., a_{ii} must be either a pure imaginary number or must be zero.

Thus, the **diagonal elements** of a skew-Hermitian matrix must be **pure imaginary** numbers or **zero**.

Examples: The matrices $\begin{bmatrix} 0 & -2-i \\ 2-i & 0 \end{bmatrix}, \begin{bmatrix} -i & 3+4i \\ -3+4i & 0 \end{bmatrix}$ are skew-Hermitian matrices.

Remarks: A skew-Hermitian matrix over the field of real numbers is nothing but a real skew-symmetric matrix.

Thus, a skew Hermitian matrix is a generalization of real skew symmetric matrix.

Obviously, a necessary and sufficient condition for a matrix A to be skew-Hermitian is that $A^{\theta} = -A$.

Some important properties of Hermitian and skew Hermitian matrices:

(1). If A is Hermitian matrix, show that iA is skew-Hermitian.

Proof: Let A be a Hermitian matrix. Then $A^{\theta} = A$.

We have $(iA)^{\theta} = \bar{i}A^{\theta}$ $\left[\because (kA)^{\theta} = \bar{k}A^{\theta}\right]$

$$=(-i)A^{\theta} \qquad \left[\because \overline{i}=-i\right]$$

$$=-\left(iA^{\theta}\right)=-(iA) \quad \left[\because A^{\theta}=A\right].$$

Since $(iA)^{\theta}=-(iA)$, therefore iA is a skew-Hermitian matrix.

(2). If A is skew-Hermitian matrix, show that iA is Hermitian.

Proof: Let A be a skew-Hermitian matrix. Then $A^{\theta}=-A$.

We have $(iA)^{\theta}=\overline{i}A^{\theta}=(-i)A^{\theta}=-\left(iA^{\theta}\right)$

$$=-\{i(-A)\}=-\{i(-A)\} \qquad \left[\because A^{\theta}=-A\right].$$

$$=-\{-(iA)\}=iA.$$

Since $(iA)^{\theta}=iA$, therefore iA is a Hermitian matrix.

(3). If A, B are Hermitian or skew-Hermitian, then so is also A + B.

Proof: (i). Let A and B be two Hermitian matrices of the same order.

Then $A^{\theta}=A$ and $B^{\theta}=B$.

Now $(A+B)^{\theta}=A^{\theta}+B^{\theta}=A+B$.

Since $(A+B)^{\theta}=A+B$, therefore A + B is a Hermitian matrix.

(ii). Let A and B be two skew-Hermitian matrices of the same order.

Then $A^{\theta}=-A$ and $B^{\theta}=-B$.

Now $(A+B)^{\theta}=A^{\theta}+B^{\theta}=-A+(-B)=-(A+B)$.

Since $(A+B)^{\theta}=-(A+B)$, therefore A + B is a skew-Hermitian matrix.

(4). A and B are Hermitian; show that $AB+BA$ is Hermitian and $AB-BA$ is skew-Hermitian.

Proof: Let A and B be two Hermitian matrices of the same order.

Then $A^{\theta}=A$ and $B^{\theta}=B$.

Now $(AB+BA)^{\theta}=(AB)^{\theta}+(BA)^{\theta}=B^{\theta}A^{\theta}+A^{\theta}B^{\theta}=BA+AB=AB+BA$.

Hence $AB+BA$ is Hermitian.

Again $(AB-BA)^{\theta}=(AB)^{\theta}-(BA)^{\theta}=B^{\theta}A^{\theta}-A^{\theta}B^{\theta}=BA-AB=-(AB-BA)$.

Hence $AB-BA$ is skew-Hermitian.

(5). If A be any square matrix, prove that $A+A^{\theta}$, AA^{θ}, $A^{\theta}A$ are all Hermitian and $A-A^{\theta}$ is skew-Hermitian.

Proof: The necessary and sufficient condition for a matrix A to be Hermitian is that A^{θ} and A are equal.

(i). $\left(A+A^{\theta}\right)^{\theta}=A^{\theta}+\left(A^{\theta}\right)^{\theta}=A^{\theta}+A=A+A^{\theta}$.

Hence $A+A^{\theta}$ is Hermitian.

(ii). $\left(AA^{\theta}\right)^{\theta}=\left(A^{\theta}\right)^{\theta}A^{\theta}$ [by the reversal law for conjugate transposes].

$=AA^{\theta}$.

Hence AA^{θ} is Hermitian.

(iii). $\left(A^{\theta}A\right)^{\theta}=A^{\theta}\left(A^{\theta}\right)^{\theta}=A^{\theta}A$.

Hence $A^{\theta}A$ is Hermitian.

(iv). The necessary and sufficient condition for a matrix A to be skew-Hermitian is that $-A$ and A^{θ} are equal.

Now $\left(A-A^{\theta}\right)^{\theta}=A^{\theta}-\left(A^{\theta}\right)^{\theta}=A^{\theta}-A=-\left(A-A^{\theta}\right)$

Hence $A-A^{\theta}$ is skew-Hermitian.

(6). Show that the matrix $B^{\theta}AB$ is Hermitian or skew-Hermitian according as A is Hermitian or skew-Hermitian.

Proof: Case I. Let A be a Hermitian matrix. Then $A^{\theta}=A$.

Now $B^{\theta}AB=B^{\theta}A^{\theta}\left(B^{\theta}\right)^{\theta}$, by reversal law for the conjugate transposes.

$=B^{\theta}A^{\theta}B=B^{\theta}AB$.

Hence $B^{\theta}AB$ is Hermitian matrix.

Case II. Let A be a skew-Hermitian matrix. Then $A^{\theta}=-A$.

Now $\left(B^{\theta}AB\right)^{\theta}=B^{\theta}A^{\theta}\left(B^{\theta}\right)^{\theta}=B^{\theta}A^{\theta}B=B^{\theta}\left(-A\right)B$

$=-\left(B^{\theta}A\right)B=-B^{\theta}AB$.

Hence $B^{\theta}AB$ is a skew-Hermitian.

(7). Show that every square matrix is uniquely expressible as the sum of a Hermitian matrix and a skew-Hermitian matrix.

Proof: If A is any square matrix, then $A + A^{\theta}$ is Hermitian matrix and $A - A^{\theta}$ is a skew-Hermitian matrix.

Therefore $\frac{1}{2}\left(A + A^{\theta}\right)$ is a Hermitian and $\frac{1}{2}\left(A - A^{\theta}\right)$ is a skew-Hermitian matrix.

Now, we have $A = \frac{1}{2}\left(A + A^{\theta}\right) + \frac{1}{2}\left(A - A^{\theta}\right) = P + Q$, say,

where P is Hermitian and Q skew-Hermitian.

Thus every square matrix can be expressed as the sum of a Hermitian and a skew-Hermitian matrix.

Let, Now, $A = R + S$ be another such representation of A, where R is Hermitian and S skew-Hermitian.

Then, $A^{\theta} = (R + S)^{\theta} = R^{\theta} + S^{\theta} = R - S$.

$\therefore R = \frac{1}{2}\left(A + A^{\theta}\right) = P$ and $S = \frac{1}{2}\left(A - A^{\theta}\right) = Q$.

Thus the representation is unique.

(8). Show that every real symmetric matrix is Hermitian.

Proof: Let $A = \left[a_{ij}\right]_{n\times n}$ be a real symmetric matrix. Then $a_{ij} = a_{ji}$.

Since a_{ji} is a real number, therefore $\overline{a}_{ji} = a_{ji}$.

Consequently $a_{ij} = \overline{a}_{ji}$. Hence A is Hermitian.

(9). Prove that $\overline{A}$ is Hermitian or skew-Hermitian according as A is Hermitian or skew-Hermitian.

Proof: Case 1: Suppose A is Hermitian. Then $A^{\theta} = A$.

We are to prove that $\overline{A}$ is Hermitian.

We have $\left(\overline{A}\right)^{\theta} = \left[\overline{\left(\overline{A}\right)}\right]'$ [by definition of conjugate transpose]

$= (A)'$ $\left[\because \overline{\left(\overline{A}\right)} = A\right]$

$= \left(A^{\theta}\right)'$ $\left[\because A \text{ is Hermitian } \Rightarrow A = A^{\theta}\right]$

$$=\left[\left(\bar{A}\right)'\right]' \qquad \left[\because (A')' = A\right]$$

Since $\left(\bar{A}\right)^{\theta} = \bar{A}$, therefore A is Hermitian.

Case 2: Now let us suppose that A is skew-Hermitian. Then $A^{\theta} = -A$.

We have $\left(\bar{A}\right)^{\theta} = \left[\overline{\left(\bar{A}\right)}\right]' = (A)' = \left(-A^{\theta}\right)' = -\left(A^{\theta}\right)' = -\left[\left(\bar{A}\right)\right]' = -\bar{A}$.

Therefore, $\bar{A}$ is also skew-Hermitian.

(10). Show that every square matrix A can be uniquely expressed as $P + iQ$ where P and Q are Hermitian matrices.

Proof: Let $P = \frac{1}{2}\left(A + A^{\theta}\right)$ and $Q = \frac{1}{2i}\left(A - A^{\theta}\right)$.

Then $A = P + iQ$. (i)

Now $P^{\theta} = \left\{\frac{1}{2}\left(A + A^{\theta}\right)\right\}^{\theta} = \frac{1}{2}\left(A + A^{\theta}\right)^{\theta} = \frac{1}{2}\left\{A^{\theta} + \left(A^{\theta}\right)^{\theta}\right\} = \frac{1}{2}\left(A^{\theta} + A\right) = \frac{1}{2}\left(A + A^{\theta}\right) = P$.

$\therefore$ P is Hermitian matrix.

Also $Q^{\theta} = \left\{\frac{1}{2i}\left(A - A^{\theta}\right)\right\}^{\theta} = \left(\overline{\frac{1}{2i}}\right)\left(A - A^{\theta}\right)^{\theta} = -\frac{1}{2i}\left\{A^{\theta} - \left(A^{\theta}\right)^{\theta}\right\} = -\frac{1}{2i}\left(A^{\theta} - A\right)$

$$= \frac{1}{2i}\left(A - A^{\theta}\right) = Q.$$

$\therefore$ Q is also a Hermitian.

Thus A can be expressed in the form (i) where P and Q are Hermitian matrices.

To show that the expression (i) for A is unique.

Let $A = R + iS$, where R and S are both Hermitian Matrices.

We have $A^{\theta} = (R + iS)^{\theta} = R^{\theta} + (iS)^{\theta} = R^{\theta} + \bar{i}\ S^{\theta} = R^{\theta} - iS^{\theta}$

$= R - iS$ [$\because$ R and S both Hermitian]

$\therefore A + A^{\theta} = (R + iS) + (R - iS) = 2R$.

This gives $R = \frac{1}{2}\left(A + A^{\theta}\right) = P$.

Also $A - A^{\theta} = (R + iS) - (R - iS) = 2iS$.

This gives $S = \frac{1}{2}\left(A - A^{\theta}\right) = Q$.

Hence expression (i) for A is unique.

(11). Prove that every Hermitian matrix A can be written as $A = B + iC$, where B is real and symmetric and C is real and skew-symmetric.

Proof: Let A be a Hermitian matrix. Then $A^\theta = A$.

Let us take $B = \frac{1}{2}\left(A + \overline{A}\right)$ and $C = \frac{1}{2i}\left(A - \overline{A}\right)$.

Then obviously both B and C are real matrices.

[Note that if $z = x + iy$ is a complex number, then $\frac{1}{2}\left(z + \overline{z}\right)$ is real and also $\frac{1}{2}\left(z - \overline{z}\right)$ is real]

Now we can write $A = \frac{1}{2}\left(A + \overline{A}\right) + i\left[\frac{1}{2i}\left(A - \overline{A}\right)\right] = B + iC$.

It remains to show that B is symmetric and C is skew-symmetric. We have

$$B' = \left[\frac{1}{2}\left(A + \overline{A}\right)\right]' = \frac{1}{2}\left(A + \overline{A}\right)' = \frac{1}{2}\left[A' + \left(\overline{A}\right)'\right] = \frac{1}{2}\left(A' + A^\theta\right) = \frac{1}{2}\left[\left(A^\theta\right)' + A\right] \left[\because A^\theta = A\right]$$

$$= \frac{1}{2}\left[\left\{\left(\overline{A}\right)'\right\}' + A\right] = \frac{1}{2}\left(\overline{A} + A\right) = B.$$

$\therefore$ B is symmetric.

Also $$C' = \left[\frac{1}{2i}\left(A - \overline{A}\right)\right]' = \frac{1}{2i}\left(A - \overline{A}\right)' = \frac{1}{2i}\left[A' - \left(\overline{A}\right)'\right] = \frac{1}{2i}\left(A' - A^\theta\right) = \frac{1}{2i}\left[\left(A^\theta\right)' - A\right]$$

$$= \frac{1}{2i}\left(\overline{A} - A\right) = -\frac{1}{2i}\left(A - \overline{A}\right) = -C.$$

$\therefore$ C is symmetric.

Hence the result.

Orthogonal matrix or Orthonormal matrix:

Definition: An **orthogonal matrix** is a square matrix with real entries whose columns (or rows) are orthogonal unit vectors (i.e., orthonormal). Because the columns are unit vectors in addition to being orthogonal, some people use the term **orthonormal** to describe such matrices.

or

A square matrix A is said to be orthogonal matrix if $AA' = I = A'A$.

Another way to define orthogonal matrix:

A square matrix A is said to be orthogonal matrix if $A' = A^{-1}$.

An orthogonal matrix is the real specialization of a unitary matrix, and thus always a normal matrix.

Normal matrix:

Definition: A square matrix A is said to be normal matrix if $AA^{\theta} = A^{\theta}A$.

Among complex matrices, all unitary, Hermitian, and skew-Hermitian matrices are normal. Likewise, among real matrices, all orthogonal, symmetric, and skew-symmetric matrices are normal.

However, it is *not* the case that all normal matrices are either unitary or (skew-) Hermitian. As an example, the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

is normal because

$$AA^{*} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} = A^{*}A$$

The matrix *A* is neither unitary, Hermitian, nor skew-Hermitian.

The sum or product of two normal matrices is not necessarily normal. If they commute, however, then this is true.

If *A* is both a triangular matrix and a normal matrix, then *A* is diagonal. This can be seen by looking at the diagonal entries of $A^{*}A$ and AA^{*}, where *A* is a normal, triangular matrix.

Unitary matrix:

Definition: A square matrix A is said to be unitary matrix if $AA^{\theta} = I = A^{\theta}A$.

Another way to define unitary matrix:

A square matrix A is said to be unitary matrix if $\overline{A}' = A^{-1}$.

This is a **generalization** of the orthogonal matrix in the complex field.

Some important properties of unitary matrix:

1. Inverse of a unitary matrix is unitary.

Proof: If U is a unitary matrix, then

$\bar{U}' = U^{-1} \Rightarrow U' = \overline{U^{-1}}$

$\therefore \left[\left(U^{-1}\right)^{-1}\right]' = \overline{U^{-1}}$

Writing $U^{-1} = V$, we have $\left[V^{-1}\right]' = \bar{V} \Rightarrow V^{-1} = \bar{V}'$

Thus $V\left(= U^{-1}\right)$ is also unitary.

Remark: Inverse of an orthogonal matrix is orthogonal.

2. Transpose of s unitary matrix is unitary.

Proof: If U is a unitary matrix, $\bar{U}' = U^{-1} \Rightarrow \left(\bar{U}'\right) = U^{-1} \Rightarrow \left[\left(\bar{U}'\right)\right]' = \left[U^{-1}\right]' = [U']^{-1}$

Writing $U' = V$, we have $\bar{V}' = V^{-1}$

Thus V (i.e. U') is also unitary.

Remark: Transpose of an orthogonal matrix is orthogonal.

3. Product of two unitary matrices is a unitary matrix.

Proof: If U and V are unitary matrices then

$U' = \bar{U}^{-1}, \quad V' = \bar{V}^{-1}$

Now, $\left(\overline{UV}\right)^{-1} = \left(\bar{U}\ \bar{V}\right)^{-1} = \bar{V}^{-1}\bar{U}^{-1}$

$= V'U' = \left(UV\right)'$

Thus UV is unitary matrix.

Remark: Product of two orthogonal matrixes is an orthogonal matrix.

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3rd Topic

Matrices

Rank, Inverse and Normal form of a Matrix

Gauss-Jordan method of finding the inverse

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Rank of a Matrix:

Definition: **A matrix is said to be of rank r, when**

(i) **it has at least one non-zero minor of order r, and**

(ii) **every minor of order higher than r vanishes.**

Briefly, the rank of a matrix is the largest order of any non-vanishing minor of the matrix.

Another definition:

A number r is said to be the rank of a matrix A if it possesses the following two properties:

1. **There is atleast one square sub-matrix of A of order r, whose determinant is not equal to zero,**
2. **If the matrix A contains any square sub-matrix of order (r +1), then the determinant of every square sub-matrix of A of order (r +1) should be zero.**

Remarks: **If a matrix has a non-zero minor of order r, its rank is** $\geq r$.
If all minors of a matrix of order (r + 1) are zero, its rank is $\leq r$.

Minor: In linear algebra, a **minor** of a matrix **A** is the determinant of some smaller square matrix, cut down from **A** by removing one or more of its rows or columns. Minors obtained by removing just one row and one column from square matrices (**first minors**) are required for calculating matrix cofactors, which in turn are useful for computing both the determinant and inverse of square matrices.

Elementary transformations of a matrix:

The following operations, three of which refer to rows and three to columns are known as elementary transformations:

- **I. The interchange of any two rows (columns).**
- **II. The multiplication of any row (column) by a non-zero number.**
- **III. The addition of a constant multiple of the elements of any row (column) to the corresponding elements of any other row (column).**

Notation:

The elementary row transformation will be denoted by the following symbols:

(i) $R_i \leftrightarrow R_j$ for the interchange of the i^{th} and j^{th} rows.

(ii) $R_i \to kR_i$ for multiplication of the i^{th} row by k.

(iii) $R_i \to R_i + pR_j$ for addition to the i^{th} row, p times the j^{th} row.

The corresponding column transformation will be denoted by writing C in place of R. Elementary transformations do not change either the order or a rank of a matrix. While the value of the minors may get changed by the transformation (i) and (ii), their zero or non-zero character remains unaffected.

Equivalent matrix:

Definition: Two matrices A and B are said to be equivalent if one can be obtained from the other by a sequence of elementary transformations. Two equivalent matrices have the same order and the same rank. The symbol ~ is used for equivalence.

Example: Determine the rank of the following matrices: $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$.

Sol.: Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$.

Operating $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - 2R_1$, we get $A \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{bmatrix}$.

Operating $R_3 \to R_3 - R_2$, we get $A \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$.

Obviously, the 3rd order minor of A vanishes.

Also its 2nd order minors formed by its 2nd and 3rd rows are all zero. But another 2nd order minor is $\begin{vmatrix} 1 & 3 \\ 0 & -1 \end{vmatrix} = -1 \neq 0$.

$\therefore \rho(A) = 2$.

Now since the rank of a matrix is the largest order of any non-vanishing minor of the matrix

Hence, the rank of the given matrix is 2.

Elementary matrices:

Definition: An elementary matrix is that, which is obtained from a unit matrix, by subjecting it to any of the elementary transformations.

Examples of elementary matrices obtained from $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

are $R_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = C_{23}$; $kR_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{bmatrix}$; $R_1 + pR_2 = \begin{bmatrix} 1 & p & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Theorem:

Statement: Every elementary row (column) transformations of a matrix A can be obtained by pre-multiplying (post-multiplying) A by the corresponding elementary matrix.

Remarks: Consider the matrix $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$.

Then $R_{23} \times A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \end{bmatrix}$.

So a pre-multiplication by R_{23} has interchanged the 2nd and 3rd rows of A. Similarly pre-multiplication by kR_2 will multiply the 2nd row of A by k and pre-multiplication by $R_1 + pR_2$ will result in the addition of p times the 2nd row of A to its 1st row.

Thus the pre-multiplication of A by elementary matrices results in the corresponding elementary row transformation of A.

Similarly, it can also be seen that post-multiplication will perform the elementary column transformations.

Normal form of a matrix:

Every non-zero matrix A of rank r, can be reduced by a sequence of elementary transformations, to the form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ called the normal form of A.

(i)

Remarks:

(i) The rank of the matrix A is r if and only if it can be reduced to the normal form (i).

(ii) Since each elementary transformation can be affected by pre-multiplication or post-multiplication with a suitable elementary matrix and each elementary matrix is non-singular, therefore, we have the following result:

Corresponding to every matrix A of rank r, there exist non-singular matrices P and Q such that PAQ = $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$.

If A be a $m \times n$ matrix, then P and Q are square matrices of orders m and n, respectively.

Example: For the matrix $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}$, find non-singular matrices P and Q such that PAQ is in the normal form.

Sol.: We write $A = I\,A\,I \Rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

We shall affect every elementary row (column) transformation of the product by subjecting the pre-factor (past factor) of A to the same.

Operating $C_2 \to C_2 - C_1,\ C_3 \to C_3 - 2C_1$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Operating $R_2 \to R_2 - R_1$, we get $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Operating $C_3 \to C_3 - C_2$, we get $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$.

Operating $R_2 \to R_2 + R_2$, we get $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$,

which is the required normal form $\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$.

Hence $P = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$, $Q = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ and $\rho(A) = 2$. Ans.

Gauss-Jordan method of finding the inverse:

Johann Carl Friedrich Gauss

30 April 1777 – 23 February 1855

German mathematician and scientist

Wilhelm Jordan

1 March 1842 – 17 April 1899

German geodesist

It is named after Carl Friedrich Gauss and Wilhelm Jordan, because it is a modification of Gaussian elimination as described by Jordan in 1887. However, the method also appears in an article by Clasen published in the same year. Jordan and Clasen probably discovered Gauss–Jordan elimination independently.

Statement: Those elementary row transformations, which reduce a given square matrix A to the unit matrix, when applied to unit matrix I, give the inverse of A.

Proof: Let the successive row transformations, which reduce A to I, result from pre-multiplication by the elementary matrices $R_1,\ R_2, \ldots\ldots\ldots, R_i$ so that

$$R_i R_{i-1} \ldots\ldots\ldots\ldots\ldots R_2 R_1 A = I$$

$$\therefore R_i R_{i-1} \ldots\ldots\ldots\ldots\ldots R_2 R_1 A A^{-1} = I A^{-1}$$

$$\Rightarrow R_i R_{i-1} \ldots\ldots\ldots\ldots\ldots R_2 R_1 I = A^{-1}. \qquad \left[\because AA^{-1} = I\right]$$

Hence the result.

For practical evaluation of A^{-1}, the two matrices A and I are written side by side and the same row transformations are performed on both. As soon as A is reduced to I, the other matrix represents A^{-1}.

Example: Using the Gauss-Jordan method, find the inverse of the matrix

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}.$$

Sol.: Writing the same matrix side by side with the unit matrix of order 3, we have

$$\left[\begin{array}{ccc:ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 1 & 3 & -3 & 0 & 1 & 0 \\ -2 & -4 & -4 & 0 & 0 & 1 \end{array}\right].$$

Operating $R_2 \to R_2 - R_1$ and $R_3 \to R_3 + 2R_1$, we get $\sim \left[\begin{array}{ccc:ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 2 & -6 & -1 & 1 & 0 \\ 0 & -2 & 2 & 2 & 0 & 1 \end{array}\right]$.

Operate $R_2 \to \frac{1}{2}R_2$ and $R_3 \to \frac{1}{2}R_3$, we get $\sim \left[\begin{array}{ccc:ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -1 & 1 & 1 & 0 & \frac{1}{2} \end{array}\right]$.

Operating $R_1 \to R_1 - R_2$ and $R_3 \to R_3 + R_2$, we get $\sim \left[\begin{array}{ccc:ccc} 1 & 0 & 6 & \frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & -3 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -2 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{array}\right]$

.

Operate $R_1 \to R_1 + 3R_3$, $R_2 \to R_2 - \frac{3}{2}R_3$ and $R_3 \to \left(-\frac{1}{2}\right)R_3$, we get

$$\sim\begin{bmatrix} 1 & 0 & 0 & : & 3 & 1 & \frac{3}{2} \\ 0 & 1 & 0 & : & -\frac{5}{4} & -\frac{1}{4} & -\frac{3}{4} \\ 0 & 0 & 1 & : & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}.$$

Hence, the inverse of the given matrix is $\begin{bmatrix} 3 & 1 & \frac{3}{2} \\ -\frac{5}{4} & -\frac{1}{4} & -\frac{3}{4} \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}$. Ans.

Problems for finding rank of a matrix:

Q.No.1.: Prove that the row equivalent matrices have the same rank.

Sol.: Let A be any $m\times n$ matrix. Let B be a matrix row equivalent to A.

Since B is obtainable from A by a finite chain of E-row operations and every E-row operation is equivalent to pre-multiplication by the corresponding E-matrix, there exist E-matrices $E_1, E_2, \ldots\ldots\ldots\ldots, E_k$ each of the type $m\times m$ such that

$B = E_k E_{k-1} \ldots\ldots\ldots\ldots E_2 E_1 A \Rightarrow B = PA$,

where $P = E_k E_{k-1} \ldots\ldots\ldots\ldots E_2 E_1$ is a non-singular matrix of the type $m\times m$.

Let us write $B = PA = \begin{bmatrix} p_{11} & p_{12} & \ldots. & p_{1m} \\ p_{21} & p_{22} & \ldots. & p_{2m} \\ \ldots. & \ldots. & \ldots. & \ldots. \\ \ldots. & \ldots. & \ldots. & \ldots. \\ p_{m1} & p_{m2} & \ldots. & p_{mm} \end{bmatrix}\begin{bmatrix} R_1 \\ R_2 \\ \ldots. \\ \ldots. \\ R_m \end{bmatrix}$, (i)

where the matrix A has been expressed as a matrix of its row sub-matrices $R_1, R_2, \ldots\ldots\ldots\ldots, R_m$.

From the product of the matrices on R. H. S. of (i) we observe that the rows of the matrix B are

$p_{11}R_1 + p_{12}R_2 + \ldots\ldots\ldots + p_{1m}R_m$,

$p_{21}R_1 + p_{22}R_2 + \dots\dots\dots + p_{2m}R_m$,

....

....

$p_{m1}R_1 + p_{m2}R_2 + \dots\dots\dots + p_{mm}R_m$.

Thus, we see that the rows of B are all linear combinations of the rows $R_1, R_2, \dots\dots\dots\dots, R_m$ of A.

Therefore, every member of the row space of B is also a member of the row space of A.

Similarly, by writing $A = P^{-1}B$ and giving the same reasoning we can prove that every member of the row space of A is also a member of the row space of B.

Therefore the row space of A and B are identical.

Thus we see that elementary row operations do not alter the row space of a matrix remains invariant under E-rows transformations.

Note: From the above theorem we also conclude that pre-multiplication by a non-singular matrix does not alter the row rank of a matrix.

Q.No.2.: Determine the rank of the following matrices: $\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

Sol.: Let $A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$.

Operating $C_3 \to C_3 - C_1,\ C_4 \to C_4 - C_1$, we get $A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 0 & 0 \\ 3 & 1 & -3 & -1 \\ 1 & 1 & -3 & -1 \end{bmatrix}$.

Operating $R_3 \to R_3 - R_1, R_4 \to R_4 - R_1$, we get $A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$.

Operating $R_3 \to R_3 - 3R_2, R_4 \to R_4 - R_2$, we get $A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Operating $C_3 \to C_3 + 3C_2, C_4 \to C_4 + C_2$, we get $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Obviously, the 4th order minor of A is zero. Also every third order minor of A is zero. But, of all the 2nd order minors, only $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -1 \neq 0$. $\therefore \rho(A) = 2$.

Hence, the rank of the given matrix is 2.

Q.No.3: Determine the rank of the matrix $\begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix}$.

Sol.: Let $A = \begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix}$.

Operating $R_3 \to 2R_3 - R_2$, $R_2 \to \left(-\frac{1}{2}R_2\right)$, we get $A = \begin{bmatrix} 3 & -1 & 2 \\ 3 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$.

Operating $R_2 + 2R_1$, we get $A = \begin{bmatrix} 3 & -1 & 2 \\ 0 & 0 & 8 \\ 0 & 0 & 4 \end{bmatrix}$.

Operating $R_2 \to R_2 - R_1$, we get $A = \begin{bmatrix} 3 & -1 & 2 \\ 0 & 0 & 8 \\ 0 & 0 & 0 \end{bmatrix}$.

$\therefore \rho(A) = 2$

Hence, the rank of the given matrix is 2.

Q.No.4: Determine the rank of the matrix **(i)** $\begin{bmatrix} 2 & -1 & 3 & 1 \\ 1 & 4 & -2 & 1 \\ 5 & 2 & 4 & 3 \end{bmatrix}$, **(ii)** $\begin{bmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 3 \\ 1 & 3 & 4 & 1 \end{bmatrix}$.

Sol.: (i) Let $A = \begin{bmatrix} 2 & -1 & 3 & 1 \\ 1 & 4 & -2 & 1 \\ 5 & 2 & 4 & 3 \end{bmatrix}$.

Operating $C_1 \to C_1 - C_4$, we get $A = \begin{bmatrix} 1 & -1 & 3 & 1 \\ 0 & 4 & -2 & 1 \\ 2 & 2 & 4 & 3 \end{bmatrix}$.

Operating $R_3 \to R_3 - 2R_1$, we get $A = \begin{bmatrix} 1 & -1 & 3 & 1 \\ 0 & 4 & -2 & 1 \\ 0 & 4 & -2 & 1 \end{bmatrix}$

Operating $R_3 \to R_3 - R_2$, we get $A = \begin{bmatrix} 1 & -1 & 3 & 1 \\ 0 & 4 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Now since all 3×3 matrices are singular $\because \begin{vmatrix} 1 & -1 & 3 \\ 0 & 4 & -2 \\ 0 & 0 & 0 \end{vmatrix} = 0$ and $\begin{vmatrix} -1 & 3 & 1 \\ 4 & -2 & 1 \\ 0 & 0 & 0 \end{vmatrix} = 0$.

Now $\because \begin{vmatrix} -1 & 3 \\ 4 & -2 \end{vmatrix} \neq 0$. Hence, the rank of the given matrix is 2.

(ii). Let $A = \begin{bmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 3 \\ 1 & 3 & 4 & 1 \end{bmatrix}$.

Operating $R_3 \to 3R_3 - R_2$, $R_2 \to \frac{1}{3}R_2$, we get $A = \begin{bmatrix} 1 & 3 & 4 & 3 \\ 1 & 3 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Operating $R_2 \to R_2 - R_1$, we get $A = \begin{bmatrix} 1 & 3 & 4 & 3 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Hence, the rank of the given matrix is 2.

Q.No.5: Determine the rank of the matrix $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$.

Sol.: Let $A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$.

Operating $R_2 \to R_2 - 2R_1, R_3 \to R_3 - 3R_1, R_4 \to R_4 - 6R_1$, we get

$$A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & -4 & -11 & 5 \end{bmatrix}.$$

Operating $R_4 \to R_4 - R_3$, we get $A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \end{bmatrix}$.

Operating $R_4 \to R_4 - R_2$, we get $A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

As 4×4 matrix is singular. But 3×3 matrix like $\begin{vmatrix} 2 & 3 & 0 \\ 0 & -3 & 2 \\ -4 & -8 & 3 \end{vmatrix}$ is non-singular.

So the rank of given matrix is 3.

Q.No.6: Determine the rank of the matrix $\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$.

Sol.: Let $A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$.

Operating $R_4 \to R_4 - 3R_1$, $R_3 \to R_3 - 3R_2$, we get $A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 0 & 4 & 9 & 10 \\ 0 & -6 & 3 & -4 \end{bmatrix}$,

Operating $R_2 = 2R_2 - R_1$, we get $A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 0 & -5 & -3 & -7 \\ 0 & 4 & 9 & 10 \\ 0 & -6 & 3 & 4 \end{bmatrix}$.

Operating $C_4 \to C_4 + C_3$, we get $A = \begin{bmatrix} 2 & 3 & -1 & 0 \\ 0 & -5 & -3 & -10 \\ 0 & 4 & 9 & 19 \\ 0 & 6 & 3 & -1 \end{bmatrix}$.

Operating $C_2 \to C_2 - 6C_4, C_3 \to C_3 + 3C_4$, we get $A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 0 & -11 & -3 & -7 \\ 0 & 22 & 9 & 10 \\ 0 & 0 & 3 & -4 \end{bmatrix}$.

Operating $R_3 \to R_3 + 2R_2$, we get $A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 0 & -11 & -3 & -7 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & 3 & -4 \end{bmatrix}$.

Operating $R_4 \to R_4 - R_3$, we get $A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 0 & -11 & -3 & -7 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

As 4×4 matrix is singular. But 3×3 matrix is non-singular.

So the rank of the matrix is 3. Ans.

Q.No.7: Determine the rank of the matrix $\begin{bmatrix} 5 & 6 & 7 & 8 \\ 6 & 7 & 8 & 9 \\ 11 & 12 & 13 & 14 \\ 16 & 17 & 18 & 19 \end{bmatrix}$.

Sol.: Let $A = \begin{bmatrix} 5 & 6 & 7 & 8 \\ 6 & 7 & 8 & 9 \\ 11 & 12 & 13 & 14 \\ 16 & 17 & 18 & 19 \end{bmatrix}$.

Operating $R_3 \to R_3 - R_1, R_4 \to R_4 - R_2$, we get $A = \begin{bmatrix} 5 & 6 & 7 & 8 \\ 6 & 6 & 8 & 9 \\ 6 & 6 & 6 & 6 \\ 10 & 10 & 10 & 10 \end{bmatrix}$.

Operating $R_2 \to R_2 - R_1$, we get $A = \begin{bmatrix} 5 & 6 & 7 & 8 \\ 1 & 1 & 1 & 1 \\ 6 & 6 & 6 & 6 \\ 10 & 10 & 10 & 10 \end{bmatrix}$.

Operating $C_1 \to C_1 - C_4, C_2 \to C_2 - C_4, C_3 \to C_3 - C_4$, we get $A = \begin{bmatrix} 3 & 2 & 1 & 8 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 10 \end{bmatrix}$.

Operating $R_4 \to R_4 - 10R_2,\ R_3 \to R_3 - 6R_2$, we get $A = \begin{bmatrix} 3 & 2 & 1 & 8 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Hence the rank of the given matrix is 2. Ans.

Q.No.8.: Find the rank of matrix

(i). $\begin{bmatrix} 1 & 2 & 3 & 4 \\ -2 & 0 & 5 & 7 \end{bmatrix}$ **(ii).** $\begin{bmatrix} 2 & 3 & 4 & -1 \\ 5 & 2 & 0 & -1 \\ -4 & 5 & 12 & -1 \end{bmatrix}$ **(iii).** $\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$.

Sol.: (i). Here $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -2 & 0 & 5 & 7 \end{bmatrix}$ is a 2×4 matrix.

$\therefore \rho(A) \le 2$, the smaller of 2 and 4.

The second order minor $\begin{vmatrix} 1 & 2 \\ -2 & 0 \end{vmatrix} = 4 \ne 0$ $\quad\therefore \rho(A) = 2$.

(ii). Here $A = \begin{bmatrix} 2 & 3 & 4 & -1 \\ 5 & 2 & 0 & -1 \\ -4 & 5 & 12 & -1 \end{bmatrix}$ is a 3×4 matrix.

$\therefore \rho(A) \le 3$.

Operating C_{14}, we get $A = \begin{bmatrix} -1 & 3 & 4 & 2 \\ -1 & 2 & 0 & 5 \\ -1 & 5 & 12 & -5 \end{bmatrix}$

Operating $R_2 - R_1$, $R_3 - R_1$, we get $A = \begin{bmatrix} -1 & 3 & 4 & 2 \\ 0 & -1 & -4 & 3 \\ 0 & 2 & 8 & -6 \end{bmatrix}$

Operating $R_1 + 3R_2$, $R_3 + 2R_2$, we get $A = \begin{bmatrix} -1 & 0 & -8 & 11 \\ 0 & -1 & -4 & 3 \\ 0 & 0 & 0 & -0 \end{bmatrix}$.

All the first order minors are zero but the second order minor

$\begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} = 1 \ne 0$. $\quad\therefore \rho(A) = 2$.

(iii). Here $A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$ is a 4×4 matrix. $\quad\therefore \rho(A) \le 4$

Operating R_{12}, we get $A = \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$.

Operating $C_2+C_1,\ \ C_3+2C_1,\ \ C_4+4C_1,$, we get $A=\begin{bmatrix}1&0&0&0\\2&5&3&7\\3&4&9&10\\6&9&12&17\end{bmatrix}$.

Operating $R_2-2R_1,\ \ R_3-3R_1,\ \ R_4-6R_1$, we get $A=\begin{bmatrix}1&0&0&0\\0&5&3&7\\0&4&9&10\\0&9&12&17\end{bmatrix}$.

Operating $R_2-R_3,\ \ \ R_4-2R_3$, we get $A=\begin{bmatrix}1&0&0&0\\0&1&-6&-3\\0&4&9&10\\0&1&-6&-3\end{bmatrix}$.

Operating $C_3+6C_2,\ \ C_4+3C_2$, we get $A=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&4&33&22\\0&1&0&0\end{bmatrix}$.

Operating $R_3-4R_2,\ \ \ R_4-R_2$, we get $A=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&33&22\\0&0&0&0\end{bmatrix}$.

Operating $\frac{1}{33}C_3$, we get $A=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&22\\0&0&0&0\end{bmatrix}$.

Operating C_4-22C_3, we get $A=\begin{bmatrix}1&0&0&:&0\\0&1&0&:&0\\0&0&1&:&0\\....&....&....&:&....\\0&0&0&:&0\end{bmatrix}=\begin{bmatrix}I_3&:&O_{3\times 2}\\....&....&....\\O_{1\times 3}&:&O_{1\times 1}\end{bmatrix}$.

$\therefore \rho(A)=3$.

Problems for inverse of a matrix by Gauss-Jordan method:

Q.No.1.: Use Gauss-Jordan method to find inverse of the following matrices:

$$\textbf{(i)} \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix}, \textbf{(ii)} \begin{bmatrix} 8 & 4 & 3 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}, \textbf{(iii)} \begin{bmatrix} 2 & 1 & 2 \\ 2 & 2 & 1 \\ 1 & 2 & 2 \end{bmatrix}, \textbf{(iv)} \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}.$$

Sol.: (i). Given matrix is $\begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix}$.

Writing the same matrix side by side with the unit matrix of order 3., we have

$$\left[\begin{array}{ccc:ccc} 2 & 1 & -1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 5 & 2 & -3 & 0 & 0 & 1 \end{array}\right].$$

Operating $R_3 \to 2R_3 - 5R_1$, we get $\left[\begin{array}{ccc:ccc} 2 & 1 & -1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & -1 & -1 & -5 & 0 & 2 \end{array}\right]$.

Operating $R_3 \to 2R_3 - R_2$, we get $\left[\begin{array}{ccc:ccc} 2 & 1 & -1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & -10 & 1 & 4 \end{array}\right]$.

Operating $R_2 \to R_2 + R_3$, $R_1 \to R_1 - R_3$, we get $\left[\begin{array}{ccc:ccc} 2 & 1 & 0 & 11 & -1 & -4 \\ 0 & 2 & 0 & -10 & 2 & 4 \\ 0 & 0 & -1 & -10 & 1 & 4 \end{array}\right]$.

Operating $R_1 \to 2R_1 - R_2$, $R_2 \to \frac{1}{2}R_2$, $R_3 \to (-1)R_3$, we get

$$\left[\begin{array}{ccc:ccc} 4 & 0 & 0 & 32 & -4 & -12 \\ 0 & 1 & 0 & -5 & 1 & 2 \\ 0 & 0 & 1 & 10 & -1 & -4 \end{array}\right].$$

Operating $R_1 \to \frac{1}{4}R_1$, we get $\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 8 & -1 & -3 \\ 0 & 1 & 0 & -5 & 1 & 2 \\ 0 & 0 & 1 & 10 & -1 & -4 \end{array}\right]$.

Hence, the inverse of the given matrix is $\begin{bmatrix} 8 & -1 & -3 \\ -5 & 1 & 2 \\ 10 & -1 & -4 \end{bmatrix}$. Ans.

(ii). Given $\begin{bmatrix} 8 & 4 & 3 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$.

Writing the same matrix side by side with the unit matrix of order 3., we have

$$\begin{bmatrix} 8 & 4 & 3 & : & 1 & 0 & 0 \\ 2 & 1 & 1 & : & 0 & 1 & 0 \\ 1 & 2 & 1 & : & 0 & 0 & 1 \end{bmatrix}.$$

Operating $R_3 \to 2R_3 - R_2$, $R_2 \to 4R_2 - R_1$, we get $\begin{bmatrix} 8 & 4 & 3 & : & 1 & 0 & 0 \\ 0 & 0 & 1 & : & -1 & 4 & 0 \\ 0 & 3 & 1 & : & 0 & -1 & 2 \end{bmatrix}$.

Operating $R_2 \to R_2 - R_3$, we get $\begin{bmatrix} 8 & 4 & 3 & : & 1 & 0 & 0 \\ 0 & -3 & 0 & : & -1 & 5 & -2 \\ 0 & 3 & 1 & : & 0 & -1 & 2 \end{bmatrix}$.

Operating $R_3 \to R_3 + R_2$, $R_2 \to \left(-\frac{1}{3}\right)R_2$, we get $\begin{bmatrix} 8 & 4 & 3 & : & 1 & 0 & 0 \\ 0 & 1 & 0 & : & \frac{1}{3} & -\frac{5}{3} & \frac{2}{3} \\ 0 & 0 & 1 & : & -1 & 4 & 0 \end{bmatrix}$.

Operating $R_1 \to R_1 - 3R_3$, we get $\begin{bmatrix} 8 & 4 & 0 & : & 4 & -12 & 0 \\ 0 & 1 & 0 & : & \frac{1}{3} & -\frac{5}{3} & \frac{2}{3} \\ 0 & 0 & 1 & : & -1 & 4 & 0 \end{bmatrix}$.

Operating $R_1 \to R_1 - 4R_2$, we get $\begin{bmatrix} 8 & 0 & 0 & : & \frac{8}{3} & -\frac{16}{3} & -\frac{8}{3} \\ 0 & 1 & 0 & : & \frac{1}{3} & -\frac{5}{3} & \frac{2}{3} \\ 0 & 0 & 1 & : & -1 & 4 & 0 \end{bmatrix}$.

Operating $R_1 \to \frac{1}{8}R_1$, we get $\begin{bmatrix} 1 & 0 & 0 : & \frac{1}{3} & -\frac{2}{3} & -\frac{1}{3} \\ 0 & 1 & 0 : & \frac{1}{3} & -\frac{5}{3} & \frac{2}{3} \\ 0 & 0 & 1 : & -1 & 4 & 0 \end{bmatrix}$

Hence the inverse of given matrix is $\begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & -\frac{5}{3} & \frac{2}{3} \\ -1 & 4 & 0 \end{bmatrix}$.

$\therefore A^{-1} = \frac{1}{9}\begin{bmatrix} 1 & -2 & -1 \\ 1 & -5 & 2 \\ -1 & 4 & 0 \end{bmatrix}$. Ans.

(iii). Let $A = \begin{bmatrix} 2 & 1 & 2 \\ 2 & 2 & 1 \\ 1 & 2 & 2 \end{bmatrix}$.

According to Gauss-Jordan method, we have $A = IA$.

$\Rightarrow \begin{bmatrix} 2 & 1 & 2 \\ 2 & 2 & 1 \\ 1 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$

Operating $R_3 \to R_3 - R_1$, we get $\begin{bmatrix} 1 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A$.

Operating $R_3 \to R_3 - R_2$, $R_2 \to R_2 - 2R_1$, we get $\begin{bmatrix} 1 & 2 & 2 \\ 0 & -2 & -3 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & -1 & 0 \end{bmatrix} A$.

Operating $R_1 \to R_1 + R_2$, $R_3 \to 2R_3 - R_2$, we get $\begin{bmatrix} 1 & 0 & -1 \\ 0 & -2 & -3 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 1 & -2 \\ 2 & -3 & -2 \end{bmatrix} A$.

Operating $R_3 \to \frac{1}{5}R_3$, we get $\begin{bmatrix} 1 & 0 & -1 \\ 0 & -2 & -3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 1 & -2 \\ \frac{2}{5} & -\frac{3}{5} & \frac{2}{5} \end{bmatrix} A$.

Operating $R_1 \to R_1 + R_3$, $R_2 \to R_2 + 3R_3$, we get, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{5} & \frac{2}{5} & -\frac{3}{5} \\ \frac{6}{5} & -\frac{4}{5} & -\frac{4}{5} \\ \frac{2}{5} & -\frac{3}{5} & \frac{2}{5} \end{bmatrix} A$.

Operating $R_2 \to -\frac{R_2}{2}$, we get, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{5} & \frac{2}{5} & -\frac{3}{5} \\ -\frac{3}{5} & \frac{2}{5} & -\frac{2}{5} \\ \frac{2}{5} & -\frac{3}{5} & \frac{2}{5} \end{bmatrix} A \Rightarrow I = A^{-1}A$.

$\therefore A^{-1} = \begin{bmatrix} \frac{2}{5} & \frac{2}{5} & -\frac{3}{5} \\ -\frac{3}{5} & \frac{2}{5} & -\frac{2}{5} \\ \frac{2}{5} & -\frac{3}{5} & \frac{2}{5} \end{bmatrix}$. Ans.

(iv). Let $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$.

According to Gauss-Jordan method, we have $A = IA \Rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$.

Operating $R_2 \to R_2 + R_1$, we get $\begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & -1 \\ 2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$.

Operating $R_3 \to -2R_1 - R_3$, we get $\begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & -1 \\ 0 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 2 & 0 & -1 \end{bmatrix} A$.

Operating $R_3 \to -R_3 + R_2$, we get $\begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & -1 \\ 0 & 0 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ -2 & -1 & 1 \end{bmatrix} A$.

Operating $R_2 \to -R_2$, we get $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & -1 & 1 \end{bmatrix} A$.

Operating $R_3 \to \frac{R_3}{-4}$, we get $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2} & \frac{1}{4} & -\frac{1}{4} \end{bmatrix} A$.

Operating $R_1 \to R_1 - R_2$, we get, $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2} & \frac{1}{4} & -\frac{1}{4} \end{bmatrix} A$.

Operating $R_1 \to R_1 - R_3$, we get, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{5}{4} & \frac{1}{4} \\ 0 & 1 & 0 \\ \frac{1}{2} & \frac{1}{4} & -\frac{1}{4} \end{bmatrix} A$.

Operating $R_2 \to R_2 - R_3$, we get, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{5}{4} & \frac{1}{4} \\ -\frac{1}{2} & \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & -\frac{1}{4} \end{bmatrix} A$.

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{5}{4} & \frac{1}{4} \\ -\frac{1}{2} & \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & -\frac{1}{4} \end{bmatrix}. \text{ Ans.}$$

Q.No.2.: Find the inverse of $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$ by elementary row operations.

Sol.: Writing the given matrix side by side with unit matrix I_3, we get

$$[A \;\; : \;\; I_3] = \begin{bmatrix} 0 & 1 & 2 & : & 1 & 0 & 0 \\ 1 & 2 & 3 & : & 0 & 1 & 0 \\ 3 & 1 & 1 & : & 0 & 0 & 1 \end{bmatrix}$$

Operating R_{12}, we get $= \begin{bmatrix} 1 & 2 & 3 & : & 0 & 1 & 0 \\ 0 & 1 & 2 & : & 1 & 0 & 0 \\ 3 & 1 & 1 & : & 1 & 0 & 1 \end{bmatrix}$

Operating $R_3 - 3R_1$, we get $= \begin{bmatrix} 1 & 2 & 3 & : & 0 & 1 & 0 \\ 0 & 1 & 2 & : & 1 & 0 & 0 \\ 0 & -5 & -8 & : & 0 & -3 & 1 \end{bmatrix}$

Operating $R_1 - 2R_2,\; R_3 + 5R_2$, we get $= \begin{bmatrix} 1 & 0 & -1 & : & -2 & 1 & 0 \\ 0 & 1 & 2 & : & 1 & 0 & 0 \\ 0 & 0 & 2 & : & 5 & -3 & 1 \end{bmatrix}$

Operating $\frac{1}{2}R_3$, we get $= \begin{bmatrix} 1 & 0 & -1 & : & -2 & 1 & 0 \\ 0 & 1 & 2 & : & 1 & 0 & 0 \\ 0 & 0 & 1 & : & \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{bmatrix}$

Operating $R_1 + R_2,\; R_2 - 2R_3$, we get $= \begin{bmatrix} 1 & 0 & 0 & : & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & : & -4 & 3 & -1 \\ 0 & 0 & 1 & : & \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{bmatrix} = [I_3 \;\; : \;\; A^{-1}]$

$$\therefore A^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{bmatrix}.$$

Problems on normal form:

Q.No.1.: If $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$ find A^{-1}. Also find two non-singular matrices P and Q such that PAQ = I, where I is the unit matrix and verify that $A^{-1} = QP$.

Sol.: Here $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$.

Part I: To find A^{-1}, $A_{11} = 1,\ A_{12} = -2,\ A_{13} = -2$,

$A_{21} = -1,\ A_{22} = 3,\ A_{23} = 3$,

$A_{31} = 0,\ A_{32} = -4,\ A_{33} = -3$.

$$\therefore \text{Adj}.A = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}.$$

Now $|A| = \begin{vmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{vmatrix} = 3(-3+4) - 2(-3+4) = 3 - 2 = 1$.

$$\therefore A^{-1} = \frac{\text{Adj}.A}{|A|} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}. \text{ Ans.}$$

Part II: Since A= PAQ, where P and Q are two non-singular unit matrices of order 3 each.

$$\Rightarrow \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Operating $R_1 \to R_1 - R_2$, we get $\begin{bmatrix} 1 & 0 & 0 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Operating $C_2 \to C_2 + C_3$, we get $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$.

Operating $R_2 \to R_2 - 2R_1$, we get $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$.

Operating $C_3 \to C_3 - 4C_2$, we get $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -4 \\ 0 & 1 & -3 \end{bmatrix}$.

$\Rightarrow I = PAQ$, where $P = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -4 \\ 0 & 1 & -3 \end{bmatrix}$.

Part III: Verification: $A^{-1} = Q\ P$.

Now $RHS = QP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -4 \\ 0 & 1 & -3 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$.

Now $LHS = A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$.

$\therefore$ L.H.S. = R.H.S.

Hence $A^{-1} = QP$.

Q.No.2.: Reduce the following matrices to the normal form and hence find their ranks.

(i) $\begin{bmatrix} 8 & 1 & 3 & 6 \\ 0 & 3 & 2 & 2 \\ -8 & -1 & -3 & 4 \end{bmatrix}$, **(ii)** $\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -2 & 1 \\ 1 & -1 & 4 & 0 \\ -2 & 2 & 8 & 0 \end{bmatrix}$.

Sol.: (i). Let $A = \begin{bmatrix} 8 & 1 & 3 & 6 \\ 0 & 3 & 2 & 2 \\ -8 & -1 & -3 & 4 \end{bmatrix}$.

Operating $C_2 \to C_2 - \frac{1}{8}C_1$, $C_3 \to C_3 - \frac{3}{8}C_1$, $C_4 \to C_4 - \frac{6}{8}C_1$, we get

$$A = \begin{bmatrix} 8 & 0 & 0 & 0 \\ 0 & 3 & 2 & 2 \\ -8 & 0 & 0 & 10 \end{bmatrix}.$$

Operating $R_1 \to \frac{1}{8}R_1$, $R_3 \to R_3 + R_1$, we get $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 2 & 2 \\ 0 & 0 & 0 & 10 \end{bmatrix}$.

Operating $R_3 \to \frac{1}{10}R_3$, we get $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 2 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Operating $C_4 \to C_4 - \frac{2}{3}C_2$, $C_3 \to C_3 - \frac{2}{3}C_3$, we get $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Operating $R_2 \to \frac{1}{3}R_2$, we get $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Operating $C_3 \leftrightarrow C_4$, we get $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$,

which is the required normal form $[I_3 \quad O]$.

Hence, rank of the matrix A is 3. Ans.

(ii). Let $A = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -2 & 1 \\ 1 & -1 & 4 & 0 \\ -2 & 2 & 8 & 0 \end{bmatrix}$.

Operating $C_4 \to C_3 - 2C_1$, $C_4 \to C_4 - C_1$, we get $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 \\ 1 & -1 & 2 & -1 \\ -2 & 2 & 12 & -2 \end{bmatrix}$.

Operating $R_3 \to R_3 - R_1$, $R_4 \to R_4 + 2R_1$, we get $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & -1 & 2 & -1 \\ 0 & 2 & 12 & -2 \end{bmatrix}$.

Operating $C_3 \to C_3 + 2C_2$, $C_4 \to C_4 - C_2$, we get $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 2 & 16 & -4 \end{bmatrix}$.

Operating $R_3 \to R_3 + R_2$, $R_4 \to R_4 - 2R_2$, we get $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 16 & -4 \end{bmatrix}$.

Operating $C_4 \to C_4 + \frac{4}{16} C_3$, we get $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 16 & 0 \end{bmatrix}$.

Operating $R_4 \to R_4 + \frac{1}{16} R_4$, we get $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$.

Operating $R_3 \leftrightarrow R_4$, we get $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$,

which is the required normal form $\begin{bmatrix} I_3 & O \\ O & O \end{bmatrix}$.

Hence, the rank of the matrix A is 3. Ans.

Q.No.3.: Find non-singular matrices P and Q such that PAQ is in the normal form for the matrices:

(i) $\begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix}$, **(ii)** $\begin{bmatrix} 1 & 2 & 3 & -2 \\ 2 & -2 & 1 & 3 \\ 3 & 0 & 4 & 1 \end{bmatrix}$.

Sol.: (i) Let $A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix}$.

Since we know that $A = I.A.I \Rightarrow \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Operating $C_2 \to C_2 + C_1, C_3 \to C_3 + C_1$, we get $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 2 \\ 3 & 4 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Operating $R_2 \to R_2 - R_1, R_3 \to R_3 - 3R_1$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 4 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Operating $R_3 \to R_3 - 2R_2$, we get $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Operating $R_2 \to \frac{R_2}{2}$, we get $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ -1 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Operating $C_3 \to C_3 - C_2$, we get $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ -1 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$.

$\therefore PAQ = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$, where $P = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ -1 & -2 & 1 \end{bmatrix}$ and $Q = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$. Ans.

Also rank of the matrix A is 2.

(ii). Let $A = \begin{bmatrix} 1 & 2 & 3 & -2 \\ 2 & -2 & 1 & 3 \\ 3 & 0 & 4 & 1 \end{bmatrix}$.

Since we know that $A = I_3 \, A \, I_4$, where I_3 and I_4 are the unit matrix of order 3 and 4 respectively.

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 & -2 \\ 2 & -2 & 1 & 3 \\ 3 & 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Operating $C_2 \to C_2 - 2C_1$, $C_3 \to C_3 - 3C_1$, $C_4 \to C_4 + 2C_1$, we get

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & -6 & -5 & 7 \\ 1 & -6 & -5 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Operating $R_3 \to R_3 - R_2$, we get $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & -6 & -5 & 7 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$

Operating $C_2 \to \left(-\frac{1}{6}\right)C_2,\ C_3 \to \left(-\frac{1}{5}\right)C_3,\ C_4 \to \left(\frac{1}{7}\right)C_4$, we get

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & \frac{1}{3} & \frac{3}{5} & \frac{2}{7} \\ 0 & -\frac{1}{6} & 0 & 0 \\ 0 & 0 & -\frac{1}{5} & 0 \\ 0 & 0 & 0 & \frac{1}{7} \end{bmatrix}.$$

Operating $C_4 \to C_4 - C_2,\quad C_3 \to C_3 - C_2$, we get

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & \frac{1}{3} & \frac{4}{15} & -\frac{1}{21} \\ 0 & -\frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & -\frac{1}{5} & 0 \\ 0 & 0 & 0 & \frac{1}{7} \end{bmatrix}.$$

Operating $R_2 \to R_2 - 2R_3,\ R_3 \to R_3 - R_1$ we get

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -2 \\ -1 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & \frac{1}{3} & \frac{4}{15} & -\frac{1}{21} \\ 0 & -\frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & -\frac{1}{5} & 0 \\ 0 & 0 & 0 & \frac{1}{7} \end{bmatrix}.$$

$$\Rightarrow \begin{bmatrix} I_2 & O \\ O & O \end{bmatrix} = PAQ, \text{ where } P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -2 \\ -1 & -1 & 1 \end{bmatrix} \text{ and } Q = \begin{bmatrix} 1 & \frac{1}{3} & \frac{4}{15} & -\frac{1}{21} \\ 0 & -\frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & -\frac{1}{5} & 0 \\ 0 & 0 & 0 & \frac{1}{7} \end{bmatrix}. \text{ Ans.}$$

Also rank of the matrix A is 2.

Home Assignments:

Q.No.1.: Find the rank of matrix

$$\textbf{(i).} \begin{bmatrix} 2 & -1 & 0 & 5 \\ 0 & 3 & 1 & 4 \end{bmatrix} \quad \textbf{(ii).} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \end{bmatrix} \quad \textbf{(iii).} \begin{bmatrix} 1 & 3 & 4 & 5 \\ 1 & 2 & 6 & 7 \\ 1 & 5 & 0 & 10 \end{bmatrix}.$$

Ans.: (i). 2 **(ii).** 2 **(iii).** 3.

Q.No.2.: Find the rank of matrix

$$\textbf{(i).} \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix} \quad \textbf{(ii).} \begin{bmatrix} 1 & -1 & 2 & -3 \\ 4 & 1 & 0 & 2 \\ 0 & 3 & 1 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix}.$$

Ans.: (i). 2 **(ii).** 4 .

Q.No.3.: Use Gauss-Jordan method to find the inverse of the matrix $\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$.

Ans.: $\begin{bmatrix} 3 & -1 & 1 \\ 15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$.

Q.No.4.: Reduce the matrices to normal form and hence find its rank

$$A = \begin{bmatrix} 2 & 1 & -3 & -6 \\ 3 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix}.$$

Ans.: 3.

Q.No.5.: Determine the rank of the matrix $A=\begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 16 & 4 & 12 & 15 \end{bmatrix}$ by reducing it to the normal form.

Ans.: 2.

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4^{th} Topic

Matrices

Vectors and Linear Dependence

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Vectors:

Any quantity having n-components is called a vector of order n. Therefore, the coefficients in linear equation or the elements in a row matrix or column matrix will form a vector. Thus, any n numbers $x_1, x_2, \ldots\ldots\ldots, x_n$ written in a particular order, constitute a vector x.

Linear dependent vectors:

The vectors $x_1, x_2, \ldots\ldots\ldots, x_n$ are said to be linearly dependent, if there exist n numbers $\lambda_1, \lambda_2, \ldots\ldots\ldots, \lambda_n$ not all zero, such that

$$\lambda_1 x_1 + \lambda_2 x_2 + \ldots\ldots\ldots + \lambda_n x_n = 0. \quad \text{(i)}$$

Linear independent vectors:

If no such numbers, other than zero, exist, then the vectors are said to be linearly independent.

Now let us suppose vectors $x_1, x_2, \ldots\ldots\ldots, x_n$ are said to be linearly dependent, then there exist r numbers $\lambda_1, \lambda_2, \ldots\ldots\ldots, \lambda_n$ not all zero.

Let us suppose $\lambda_1 \neq 0$, then we write (i) in the form

$$x_1 = -\frac{1}{\lambda_1}[\lambda_2 x_2 + \lambda_3 x_3 + \ldots\ldots\ldots\ldots\ldots + \lambda_n x_n]$$

$\Rightarrow x_1 = \mu_2 x_2 + \mu_3 x_3 + \dots\dots\dots\dots + \mu_n x_n$.

This means that the vector x_1 is said to be a linear combination of the vectors $x_2, \dots\dots\dots, x_n$.

Now let us solve some problems:

Q.No.1.: Are the vectors $x_1 = (1, 3, 4, 2)$, $x_2 = (3, -5, 2, 2)$ and $x_3 = (2, -1, 3, 2)$ linearly dependent? If so express one of these as a linear combination of the others.

Sol.: Since we know that, the vectors x_1, x_2, x_3 are said to be L.D., if $\exists$ numbers $\lambda_1, \lambda_2, \lambda_3$ not all zero s.t. $\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 = 0$.

The relation $\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 = 0$.

$\Rightarrow \lambda_1(1, 3, 4, 2) + \lambda_2(3, -5, 2, 2) + \lambda_3(2, -1, 3, 2) = 0$,

$\Rightarrow \lambda_1 + 3\lambda_2 + 2\lambda_3 = 0$, $3\lambda_1 - 5\lambda_2 - \lambda_3 = 0$, $4\lambda_1 + 2\lambda_2 + 3\lambda_3 = 0$, $2\lambda_1 + 2\lambda_2 + 2\lambda_3 = 0$.

$$\approx \begin{bmatrix} 1 & 3 & 2 \\ 3 & -5 & -1 \\ 4 & 2 & 3 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Operating $R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 4R_1$, $R_4 \to R_4 - 2R_1$, we get

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & -14 & -7 \\ 0 & -10 & -5 \\ 0 & -4 & -2 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Operating $R_3 \to 7R_3 - 5R_2$, $R_4 \to 5R_4 - 2R_2$, we get $\begin{bmatrix} 1 & 3 & 2 \\ 0 & -14 & -7 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

Operating $R_2 \to \left(-\frac{1}{7}\right) R_2$, we get $\begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

$\Rightarrow \lambda_1 + 3\lambda_2 + 2\lambda_3 = 0$,

$$2\lambda_2 + \lambda_3 = 0.$$

$\Rightarrow \lambda_3 = -2\lambda_2$ and $\lambda_1 = \lambda_2$

$\Rightarrow \lambda_1 = \lambda_2 = -\frac{1}{2}\lambda_3.$

Now these are satisfied by the values $\lambda_1 = 1$, $\lambda_2 = 1$ and $\lambda_3 = -2$, which are not zero.

Thus, the given vectors are linearly dependent.

Relation: Substituting these values in (i), we get $x_1 + x_2 - 2x_3 = 0$,

$\Rightarrow$ Any of the given vectors can be expressed as a linear combination of the others.

e.g. $x_1 = 2x_3 - x_2$.

Thus x_1 is a linear combination of x_2 and x_3

Remarks:

Applying elementary row operations to the vectors x_1, x_2, x_3, we see that the matrices

$$A = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ and } B = \begin{bmatrix} x_1 \\ x_2 \\ x_1 + x_2 - 2x_3 \end{bmatrix},$$

have the same rank. The rank of B being 2, the rank of A is also 2.

Moreover, x_1, x_2 are linearly independent and x_3 can be expressed as a linear combination of x_1 and x_2 $\left[\because x_3 = \frac{1}{2}(x_1 + x_2)\right]$.

Similar results will hold for column operations and for any matrix. In general, we have the following results:

If a given matrix has r linearly independent vectors (rows or columns) and the remaining vectors are linear combinations of these r vectors, then rank of the matrix is r. Conversely, if a matrix is of rank r, it contains r linearly independent vectors and remaining vectors (if any) can be expressed as a linear combination of these vectors.

Q.No.2: Are the following vectors linearly dependent. If so, find the relation between them:

(i) $(3, 2, 7)$, $(2, 4, 1)$, $(1, -2, 6)$,

(ii) (1, 1, 1, 3), (1, 2, 3, 4), (2, 3, 4, 9),

(iii) $x_1 = (1, 2, 4)$, $x_2(2, -1, 3)$, $x_3 = (0, 1, 2)$, $x_4 = (-3, 7, 2)$.

Sol.: (i). Let $x_1 = (3, 2, 7)$, $x_2 = (2, 4, 1)$, $x_3 = (1, -2, 6)$.

Then $\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 = 0$. (i)

$\Rightarrow \lambda_1(3, 2, 7) + \lambda_2(2, 4, 1) + \lambda_3(1, -2, 6) = 0$,

which is equivalent to

$3\lambda_1 + 2\lambda_2 + \lambda_3 = 0$, $2\lambda_1 + 4\lambda_2 - 2\lambda_3 = 0$, $7\lambda_1 + \lambda_2 + 6\lambda_3 = 0$.

$$\Rightarrow \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & -2 \\ 7 & 1 & 6 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Operating $R_2 \to R_2 - \frac{2}{3}R_1$, $R_3 \to R_3 - \frac{7}{3}R_1$, we get $\begin{bmatrix} 3 & 2 & 1 \\ 0 & \frac{8}{3} & -\frac{8}{3} \\ 0 & -\frac{11}{3} & \frac{11}{3} \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Operating $R_3 \to R_3 + \frac{11}{8}R_2$, we get $\begin{bmatrix} 3 & 2 & 1 \\ 0 & \frac{8}{3} & -\frac{8}{3} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

$\Rightarrow 3\lambda_1 + 2\lambda_2 + \lambda_3 = 0$ and $\frac{8}{3}\lambda_2 - \frac{8}{3}\lambda_3 = 0 \Rightarrow \lambda_2 = \lambda_3$.

Thus $3\lambda_1 + 3\lambda_3 = 0 \Rightarrow \lambda_1 = -\lambda_3$.

$\therefore \lambda_1 = -\lambda_2 = -\lambda_3$.

Putting $\lambda_1 = 1$, $\lambda_2 = -1$, $\lambda_3 = -1$, which are not zero.

Thus, the given vectors are linearly dependent.

Relation: Putting these values in (i), we get $x_1 - x_2 - x_3 = 0$.

Hence $x_1 = x_2 + x_3$.

Hence, x_1 can be expressed in terms of x_2 and x_3.

(ii). Let $x_1 = (1, 1, 1, 3)$, $x_2 = (1, 2, 3, 4)$, $x_3 = (2, 3, 4, 9)$

Then $\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 = 0$. (i)

$\lambda_1(1,1,1,3)+\lambda_2(1,2,3,4)+\lambda_3(2,3,4,9)=0$

which is equivalent to

$\lambda_1+\lambda_2+2\lambda_3=0$,

$\lambda_1+2\lambda_2+3\lambda_3=0$,

$\lambda_1+3\lambda_2+4\lambda_3=0$,

$3\lambda_1+4\lambda_2+9\lambda_3=0$.

$$\Rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \\ 3 & 4 & 9 \end{bmatrix}\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Operating $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, $R_4 \to R_4 - R_1$, we get

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 1 & 3 \end{bmatrix}\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Operating $R_3 \to R_3 - 2R_2$, $R_4 \to R_4 - R_2$, we get $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

$\Rightarrow \lambda_1+\lambda_2+2\lambda_3=0$, $\lambda_2+\lambda_3=0$ and $2\lambda_3=0$.

$\Rightarrow \lambda_3=0$, $\lambda_2=0$ and $\lambda_1=0$.

Thus λ_1, λ_2 and λ_3 have value equal to 0.

Hence, it is linearly independent.

Relation: Since vectors are linearly independent, so there is no relation between them.

(iii). $x_1=(1,2,4)$, $x_2=(2,-1,3)$, $x_3=(0,1,2)$, $x_4=(-3,7,2)$.

Then $\lambda_1x_1+\lambda_2x_2+\lambda_3x_3+\lambda_4x_4=0$. (i)

$\Rightarrow \lambda_1(1,2,4)+\lambda_2(2,-1,4)+\lambda_3(0,1,2)+\lambda_4(-3,7,2)=0$,

which is equivalent to

$\lambda_1+2\lambda_2-3\lambda_4=0$,

$2\lambda_1 - \lambda_2 + \lambda_3 + 7\lambda_4 = 0$,

$4\lambda_1 + 3\lambda_2 + 2\lambda_3 + 2\lambda_4 = 0$.

$$\Rightarrow \begin{bmatrix} 1 & 2 & 0 & -3 \\ 2 & -1 & 1 & 7 \\ 4 & 3 & 2 & 2 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Operating $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 4R_1$, we get $\begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -5 & 1 & 13 \\ 0 & -5 & 2 & 14 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Operating $R_3 \to R_3 - R_2$, we get $\begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -5 & 1 & 13 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

$\Rightarrow \lambda_1 + 2\lambda_2 - 3\lambda_4 = 0, \quad -5\lambda_2 + \lambda_3 - 13\lambda_4 = 0, \quad \lambda_3 + \lambda_4 = 0$.

By solving $\lambda_1 = \frac{9}{5}\lambda_4, \quad \lambda_2 = -\frac{12}{5}\lambda_4, \quad \lambda_3 = -\lambda_4$.

Putting $\lambda_4 = 1$, we get $\lambda_1 = \frac{9}{5}, \quad \lambda_2 = -\frac{12}{5}, \quad \lambda_3 = -1$.

Thus, the given vectors are linearly dependent.

Relation: Now $\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 + \lambda_4 x_4 = 0$

$$\Rightarrow \frac{9}{5}x_1 + \left(-\frac{12}{5}\right)x_2 + (-1)x_3 + (1)x_4 = 0$$

$$\Rightarrow 9x_1 - 12x_2 - 5x_3 + 5_4 = 0,$$

which is the required relation between these vectors.

Home Assignments

Q.No.1.: Are the following vectors linearly dependent? If so, find a relation between them.

(i). $x_1 = (1,3,2)$, $x_2 = (5,-2,1)$, $x_3 = (-7,13,4)$

(ii). $x_1 = (1,-1,3,2)$, $x_2 = (1,3,4,2)$, $x_3 = (3,-5,2,2)$

(iii). $x_1 = (2,3,1,-1)$, $x_2 = (2,3,1,-2)$, $x_3 = (4,6,2,1)$.

Ans.: (i). Yes, Relation: $3x_1 - 2x_2 - x_3 = 0$.

(ii). Yes, Relation: $2x_1 - x_2 - x_3 = 0$.

(iii). Yes, Relation: $5x_1 - 3x_2 - x_3 = 0$.

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5th Topic

Matrices

Consistency of linear system of equations

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Consistency of linear system of equations:

Let

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \dots\dots\dots + a_{1n}x_n = k_1 \\ a_{21}x_1 + a_{22}x_2 + \dots\dots + a_{2n}x_n = k_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots\dots + a_{mn}x_n = k_m \end{array}\right\}, \qquad \text{(i)}$$

be a system of m-non-homogenous equations in n-unknowns $x_1, x_2, \dots\dots, x_n$.

If we write $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}_{m\times n}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix}_{n\times 1}$, $B = \begin{bmatrix} k_1 \\ k_2 \\ \dots \\ k_m \end{bmatrix}_{m\times 1}$

$\therefore$ (i) $\Rightarrow AX = B$.

Solution of linear system of equations:

Any set of values of $x_1, x_2, \dots\dots, x_n$ which simultaneously satisfy all these equations is called a solution of the system of equations (i).

Consistent and inconsistent:

When the system of equations has one or more solutions, then the equations are said to be consistent, otherwise, they are said to be inconsistent.

Augmented matrix:

The matrix $K=[A \vdots B]=\begin{bmatrix} a_{11} & a_{12} & .... & a_{1n} & \vdots & k_1 \\ a_{21} & a_{22} & .... & a_{2n} & \vdots & k_2 \\ ..... & ..... & .... & .... & \vdots & .... \\ a_{m1} & a_{m2} & .... & a_{mn} & \vdots & k_m \end{bmatrix}$ **is called the augmented matrix of the given system of equations.**

Theorem of consistency

Statement: The system of equations AX = B is consistent, i.e., possesses a solution iff the coefficient matrix A and the augmented matrix K = (A:B) are of the same rank. Otherwise, the system is inconsistent.

Proof: Let

$$\left.\begin{array}{l} a_{11}x_1+a_{12}x_2+..........+a_{1n}x_n=k_1 \\ a_{21}x_1+a_{22}x_2+........+a_{2n}x_n=k_2 \\ .... \\ a_{m1}x_1+a_{m2}x_2+.....+a_{mn}x_n=k_m \end{array}\right\}, \qquad \text{(i)}$$

be a system of m-non-homogenous equations in n-unknowns $x_1, x_2, ..., x_n$.

If we write $A=\begin{bmatrix} a_{11} & a_{12} & .... & a_{1n} \\ a_{21} & a_{22} & .... & a_{2n} \\ .... & .... & .... & .... \\ a_{m1} & a_{m2} & .... & a_{mn} \end{bmatrix}_{m\times n}$, $X=\begin{bmatrix} x_1 \\ x_2 \\ .... \\ x_n \end{bmatrix}_{n\times 1}$, $B=\begin{bmatrix} k_1 \\ k_2 \\ .... \\ k_m \end{bmatrix}_{m\times 1}$

$\therefore$ (i) $\Rightarrow AX=B$.

Now we consider the following two possible cases:

Case I.: When the rank of A = the rank of K = r (r$\leq$ the smaller of numbers m and n).

Then by, suitable row operations, the system of equations AX = B can be reduced to

$$\left.\begin{array}{l} b_{11}x_1+b_{12}x_2+.......+b_{1n}x_n=\ell_1 \\ 0x_1+b_{22}x_2+..........+b_{2n}x_n=\ell_2 \\ .... \\ 0x_1+0x_2+...............+b_{rn}x_n=\ell_r \end{array}\right\} \qquad \text{(ii)}$$

and the remaining $(m-r)$ equations being all of the form

$0.x_1+0.x_2+..........+0.x_n=0$.

The equations (ii) will have a solution, by choosing (n – r) unknowns arbitrary.

The solution will be unique only when r = n.

Hence, the equations (i) are consistent, i.e., possesses solution.

Case II.: When the rank of A (i.e. r) < the rank of K.

In Particular, let the rank of K be r + 1.

Then, by suitable row operations, the esystem of equations AX = B can reduced to

$$\left.\begin{array}{l} b_{11}x_1 + b_{12}x_2 + \ldots\ldots + b_{1n}x_n = \ell_1, \\ 0x_1 + b_{22}x_2 + \ldots\ldots\ldots + b_{2n}x_n = \ell_2, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ 0x_1 + 0x_2 + \ldots\ldots\ldots\ldots + b_{rn}x_n = \ell_r, \\ 0x_1 + 0x_2 + \ldots\ldots\ldots\ldots\ldots + 0x_n = \ell_{r+1}, \end{array}\right\}$$

and the remaining $[m-(r+1)]$ equations are of the form

$$0.x_1 + 0.x_2 + \ldots\ldots\ldots + 0.x_n = 0.$$

Clearly, the $(r+1)^{th}$ equation can not be satisfied by any set of values for the unknowns.

Hence, the equations (i) are inconsistent, i.e., does not possess a solution.

This completes the proof.

Procedure to test the consistency of a system of equations in n-unknown:

Find the ranks of the coefficient matrix A and the augmented matrix K, by reducing A to the triangular form by elementary row operations. Let the rank of A = r and rank of K = r'.

(i) If $r \neq r'$, then the equations are inconsistent, i.e., there is no solution.

(ii) $r = r' = n$, then the equations are consistent and there is a unique solution.

(iii) $r = r' < n$, then the equations are consistent and there are infinite number of solutions, by choosing to (n – r) unknowns arbitrary.

System of linear homogeneous equations:

Consider the homogeneous linear equations

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots\ldots\ldots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \ldots\ldots\ldots + a_{2n}x_n = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots + a_{mn}x_n = 0 \end{array}\right\}. \qquad \text{(i)}$$

Find the rank r of the coefficient matrix A by reducing it to the triangular form by elementary operations.

Case (i):If the rank of A = n:

Then the equations (i) have only a trivial solution

$$x_1 = x_2 = ... = x_n = 0.$$

If the rank of A $< n$:

Then the equations (i) have $(n-r)$ independent solutions and r cannot be $> n$.

Remarks: The number of linearly independent solutions is $(n-r)$ means, if arbitrary values are assigned to $(n-r)$ of the variables, the values of the remaining variables can be uniquely found.

Case (ii):When m < n:

(i.e. the number of equations is less than the number of variables)

Then the solution is always other than $x_1 = x_2 = ... = x_n = 0$.

Case (iii):When m = n:

(i.e. the number of equations = the number of variables).

Then the necessary and sufficient condition for solutions other than $x_1 = x_2 = ... = x_n = 0$, is that the determinant of the coefficient matrix is zero. In this case the equations are said to be consistent and such a solution is called non-trivial solution. The determinant is called the eliminant of the equation.

Now let us examine the consistency of the following system of equations:

System of homogenous equations

Q.No.1.: Solve the equations:

(i) $x + 2y + 3z = 0$, $3x + 4y + 4z = 0$, $7x + 10y + 12z = 0$,

(ii) $4x + 2y + z + 3w = 0$, $6x + 3y + 4z + 7w = 0$, $2x + y + w = 0$.

Sol.: (i). Here coefficient matrix A = $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix}$.

Find the rank of the coefficient matrix A:

Operating $R_2 \to R_2 - 3R_1$, we get $A \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 7 & 10 & 12 \end{bmatrix}$.

Operating $R_3 \to R_3 - 7R_1 - 2R_2$, we get $A \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 0 & 0 & 1 \end{bmatrix}$.

Thus the rank of A = 3= the number of variables (i.e. r = n).

$\therefore$ The equations have only a trivial solution x = y = z = 0.

(ii). Here coefficient matrix A is $A = \begin{bmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{bmatrix}$.

Find the rank of the coefficient matrix:

Operating $R_2 \to R_2 - \frac{3}{2}R_1$, $R_3 \to R_3 - \frac{1}{2}R_1$, we get $A \sim \begin{bmatrix} 4 & 2 & 1 & 3 \\ 0 & 0 & \frac{5}{2} & \frac{5}{2} \\ 0 & 0 & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$.

Operating $R_3 \to R_3 + \frac{1}{5}R_2$, we get $A \sim \begin{bmatrix} 4 & 2 & 1 & 3 \\ 0 & 0 & \frac{5}{2} & \frac{5}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Thus the rank of A = 2 < the number of variable (i.e. $r < n$)

$\therefore$ Number of independent solutions = $4 - 2 = 2$. Also the given system is equivalent to $4x + 2y + z + 3w = 0$, $z + w = 0$.

$\therefore$ We have $z = -w$ and $y = -2x - w$,

which give an infinite number of non-trivial solutions, by choosing the values of x and w arbitrary.

Q.No.2.: Find the values of λ for which the equations

$(\lambda - 1)x + (3\lambda + 1)y + 2\lambda z = 0$,

$(\lambda - 1)x + (4\lambda - 2)y + (\lambda + 3)z = 0$,

$2x + (3\lambda + 1)y + 3(\lambda - 1)z = 0$, are consistent, and find the ratios of x : y : z, when λ has the smallest of these values. What happens when λ has the greater of these values.

Sol.: The given equations will be consistent, if $\begin{vmatrix} \lambda-1 & 3\lambda+1 & 2\lambda \\ \lambda-1 & 4\lambda-2 & \lambda+3 \\ 2 & 3\lambda+1 & 3(\lambda-1) \end{vmatrix} = 0$.

Operating $R_2 \to R_2 - R_1$, we get $\begin{vmatrix} \lambda-1 & 3\lambda+1 & 2\lambda \\ 0 & \lambda-3 & 3-\lambda \\ 2 & 3\lambda+1 & 3(\lambda-1) \end{vmatrix} = 0$.

Operating $C_3 \to C_3 + C_2$, we get $\begin{vmatrix} \lambda-1 & 3\lambda+1 & 5\lambda+1 \\ 0 & \lambda-3 & 0 \\ 2 & 3\lambda+1 & 6\lambda-2 \end{vmatrix} = 0$.

$\Rightarrow (\lambda-3)\begin{vmatrix} \lambda-1 & 5\lambda+1 \\ 2 & 2(3\lambda+1) \end{vmatrix} = 0 \Rightarrow 2(\lambda-3)[(\lambda-1)(3\lambda-1)-(5\lambda+1)] = 0 \Rightarrow 6\lambda(\lambda-3)^2 = 0$

$\Rightarrow \lambda = 0$ or 3.

(a). When $\lambda = 0$, the equations become $-x + y = 0$ (i)

$-x - 2y + 3z = 0$ (ii)

$2x + y - 3z = 0$ (iii)

Solving (ii) and (iii), we get $\dfrac{x}{6-3} = \dfrac{y}{6-3} = \dfrac{z}{-1+4}$.

Hence x = y = z.

(b). When $\lambda = 3$, equations become identical.

Q.No.3.: Determine the values of λ for which the following set of equations may possess non-trivial solution:

$3x_1 + x_2 - \lambda x_3 = 0$, $4x_1 - 2x_2 - 3x_3 = 0$, $2\lambda x_1 + 4x_2 + \lambda x_3 = 0$.

For each permissible value of λ, determine the general solution.

Sol.: Given equations are $3x_1 + x_2 - \lambda x_3 = 0$, (i)

$4x_1 - 2x_2 - 3x_3 = 0$, (ii)

$2\lambda x_1 + 4x_2 + \lambda x_3 = 0$. (iii)

The given system of equations will be consistent, if $|A| = 0 \Rightarrow \begin{vmatrix} 3 & 1 & -\lambda \\ 4 & -2 & -3 \\ 2\lambda & 4 & \lambda \end{vmatrix} = 0$.

$\Rightarrow 3(-2\lambda + 12) - 1(4 + 6\lambda) - \lambda(16 + 4\lambda) = 0$

$\Rightarrow \lambda^2 - \lambda + 9\lambda - 9 = 0 \Rightarrow \lambda = 1, \quad -9$.

For $\lambda = 1$, equations (i), (ii) and (iii), becomes

$3x_1 + x_2 - x_3 = 0$ (iv)

$4x_1 - 2x_2 - 3x_3 = 0$ (v)

$2x_1 + 4x_2 + x_3 = 0$ (vi)

By (iv) and (vi), we get

$5x_1 + 5x_2 = 0$

$\Rightarrow x_1 = -x_2 = k$ (say)

$\therefore x_1 = k, \quad x_2 = -k$

Value of k put in equation (iv), we get

$3k - k - x_3 = 0 \Rightarrow x_3 = 2k$.

When $\lambda = 1$. Solution is $x_1 = k$, $x_2 = -k$ and $x_3 = 2k$. Ans.

For $\lambda = -9$, equations (i), (ii) and (iii), becomes

$3x_1 + x_2 + 9x_3 = 0$ (vii)

$4x_1 - 2x_2 - 3x_3 = 0$ (viii)

$-18x_1 + 4x_2 - 9x_3 = 0$. (ix)

By equation (vii) and (viii), we get

$$\frac{x_1}{-3+18} = \frac{x_2}{36+9} = \frac{x_3}{-6-4} = k$$

$$\Rightarrow \frac{x_1}{3} = \frac{x_2}{9} = \frac{x_3}{-2} = k \Rightarrow x_1 = 3k, \quad x_2 = 9k, \quad x_3 = -2k.$$

Hence we calculated

For $\lambda = -9$, $x_1 = 3k$, $x_2 = 9k$, $x_3 = -2k$,

For $\lambda = 1$, $x_1 = k$, $x_2 = -k$ and $x_3 = 2k$, be the required general solution.

Q.No.4.: Solve completely the system of equations

$x+y-2z+3w=0$, $x-2y+z-w=0$,

$4x+y-5z+8w=0$, $5x-7y+2z-w=0$.

Sol.: The matrix form of the given system of equations is $\begin{bmatrix}1 & 1 & -2 & 3\\ 1 & -2 & 1 & -1\\ 4 & 1 & -5 & 8\\ 5 & -7 & 2 & -1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\\ w\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\\ 0\end{bmatrix}$.

Operating $R_2 \to R_2 - R_1$, $R_3 \to R_3 - 4R_1$, $R_4 \to R_4 - 5R_1$, we get

$$\begin{bmatrix}1 & 1 & -2 & 3\\ 0 & -3 & 3 & -4\\ 0 & -3 & 3 & -4\\ 0 & -12 & 12 & -16\end{bmatrix}\begin{bmatrix}x\\ y\\ z\\ w\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\\ 0\end{bmatrix}.$$

Operating $R_3 \to R_3 - R_2$, $R_4 \to R_4 - 4R_2$, we get $\begin{bmatrix}1 & 1 & -2 & 3\\ 0 & -3 & 3 & -4\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\ y\\ z\\ w\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\\ 0\end{bmatrix}$

$$\Rightarrow x+y-2z+3w=0, \tag{i}$$

$$-3y+3z-4w=0. \tag{ii}$$

Suppose $z=\lambda$ and $w=\mu$

Now put the values of z and w in equation (ii), we get

$$-3y+3\lambda-4\mu=0 \Rightarrow -3y=4\mu-3\lambda \Rightarrow y=\frac{3\lambda-4\mu}{3} \Rightarrow y=\lambda-\frac{4}{3}\mu.$$

Put the value of y in equation (i), we get

$$x+\frac{(3\lambda-4\mu)}{3}-2\lambda+3\mu=0 \Rightarrow 3x+3\lambda-4\mu-6\lambda+9\mu=0$$

$$\Rightarrow 3x-3\lambda+5\mu=0 \Rightarrow x=\lambda-\frac{5}{3}\mu.$$

Thus $x=\lambda-\frac{5}{3}\mu$, $y=\lambda-\frac{4}{3}\mu$, $z=\lambda$ and $w=\mu$ be the required solution.

Q.No.5.: Solve the equations

$x_1+3x_2+2x_3=0$, $2x_1-3x_2+3x_3=0$,

$3x_1 - 5x_2 + 4x_3 = 0$, $x_1 + 17x_2 + 4x_3 = 0$.

Sol.: In matrix notation, the given system of equations can be written as AX = 0

where $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & -1 & 3 \\ 3 & -5 & 4 \\ 1 & 17 & 4 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

Operating $R_1 - 2R_1$, $R_3 - 3R_1$, $R_4 - R_1$, we get $A \approx \begin{bmatrix} 1 & 3 & 2 \\ 0 & -7 & -1 \\ 0 & -14 & -2 \\ 0 & 14 & 2 \end{bmatrix}$.

Operating $R_3 - 2R_2$, $R_4 + 2R_2$, we get $A \approx \begin{bmatrix} 1 & 3 & 2 \\ 0 & -7 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Operating $R_1 + 2R_2$, we get $A \approx \begin{bmatrix} 1 & -11 & 0 \\ 0 & -7 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

$\rho(A) = 2 <$ number of unknowns.

$\Rightarrow$ The system has an infinite number of non-trivial solutions given by

$x_1 - 11x_2 = 0$, $-7x_2 - x_3 = 0$

i.e., $x_1 = 11k$, $x_2 = k$, $x_3 = 7k$, where k is any number. Different values of k give different solutions.

System of non-homogenous equations

Q.No.1.: Test the consistency and solve

$5x + 3y + +7z = 4$, $3x + 26y + 2z = 9$, $7x + 2y + 10z = 5$.

Sol.: The given set of equations can be written as $\begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 5 \end{bmatrix}$.

Operating $R_1 \to 3R_1,\ R_2 \to 5R_2$, we get $\begin{bmatrix} 15 & 9 & 21 \\ 15 & 130 & 10 \\ 7 & 2 & 10 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 45 \\ 5 \end{bmatrix}$.

Operating $R_2 \to R_2 - R_1$, we get $\begin{bmatrix} 15 & 9 & 21 \\ 0 & 121 & -11 \\ 7 & 2 & 10 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 33 \\ 5 \end{bmatrix}$.

Operating $R_1 \to \frac{7}{3}R_1, R_3 \to 5R_3, R_2 \to \frac{1}{11}R_2$, we get $\begin{bmatrix} 35 & 21 & 49 \\ 0 & 11 & -1 \\ 35 & 10 & 50 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 28 \\ 3 \\ 25 \end{bmatrix}$.

Operating $R_3 \to R_3 - R_1 + R_2, R_1 \to \frac{1}{7}R_1$, we get $\begin{bmatrix} 5 & 3 & 7 \\ 0 & 11 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}$.

Here the ranks of coefficient matrix A = the rank the augmented matrix K = 2.

Hence, the equations are consistent.

Also the given system is equivalent to

$5x + 3y + 7z = 4$, $11y - z = 3$.

$\therefore\ y = \frac{3}{11} + \frac{z}{11}$ and $x = \frac{7}{11} - \frac{16}{11}z$, where z is parameter.

Thus, we have infinite number of solutions by choosing one unknown arbitrary.

If we put z = 0, we get

$x = \frac{7}{11},\ y = \frac{3}{11}$, which is a particular solution.

Q.No.2.: Investigate for consistency of the following equations and if possible find the solutions:

$4x - 2y + 6z = 8$, $x - y + 3z = -1$, $15x - 3y + 9z = 21$.

Sol.: Here $\begin{bmatrix} 4 & -2 & 6 \\ 1 & 1 & -3 \\ 15 & -3 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ -1 \\ 21 \end{bmatrix}$ is the matrix representation of the given equations.

Now operating $R_1 \to \frac{R_1}{2},\ R_3 \to \frac{R_3}{2}$, we get $\begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & -3 \\ 5 & -1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 7 \end{bmatrix}$.

Operating $R_2 \to 2R_2 - R_1$, we get $\begin{bmatrix} 2 & -1 & 3 \\ 0 & 3 & -9 \\ 5 & -1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -6 \\ 7 \end{bmatrix}$.

Operating $R_2 \to \frac{R_2}{3}$, $R_3 \to 2R_3 - 5R_1$, we get $\begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 0 \end{bmatrix}$. (i)

Here the rank of coefficient matrix A = 2 = the rank of augmented matrix $K < 3$.

Hence the given system of equations is consistent and we have infinite number of solutions.

Now (i) $\Rightarrow 2x - y + 3z = 4$, $y - 3z = -2$.

Let z = k arbitrary number, hence

$y = 3k - 2$ and $2x - 3k + 2 + 3z = 4$

$\Rightarrow 2x - 3k + 2 + 3k = 4 \Rightarrow 2x = 2 \Rightarrow x = 1$.

Hence $x = 1$, $y = 3k - 2$ and $z = k$ for all k,

which gives an infinite no. of non-trivial solutions.

Q.No.3.: Test for consistency and solve:

(i) $2x - 3y + 7x = 5$, $3x + y - 3z = 13$, $2x + 19y - 47z = 32$,

(ii) $x + 2y + z = 3$, $2x + 3y + 2z = 5$, $3x - 5y + 5z = 2$, $3x + 9y - z = 4$,

(iii) $2x + 6y + 11 = 0$, $6x + 20y - 6z + 3 = 0$, $6y - 18z + 1 = 0$.

Sol.: (i) We have $AX = B \Rightarrow \begin{bmatrix} 2 & -3 & -7 \\ 3 & -1 & -3 \\ 2 & 19 & -47 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 13 \\ 32 \end{bmatrix}$.

Operating $R_1 \to 3R_1 - 2R_2$, $R_3 \to R_3 - R_1$, we get $\begin{bmatrix} 0 & -11 & 27 \\ 3 & 1 & -3 \\ 0 & 22 & -54 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -11 \\ 13 \\ 27 \end{bmatrix}$.

Operating $R_3 \to R_3 + 2R_1$, we get $\begin{bmatrix} 0 & -11 & 27 \\ 3 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -11 \\ 13 \\ 27 \end{bmatrix}$.

Here $\rho(A) = 2 \neq \rho(K) = 3$.

This shows that the given system of equations is not consistent, i.e., no solution for these equations.

(ii). Given equations are

$x+2y+z=3$, $2x+3y+2z=5$, $3x-5y+5z=2$, $3x+9y-z=4$.

Now we have $AX=B \Rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 3 & -5 & 5 \\ 3 & 9 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 2 \\ 4 \end{bmatrix}$.

Operating $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$ and $R_4 \to R_4 - 3R_1$, we get

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & -11 & 2 \\ 0 & 3 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ -7 \\ -5 \end{bmatrix}.$$

Operating $R_3 \to R_3 - 11R_1$, $R_4 \to R_4 + 3R_1$, we get $\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 4 \\ -8 \end{bmatrix}$

Operating $R_4 \to R_4 + 2R_3$, we get $\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 4 \\ 0 \end{bmatrix}$. (i)

Here $\rho(A) = 3 = \rho(K) =$ no. of unknowns.

Hence, the given system of equations is consistent and there is only unique solution.

Now (i) $\Rightarrow x+2y+z=3$,

$-y=-1 \Rightarrow y=1$,

$2z=4 \Rightarrow z=2$.

Now putting y and z in the equation, we get $x=-1$.

Hence, solution is $x=-1$, $y=1$ and $z=2$. Ans.

(iii). Given equation are

$2x+6y+11=0$, $6x+20y-6z+3=0$, $6y-18z+1=0$.

Now we have AX= B $\Rightarrow \begin{bmatrix} 2 & 6 & 0 \\ 6 & 20 & -6 \\ 0 & 6 & -18 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -11 \\ 30 \\ -1 \end{bmatrix}$.

Operating $R_2 \to R_2 - 3R_1$, we get $\begin{bmatrix} 2 & 6 & 0 \\ 0 & 2 & -6 \\ 0 & 6 & -18 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -11 \\ 30 \\ -1 \end{bmatrix}$.

Operating $R_3 \to R_3 - 3R_2$, we get $\begin{bmatrix} 2 & 6 & 0 \\ 0 & 2 & -6 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -11 \\ 30 \\ -91 \end{bmatrix}$.

Here $\rho(A) = 2 \neq \rho(K) = 3$.

This shows that the given system of equations is not consistent, i.e. no solution for these equations.

Q.No.4.: Test for consistency and solve:

$2x_1 - 2x_2 + 4x_3 + 3x_4 = 9$, $x_1 - x_2 + 2x_3 + 2x_4 = 6$,

$2x_1 - 2x_2 + x_3 + 2x_4 = 3$, $x_1 - x_2 + x_4 = 2$

Sol.: Apply elementary row operation on $[A|B]$.

Since $[A|B] = \left[\begin{array}{cccc|c} 2 & -2 & 4 & 3 & 9 \\ 1 & -1 & 2 & 2 & 6 \\ 2 & -2 & 1 & 2 & 3 \\ 1 & -1 & 0 & 1 & 2 \end{array}\right]$.

Operating R_{12}, $R_{21(-2)}$, $R_{41(-1)}$, $R_{31(-2)}$, $R_{2(-1)}$, $R_{3(-1)}$, $R_{4(-1)}$, we get

$$[A|B] = \left[\begin{array}{cccc|c} 1 & -1 & 2 & 2 & 6 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 3 & 2 & 9 \\ 0 & 0 & 2 & 1 & 4 \end{array}\right].$$

Operating $R_{34(-1)}$, we get $[A|B] = \left[\begin{array}{cccc|c} 1 & -1 & 2 & 2 & 6 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 1 & 5 \\ 0 & 0 & 2 & 1 & 4 \end{array}\right]$.

Operating R_{32}, R_{43}, we get $[A|B]=\left[\begin{array}{cccc|c}1 & -1 & 2 & 2 & 6\\ 0 & 0 & 0 & 1 & 5\\ 0 & 0 & 2 & 1 & 4\\ 0 & 0 & 0 & 1 & 3\end{array}\right]$.

Operating $R_{32(-2)}$, $R_{3(-1)}$, we get $[A|B]=\left[\begin{array}{cccc|c}1 & -1 & 2 & 2 & 6\\ 0 & 0 & 1 & 1 & 5\\ 0 & 0 & 0 & 1 & 6\\ 0 & 0 & 0 & 1 & 3\end{array}\right]$.

Operating $R_{43(-1)}$, $R_{4(-1)}$, we get $[A|B]=\left[\begin{array}{cccc|c}1 & -1 & 2 & 2 & 6\\ 0 & 0 & 1 & 1 & 5\\ 0 & 0 & 0 & 1 & 6\\ 0 & 0 & 0 & 0 & 3\end{array}\right]$.

Rank of (A) $=3\neq 4$ = rank of $[A|B]$.

So the given system in inconsistent and therefore has no solution.

Q.No.5.: Solve the system of equations:

$2x_1+x_2+2x_3+x_4=6,\ 6x_1-6x_2+6x_3+12x_4=36,$

$4x_1+3x_2+3x_3-3x_4=-1,\ 2x_1+2x_2-x_3+x_4=10.$

Sol.: In matrix notation, the given system of equations can be written as AX = B

where $A=\begin{bmatrix}2 & 1 & 2 & 1\\ 6 & -6 & 6 & 12\\ 4 & 3 & 3 & -3\\ 2 & 2 & -1 & 1\end{bmatrix}$, $X=\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}$, $B=\begin{bmatrix}6\\ 36\\ -1\\ 10\end{bmatrix}$.

Augmented matrix $[A\vdots B]=\left[\begin{array}{cccccc}2 & 1 & 2 & 1 & \vdots & 6\\ 6 & -6 & 6 & 12 & \vdots & 35\\ 4 & 3 & 3 & -3 & \vdots & -1\\ 2 & 2 & -1 & 1 & \vdots & 10\end{array}\right]$.

Operating $R_2-3R_1, R_3-2R_1, R_4-R_1$, we get $[A\vdots B]=\left[\begin{array}{cccccc}2 & 1 & 2 & 1 & \vdots & 6\\ 0 & -9 & 0 & 9 & \vdots & 18\\ 0 & 1 & -1 & -5 & \vdots & -13\\ 0 & 1 & -3 & 1 & \vdots & 4\end{array}\right]$.

Operating $-\frac{1}{9}R_2$, we get $[A \vdots B] = \begin{bmatrix} 2 & 1 & 2 & 1 & \vdots & 6 \\ 0 & 1 & 0 & -1 & \vdots & -2 \\ 0 & 1 & -1 & -5 & \vdots & -13 \\ 0 & 1 & -3 & 0 & \vdots & 4 \end{bmatrix}$.

Operating $R_1 - R_2, R_3 - R_2, R_4 - R_2$, we get $[A \vdots B] = \begin{bmatrix} 2 & 0 & 2 & 1 & \vdots & 8 \\ 0 & 1 & 0 & -1 & \vdots & -2 \\ 0 & 0 & -1 & -4 & \vdots & -11 \\ 0 & 0 & -3 & 1 & \vdots & 6 \end{bmatrix}$.

Operating $R_4 - 3R_3, \frac{1}{2}R_1$, we get $[A \vdots B] = \begin{bmatrix} 1 & 0 & 1 & 1 & \vdots & 4 \\ 0 & 1 & 0 & -1 & \vdots & -2 \\ 0 & 0 & -1 & -4 & \vdots & -11 \\ 0 & 0 & 0 & 13 & \vdots & 39 \end{bmatrix}$.

Operating $R_1 + R_3, \frac{1}{13}R_4$, we get $[A \vdots B] = \begin{bmatrix} 1 & 0 & 1 & -3 & \vdots & -7 \\ 0 & 1 & 0 & -1 & \vdots & -2 \\ 0 & 0 & -1 & -4 & \vdots & -11 \\ 0 & 0 & 0 & 1 & \vdots & 3 \end{bmatrix}$.

Operating $R_1 + 3R_4, R_2 + R_4,\ R_3 + 4R_4$, we get $[A \vdots B] = \begin{bmatrix} 1 & 0 & 1 & 0 & \vdots & 2 \\ 0 & 1 & 0 & 0 & \vdots & 1 \\ 0 & 0 & -1 & 0 & \vdots & 1 \\ 0 & 0 & 0 & 1 & \vdots & 3 \end{bmatrix}$.

Operating $(-1)R_3$, we get $[A \vdots B] = \begin{bmatrix} 1 & 0 & 1 & 0 & \vdots & 2 \\ 0 & 1 & 0 & 0 & \vdots & 1 \\ 0 & 0 & 1 & 0 & \vdots & -1 \\ 0 & 0 & 0 & 1 & \vdots & 3 \end{bmatrix}$.

Hence $x_1 = 2$, $x_2 = 1$, $x_3 = 1$, $x_4 = 3$.

Q.No.6.: Using matrix method, show that the equations:

$3x + 3y + 2z = 1,\ x + 2y = 4,\ 10y + 3z = 2,\ 2x - 3y - z = 5$

are consistent and hence obtain the solutions for x, y and z.

Sol.: In matrix notation, the given system of equations can be written as AX = B where $A = \begin{bmatrix} 3 & 3 & 2 \\ 1 & 2 & 0 \\ 0 & 10 & 3 \\ 2 & -3 & -1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 4 \\ -2 \\ 5 \end{bmatrix}$.

Augmented matrix $[A \vdots B] = \begin{bmatrix} 3 & 3 & 2 & \vdots & 1 \\ 1 & 2 & 0 & \vdots & 4 \\ 0 & 10 & 3 & \vdots & -2 \\ 2 & -3 & -1 & \vdots & 5 \end{bmatrix}$.

Operating R_{12}, we get $[A \vdots B] = \begin{bmatrix} 1 & 2 & 0 & \vdots & 4 \\ 3 & 3 & 2 & \vdots & 1 \\ 0 & 10 & 3 & \vdots & -2 \\ 2 & -3 & -1 & \vdots & 5 \end{bmatrix}$.

Operating $R_2 - 3R_1$, $R_4 - 2R_1$ we get $[A \vdots B] = \begin{bmatrix} 1 & 2 & 0 & \vdots & 4 \\ 0 & -3 & 2 & \vdots & -11 \\ 0 & 10 & 3 & \vdots & -2 \\ 0 & -7 & -1 & \vdots & -3 \end{bmatrix}$.

Operating $R_3 + 3R_2$, $R_4 - 2R_2$ we get $[A \vdots B] = \begin{bmatrix} 1 & 2 & 0 & \vdots & 4 \\ 0 & -3 & 2 & \vdots & -11 \\ 0 & 1 & 9 & \vdots & -35 \\ 0 & -1 & -5 & \vdots & 19 \end{bmatrix}$.

Operating $R_1 - 2R_3$, $R_2 + 3R_3$, $R_4 + R_3$ we get $[A \vdots B] = \begin{bmatrix} 1 & 0 & -18 & \vdots & 74 \\ 0 & 0 & 29 & \vdots & -116 \\ 0 & 1 & 9 & \vdots & -35 \\ 0 & 0 & 4 & \vdots & -16 \end{bmatrix}$.

Operating R_{23}, $\frac{1}{4}R_4$, we get $[A \vdots B] = \begin{bmatrix} 1 & 0 & -18 & \vdots & 74 \\ 0 & 0 & 9 & \vdots & -35 \\ 0 & 1 & 29 & \vdots & -116 \\ 0 & 0 & 1 & \vdots & -4 \end{bmatrix}$

Operating $R_1 + 18R_4$, $R_2 - 9R_4$, $R_3 - 29R_4$ we get $[A \vdots B] = \begin{bmatrix} 1 & 0 & 0 & \vdots & 2 \\ 0 & 1 & 0 & \vdots & 1 \\ 0 & 0 & 0 & \vdots & 0 \\ 0 & 0 & 1 & \vdots & -4 \end{bmatrix}$

Operating R_{34}, we get $[A \vdots B] = \begin{bmatrix} 1 & 0 & 0 & \vdots & 2 \\ 0 & 1 & 0 & \vdots & 1 \\ 0 & 0 & 1 & \vdots & -4 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$

$\rho(A) = \rho(A : B) = 3 =$ number of unknowns.

$\Rightarrow$ The given system of equations is consistent and the solution is $x = 2,\ y = 1,\ z = -4$.

Q.No.7.: Test for consistency and solve:

$3x + 3y + 2z = 1,\ x + 2y = 4, 10y + 3z = -2,\ 2x - 3y - z = 5$.

Sol.: $[A|B] = \left[\begin{array}{ccc|c} 3 & 3 & 2 & 1 \\ 1 & 2 & 0 & 4 \\ 0 & 10 & 3 & -2 \\ 2 & -3 & -1 & 5 \end{array}\right]$.

Operating R_{12}, we get $[A|B] = \left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 3 & 3 & 2 & 1 \\ 0 & 10 & 3 & -2 \\ 2 & -3 & -1 & 5 \end{array}\right]$.

Operating $R_{21(-3)}$, $R_{41(-2)}$, we get $[A|B] = \left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & -3 & 2 & -11 \\ 0 & 10 & 3 & -2 \\ 0 & -7 & -1 & 3 \end{array}\right]$.

Operating $R_{2\left(-\frac{1}{3}\right)}$, $R_{32(-10)}$, $R_{42(7)}$, we get $[A|B] = \left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & -\frac{2}{3} & \frac{11}{3} \\ 0 & 0 & \frac{29}{3} & -\frac{116}{3} \\ 0 & 0 & -\frac{17}{3} & \frac{68}{3} \end{array}\right]$.

Operating $R_{3\left(\frac{3}{29}\right)}$, $R_{43\left(\frac{17}{3}\right)}$, we get $[A|B] = \left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & -\frac{2}{3} & \frac{11}{3} \\ 0 & 0 & 1 & -\frac{116}{3} \\ 0 & 0 & 0 & 0 \end{array}\right]$.

r(A) = 3 = $[A|B]$ = n = number of variables.

The system is consistent and has unique solution.

Solving, we get $z = -\frac{116}{29} = -4$.

$y - \frac{2}{3}z = \frac{11}{3} \Rightarrow y = \frac{11}{3} + \frac{2}{3}(-4) = 1$

$x + 2y + 0 \Rightarrow x = 4 - 2 = 2$.

i.e., $x = 2,\ y = 1,\ z = -4$.

Q.No.8.: Solve $x_1 + x_2 - x_3 = 0$, $2x_1 - x_2 + x_3 = 3$, $4x_1 + 2x_2 - 2x_3 = 2$.

Sol.: By applying elementary row operation

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 2 & -1 & 1 & 3 \\ 4 & 2 & -2 & 2 \end{array}\right].$$

Operating $R_{21(-2)}, R_{31(-4)}$, we get $[A|B] = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & -3 & 3 & 3 \\ 0 & -2 & 2 & 2 \end{bmatrix}$.

Operating $R_{2\left(-\frac{1}{3}\right)}, R_{3\left(-\frac{1}{2}\right)}$, we get $[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 1 & -1 & -1 \end{array}\right]$.

Operating $R_{32(-1)}$, we get $[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$.

r(A) = 2 = $[A|B]$< 3 = n = number of variables.

The system is consistent but has infinite numbers of solutions in terms of $n - r = 3 - 2 = 1$ variable.

Choose $x_3 = k$ = arbitrary constant

Solving $x_2 - x_3 = 1 \Rightarrow x_3 = x_3 - 1 = k - 1$.

$x_1 + x_2 - x_3 = 0 \Rightarrow x_1 = -x_2 + x_3 = -k + 1 + k = 1$.

Thus the solutions are

$x_1 = 1,\ x_2 = k - 1,\ x_3 = k$, where k is arbitrary.

Q.No.9.: Solve, with the help of matrices, the simultaneous equations:

$x+y+z=3$, $x+2y+3z=4$, $x+4y+9z=6$.

Sol.: In this question, there is no restriction that the solution must be obtained by finding A^{-1}.

Now here augmented matrix $[A:B]=\begin{bmatrix}1 & 1 & 1 & : & 3\\ 1 & 2 & 3 & : & 4\\ 1 & 4 & 9 & : & 6\end{bmatrix}$.

Operating R_2-R_1, R_3-R_1, we get $[A:B]=\begin{bmatrix}1 & 1 & 1 & : & 3\\ 0 & 1 & 2 & : & 1\\ 0 & 3 & 8 & : & 3\end{bmatrix}$.

Operating R_3-3R_2, we get $[A:B]=\begin{bmatrix}1 & 1 & 1 & : & 3\\ 0 & 1 & 2 & : & 1\\ 0 & 0 & 2 & : & 0\end{bmatrix}$.

Operating $\frac{1}{2}R_3$, we get $[A:B]=\begin{bmatrix}1 & 1 & 1 & : & 3\\ 0 & 1 & 2 & : & 1\\ 0 & 0 & 1 & : & 0\end{bmatrix}$.

Operating R_2-R_3, R_2-2R_3, we get $[A:B]=\begin{bmatrix}1 & 1 & 0 & : & 3\\ 0 & 1 & 0 & : & 1\\ 0 & 0 & 1 & : & 0\end{bmatrix}$.

Operating R_1-R_2, we get $[A:B]=\begin{bmatrix}1 & 0 & 0 & : & 2\\ 0 & 1 & 0 & : & 1\\ 0 & 0 & 1 & : & 0\end{bmatrix}$.

$\therefore x=2$, $y=1$, $z=0$.

This method is especially useful when the number of unknown is 4, since $|A|$ is order of 4 and the co-factor of its various elements are determinants of order 3.

Q.No.10.: Investigate the values of λ and μ so that the equations

$2x+3y+5z=9$, $7x+3y-2z=8$, $2x+3y+\lambda z=\mu$, have

(i) no solution, **(ii)** a unique solution and **(iii)** an infinite number of solutions.

Sol.: The given set of equations can be written as $\begin{bmatrix}2 & 3 & 5\\ 7 & 3 & -2\\ 2 & 3 & \lambda\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}9\\ 8\\ \mu\end{bmatrix}$.

$$\Rightarrow AX = B \Rightarrow \begin{bmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & \lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ \mu \end{bmatrix}.$$

The augmented matrix $K = [A : B] \begin{bmatrix} 2 & 3 & 5 & \vdots & 9 \\ 7 & 3 & -2 & \vdots & 8 \\ 2 & 3 & \lambda & \vdots & \mu \end{bmatrix}$.

Operating $R_3 \to R_3 - R_1$, $R_2 \to 2R_2 - 7R_1$, we get $K \sim \begin{bmatrix} 2 & 3 & 5 & \vdots & 9 \\ 0 & -15 & -39 & \vdots & -47 \\ 0 & 0 & \lambda - 5 & \vdots & \mu - 9 \end{bmatrix}$

(i). If $\lambda \neq 5$, we have rank of K = 3 = rank of A [i.e. $r = r'$]

$\Rightarrow$ The given system of equations is consistent.

Also the rank of A = the number of unknowns.

$\Rightarrow$ The given system of equations posses a unique solution.

Thus, $\lambda \neq 5$, the given equations possesses a unique solution for any value of μ.

(ii). If $\lambda = 5$ and $\mu = 9$, we have rank K = rank A.

$\Rightarrow$ The given system of equations is again consistent.

Also the rank of A < the numbers of unknowns.

$\Rightarrow$ The given system of equations possesses an infinite number of solutions.

(iii). If $\lambda = 5$ and $\mu \neq 9$, we have rank of K = 3, and rank of A = 2 $\Rightarrow$ rank K $\neq$ rank A.

$\Rightarrow$ The given system of equations is inconsistent and possesses no solution.

Q.No.11.: Investigate for what values of λ and μ the simultaneous equations

$x + y + z = 6$, $x + 2y + 3z = 10$, $x + 2y + \lambda z = \mu$, have

(i) no solution, **(ii)** a unique solution **(iii)** an infinite number of solutions.

Sol.: We have $AX = B \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$.

The augmented matrix $K = [A : B] = \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 1 & 2 & 3 & : & 10 \\ 1 & 2 & \lambda & : & \mu \end{bmatrix}$.

Operating $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, we get $K = [A : B] = \begin{bmatrix} 1 & 1 & 1 : & 6 \\ 0 & 1 & 2 : & 4 \\ 0 & 1 & \lambda - 1 : & \mu - 6 \end{bmatrix}$.

Operating $R_3 \to R_3 - R_2$, we get $\begin{bmatrix} 1 & 1 & 1 : & 6 \\ 0 & 1 & 2 \ : & 4 \\ 0 & 0 & \lambda - 3 : & \mu - 10 \end{bmatrix}$.

(i). If $\lambda = 3$ and $\mu \neq 10$, then $\rho(A) = 2 \neq \rho(K) = 3$

$\Rightarrow$ the given system of equations is inconsistent i.e., possesses no solution.

(ii). If $\lambda \neq 3$ and $\forall \mu$, then $\rho(A) = \rho(K) = 3$ = the number of unknowns.

$\Rightarrow$ The given system of equations is consistent, and possesses a unique solution.

Thus if $\lambda \neq 3$, $\forall \mu$, the given system of equations possesses a unique solution.

(iii). If $\lambda = 3$ and $\mu = 10$, then $\rho(A) = \rho(K) = 2 <$ the number of unknowns.

$\Rightarrow$ The given system of equations is again consistent and possesses an infinite number of solutions.

Q.No.12.: For what values of k the equations $x + y + z = 1$, $2x + y + 4z = k$, $4x + y + 10z = k^2$ have a solution and solve them completely in each case.

Sol.: Here the matrix form of the given system of equations is $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 4 \\ 4 & 1 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ k \\ k^2 \end{bmatrix}$.

Operating $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 4R_1$, we get $\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ 0 & -3 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ k - 2 \\ k^2 - 4 \end{bmatrix}$.

Operating $R_3 \to \frac{1}{3} R_3$, we get $\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ k - 2 \\ \frac{(k^2 - 4)}{3} \end{bmatrix}$.

Operating $R_3 \to R_3 - R_2$, we get $\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ k - 2 \\ \frac{(k^2 - 4)}{3} - k + 2 \end{bmatrix}$.

$\Rightarrow x+y+z=1,\ -y+2z=k-2$ and $0=\dfrac{k^2-4}{3}-k+2$.

This is only possible i. e. have solution if $\dfrac{\left(k^2-4\right)}{3}-k+2=0$

$\Rightarrow k^2-3k+2=0 \Rightarrow\ k=2, 1.$

Case 1: Let k = 2.

We have $x+y+z=1,\ -y+2z=k-2=0 \Rightarrow y=2z$

If z = c, then $-y+2c=0 \Rightarrow y=2c$

and $x=1-3c$.

$\therefore$ At k = 2, $x=1-3c$, $y=2c$, $z=c$,

which is the required solution when k = 1.

Case 2: Let k = 1, then $-y+2c=-1 \Rightarrow y=1+2c$

and $x=1-1-2c=-3c$.

$\therefore$ At k = 1, $x=-3c$, $y=1+2c$, $z=c$,

which is the required solution when k = 1.

Q.No.13.: Find the values of a and b for which the equations:

$x+ay+z=3,\ x+2y+2z=b,\ x+5y+3z=9$ are consistent.

Determine the solution in each case.

When will these equations have unique solution ?

Sol.: The matrix form of the given system of equations is $\begin{bmatrix}1 & a & 1\\1 & 2 & 2\\1 & 5 & 3\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}3\\b\\9\end{bmatrix}$.

Operating $R_3 \to R_3-R_2$, we get $\begin{bmatrix}1 & a & 1\\1 & 2 & 2\\0 & 3 & 1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}3\\b\\9-b\end{bmatrix}$.

Operating $R_2 \to R_2-R_1$, we get $\begin{bmatrix}1 & a & 1\\0 & 2-a & 1\\0 & 3 & 1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}3\\b-3\\9-b\end{bmatrix}$.

Operating $R_2 \to R_3 + R_2$, we get $\begin{bmatrix} 1 & a & 1 \\ 0 & 2-a & 0 \\ 0 & 1+a & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ b-3 \\ 12-2b \end{bmatrix}$.

Case (i): When $a = -1,\ b = 6$, then equations will be consistent and have infinite number of solutions.

Case (ii): When $a = -1,\ b \neq 6$, then equations will be inconsistent.

Case (iii): When $a \neq -1\ \ \forall\ b$, then equations will be consistent and have a unique solutions.

Q.No.14.: Determine the values of a and b for which the system

$x + 2y + 3z = 6,\ x + 3y + 5z = 9,\ 2x + 5y + az = b$

has (i) no solution (ii) unique solution (iii) infinite number of solutions.

Find the solution in case (ii) and (iii).

Sol.: $[A|B] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 1 & 3 & 5 & 9 \\ 2 & 5 & a & b \end{array}\right]$

Operating $R_{21(-1)},\ R_{31(-2)}$, we get $= \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & a-6 & b-12 \end{array}\right]$

Operating $R_{32(-1)}$, we get $= \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & a-8 & b-15 \end{array}\right]$

Case 1: $a = 8,\ b \neq 15$, $r(A) = 2 \neq 3 = r[A|B]$, inconsistent, no solution.

Case 2: $a \neq 8$**,** b any value. r(A) = 3 = $[A|B]$ = n = number of variables, unique solution,

$z = \dfrac{b-15}{a-8}$.

$y = \dfrac{(3a-2b+6)}{(a-8)}$**,** $x = z = \dfrac{(b-15)}{(a-8)}$**.**

Case 3: a = 8, b = 15, r(A) = 2 = $[A|B]$< 3 = n. Infinite solutions with $n - r = 3 - 2 = 1$ arbitrary variable. x = k, $y = 3 - 2k$, z = k, with k arbitrary.

Q.No.15.: Show that the equations $3x+4y+5z=a$, $4x+5y+6z=b$, $5x+6y+7z=c$ do not have a solution unless $a+c=2b$.

Sol.: Let $A=\begin{bmatrix}3&4&5\\4&5&6\\5&6&7\end{bmatrix}$, $B=\begin{bmatrix}a\\b\\c\end{bmatrix}$, $X=\begin{bmatrix}x\\y\\z\end{bmatrix}$.

Then the matrix form of the equations is $AX=B \Rightarrow \begin{bmatrix}3&4&5\\4&5&6\\5&6&7\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}a\\b\\c\end{bmatrix}$.

Operating, $R_2 \to 3R_2-4R_1$, we get $\begin{bmatrix}3&4&5\\0&-1&-2\\5&6&7\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}a\\3b-4a\\c\end{bmatrix}$.

Operating, $R_3 \to 3R_3-5R_1$, we get $\begin{bmatrix}3&4&5\\0&-1&-2\\0&-2&-4\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}a\\3b-4a\\3c-5a\end{bmatrix}$.

Operating, $R_3 \to \frac{1}{2}R_3-R_2$, we get $\begin{bmatrix}3&4&5\\0&-1&-2\\0&0&0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}a\\3b-4a\\\dfrac{3c-6b+3a}{2}\end{bmatrix}$.

If $\dfrac{3c-6b+3a}{2}\neq 0$, then equations are inconsistent.

If $\rho(A)=\rho(K)$, then equations are consistent. This is possible only when $\dfrac{3c-6b+3a}{2}=0 \Rightarrow 3c-6b+3a=0 \Rightarrow a+c=2b$.

Thus the given equations do not have a solution unless a + c =2b.

Q.No.16.: Show that if $\lambda \neq -5$, the system of equations

$3x-y+4z=3$, $x+2y-3z=-2$, $6x+5y+\lambda z=-3$ have a unique solution.

If $\lambda=-5$, show that the equations are consistent.

Determine the solution in each case.

Sol.: The matrix form of the given system of equations is $\begin{bmatrix}3&-1&4\\1&2&-3\\6&5&\lambda\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}3\\-2\\-3\end{bmatrix}$.

Operating $R_2 \to 3R_2 - R_1$, $R_3 \to R_3 - 2R_1$, we get $\begin{bmatrix} 3 & -1 & 4 \\ 0 & 7 & -13 \\ 0 & 7 & \lambda - 8 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -9 \\ -9 \end{bmatrix}$.

Operating, $R_3 \to R_3 - R_2$, we get $\begin{bmatrix} 3 & -1 & 4 \\ 0 & 7 & -13 \\ 0 & 0 & \lambda + 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -9 \\ 0 \end{bmatrix}$.

Case 1. If $\lambda \neq -5$. Then $\rho(A) = 3 = \rho(K) =$ number of unknowns.

$\Rightarrow$ The system of equations is consistent and have a unique solution.

Then the unique solution is $z = 0, \quad y = -\frac{9}{7}, \quad x = \frac{4}{7}$.

Case 2. If $\lambda = -5$, then $\rho(A) = 2 = \rho(K) <$ number of unknowns = 3.

$\Rightarrow$The system of equations is consistent and have infinite number of solutions.

Put $z = k$ for all values of k, then

$$y = \frac{1}{7}(13k - 9), \quad x = \frac{1}{3}(y - 4z + 3) \Rightarrow x = \frac{1}{7}(4 - 5k).$$

Hence when $\lambda = -5$, then $x = \frac{1}{7}(4 - 5k)$, $y = \frac{1}{7}(13k - 9)$ and $z = k$ for all values of k, be the required solution.

Q.No.17.: Find the values of λ for which the equations $(2-\lambda)x + 2y + 3 = 0$, $2x + (4-\lambda)y + 7 = 0$, $2x + 5y + (6-\lambda) = 0$ are consistent and find the values of x and y corresponding to each of these values of λ.

Sol.: Here coefficient matrix $A = \begin{bmatrix} 2-\lambda & 2 & 3 \\ 2 & 4-\lambda & 7 \\ 2 & 5 & 6-\lambda \end{bmatrix}$.

The given equations are consistent if $|A| = 0 \Rightarrow \begin{vmatrix} 2-\lambda & 2 & 3 \\ 2 & 4-\lambda & 7 \\ 2 & 5 & 6-\lambda \end{vmatrix} = 0$.

Operating $R_3 \to R_3 - R_2$, we get $\begin{vmatrix} 2-\lambda & 2 & 3 \\ 2 & 4-\lambda & 7 \\ 0 & 1+\lambda & -1-\lambda \end{vmatrix} = 0$.

Operating $R_1 \to R_1 - R_2$, we get $\begin{vmatrix} -\lambda & -2+\lambda & -4 \\ 2 & 4-\lambda & 7 \\ 0 & 1+\lambda & -1-\lambda \end{vmatrix} = 0$.

Operating $C_2 \to C_2 + C_3$, we get $\begin{vmatrix} -\lambda & 6+\lambda & -4 \\ 2 & 11-\lambda & 7 \\ 0 & 0 & -1-\lambda \end{vmatrix} = 0$.

Operating $R_1 \to R_1 + R_2$, we get $\begin{vmatrix} 2-\lambda & 5 & 3 \\ 2 & 11-\lambda & 7 \\ 0 & 0 & 1+\lambda \end{vmatrix} = 0$.

Now expanding the determinant, we get

$(1+\lambda)\{(2-\lambda)(11-\lambda)-10\} = 0 \Rightarrow (1+\lambda)(\lambda^2 - 13\lambda + 12) = 0$

$\Rightarrow$ Either $(\lambda+1) = 0$ or $\lambda^2 - 13\lambda + 12 = 0$.

$\Rightarrow \lambda^2 - 13\lambda + 12 = 0 \Rightarrow \lambda = 12,\ 1$.

Therefore the values of $\lambda = -1,\ 1,\ 12$.

Case 1. When $\lambda = -1$, the equations become

$3x + 2y + 3 = 0$,

$2x + 5y + 7 = 0$,

$2x + 5y + 7 = 0$.

On solving these equations, we get $x = -\frac{1}{11}$, $y = -\frac{15}{11}$. Ans.

Case 2. When $\lambda = 1$, the equations become

$x + 2y + 3 = 0$,

$2x + 3y + 7 = 0$,

$2x + 5y + 5 = 0$.

On solving these equations, we get $x = -5$, $y = 1$. Ans

Case 3. When $\lambda = 12$, the equations become

$-10x + 2y + 3 = 0$,

$2x + (-8y) + 7 = 0$,

$2x + 5y - 6 = 0$.

On solving these equations, we get $x = \frac{1}{2}$, $y = 1$. Ans.

Q.No.18.: Show that there are three real values of λ for which the equations

$(a-\lambda)x + by + cz = 0$, $bx + (c-\lambda)y + az = 0$, $cx + ay + (b-\lambda)z = 0$

are simultaneously true and that the product of these values of λ is

$$D = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}.$$

Sol.: Here the coefficient matrix $A = \begin{bmatrix} a-\lambda & b & c \\ b & c-\lambda & a \\ c & a & b-\lambda \end{bmatrix}$.

These equations will be consistent if $|A| = 0 \Rightarrow \begin{vmatrix} a-\lambda & b & c \\ b & c-\lambda & a \\ c & a & b-\lambda \end{vmatrix} = 0$.

Operating $C_1 \to C_1 + C_2 + C_3$, we get $\begin{vmatrix} a+b+c-\lambda & b & c \\ a+b+c-\lambda & c-\lambda & a \\ a+b+c-\lambda & a & b-\lambda \end{vmatrix} = 0$.

Taking $(a+b+c-\lambda)$ out side, we get $(a+b+c-\lambda)\begin{vmatrix} 1 & b & c \\ 1 & c-\lambda & a \\ 1 & a & b-\lambda \end{vmatrix} = 0$.

Operating $R_1 \to R_1 - R_3$, $R_2 \to R_2 - R_3$, we get

$$(a+b+c-\lambda)\begin{vmatrix} 0 & b-a & c-b+\lambda \\ 0 & c-\lambda-a & a-b+\lambda \\ 1 & a & b-\lambda \end{vmatrix} = 0.$$

On expanding, we get

$$(a+b+c-\lambda)\{(b-a)(a-b+\lambda) - (c-b+\lambda)(c-\lambda-a)\} = 0$$

$$\Rightarrow (a+b+c-\lambda)\{(ab - b^2 + b\lambda - a^2 + ab - a\lambda) - c^2 + c\lambda + ac + cb - b\lambda - ab - c\lambda + \lambda^2 + a\lambda\} = 0$$

$$\Rightarrow (a+b+c-\lambda)\{\lambda^2 - 2(a^2 + b^2 + c^2 - ab - ca - bc)\} = 0$$

Either $\lambda = a+b+c$ or $\lambda^2 - 2\{a^2 + b^2 + c^2 - (ab + ca + bc)\} = 0$.

$$\Rightarrow \lambda = \frac{\pm\sqrt{4\left(a^2+b^2+c^2-ab-bc-ca\right)}}{2} = \pm\sqrt{\left(a^2+b^2+c^2-ab-bc-ca\right)}.$$

Thus three roots are $\lambda_1 = a+b+c$, $\lambda_2 = \sqrt{a^2+b^2+c^2-ab-bc-ca}$ and

$\lambda_3 = -\sqrt{a^2+b^2+c^2-ab-bc-ca}$

Product of three roots of equation

$$\lambda_1\lambda_2\lambda_3 = -(a+b+c)\left[a^2+b^2+c^2-ab-bc-ca\right]. \qquad \text{(i)}$$

Now we have given $D = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$.

Operating $R_1 \to R_1+R_2+R_3$, we get $D = \begin{vmatrix} a+b+c & b & c \\ a+b+c & c & a \\ a+b+c & a & b \end{vmatrix} = (a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix}$.

Operating $R_1 \to R_1-R_3$, $R_2 \to R_2-R_3$, we get $(a+b+c)\begin{vmatrix} 0 & b-a & c-b \\ 0 & c-a & a-b \\ 1 & a & b \end{vmatrix}$.

On expanding D, we get $D = (a+b+c)\begin{vmatrix} 0 & b-a & c-b \\ 0 & c-a & a-b \\ 1 & a & b \end{vmatrix}$

$= (a+b+c)\{(b-a)(a-b)-(c-a)(c-b)\}$

$= (a+b+c)\left[\left(ba-b^2-a^2+ab\right)-\left(c^2-cb-ac+ab\right)\right]$

$= (a+b+c)\left(ab+bc+ca-a^2-b^2-c^2\right)$

$= -(a+b+c)\left(a^2+b^2+c^2-ab-bc-ca\right) \qquad \text{(ii)}$

Hence we have found that (i) and (ii) are equal.

Hence, it is proved that product of 3 values of λ is equal to the $|D|$.

Q.No.19.: Show that the system of the equations $2x_1-2x_2+x_3=\lambda x_1$, $2x_1-3x_2+2x_3=\lambda x_2$, $-x_1+2x_2=\lambda x_3$ can posses a non-trivial solution only if $\lambda=1$, $\lambda=-3$. Obtain the general solution in each case.

Sol.: Given equations are $(2-\lambda)x_1-2x_2+x_3=0$, $2x_1-(3+\lambda)x_2+2x_3=0$,

$-x_1 + 2x_2 - \lambda x_3 = 0$.

The given system of equations will be consistent, if $\begin{vmatrix} 2-\lambda & -2 & 1 \\ 2 & -(3+\lambda) & 2 \\ -1 & 2 & -\lambda \end{vmatrix} = 0$.

$\Rightarrow (2-\lambda)(\lambda^2 + 3\lambda - 4) + 2(-2\lambda + 2) + (4 - \lambda - 3) = 0$

$\Rightarrow 4 + 2\lambda^2 + 6\lambda - 8 - \lambda^3 - 3\lambda^2 + 4\lambda(-1) + 4 - \lambda - 3 = 3$

$\Rightarrow -\lambda^3 - \lambda^2 + 5\lambda + (-3) = 0 \Rightarrow \lambda^3 + \lambda^2 - 5\lambda + 3 = 0$

$\Rightarrow (\lambda - 1)(\lambda^2 + 2\lambda - 3) = 0$

Thus $\lambda = 1$ and $\lambda^2 + 2\lambda - 3 = 0 \Rightarrow \lambda = -3,\ 1$.

For $\lambda = 1$ and $\lambda = -3$ the given system of the equations are consistent and posses a non-trivial solution.

If we put $\lambda = 1$ in the given equations, we get

$x_1 - 2x_2 + x_3 = 0$,

$2x_1 - 4x_2 + 2x_3 = 0$,

$-x_1 + 2x_2 - x_3 = 0$.

Let $x_1 = a$, $x_3 = b$, $\Rightarrow x_2 = \dfrac{a+b}{2}$.

If we put $\lambda = -3$ in the given equations, we get

$5x_1 - 2x_2 + x_3 = 0$,

$2x_1 + 2x_3 = 0$,

$-x_1 + 2x_2 + 3x_3 = 0$.

$\Rightarrow x_2 = -2x_3 \Rightarrow x_3 = -\dfrac{x_2}{2}$

$x_1 = \dfrac{x_2}{2} = x_3 = t$.

$x_1 = t$, $x_2 = -2t$, $x_3 = t$ is the general solution.

Q.No.20.: Prove that the equations $5x + 3y + 2z = 12$, $2x + 4y + 5z = 2$,

$39x + 43y + 45z = c$ are incompatible unless $c = 74$; and in that case the equations are satisfied by $x = 2 + t$, $y = 2 - 3t$, $z = -2 + 2t$, where t is any arbitrary quantity.

Sol.: The equations are $5x + 3y + 2z = 12$, $2x + 4y + 5z = 2$, $39x + 43y + 45z = c$.

The matrix form of these equations is $AX = B \Rightarrow \begin{bmatrix} 5 & 3 & 2 \\ 2 & 4 & 5 \\ 39 & 43 & 45 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 2 \\ c \end{bmatrix}$.

Operating, $R_2 \to 5R_2 - 2R_1$, we get $\begin{bmatrix} 5 & 3 & 2 \\ 0 & 14 & 21 \\ 39 & 43 & 45 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ -14 \\ c \end{bmatrix}$.

Operating, $R_2 \to \frac{R_2}{7}$, we get $\begin{bmatrix} 5 & 3 & 2 \\ 0 & 2 & 3 \\ 39 & 43 & 45 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ -2 \\ c \end{bmatrix}$.

Operating, $R_3 \to 5R_3 - 39R_3$, $R_2 \to \frac{R_2}{7}$ we get $\begin{bmatrix} 5 & 3 & 2 \\ 0 & 2 & 3 \\ 0 & 98 & 147 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ -2 \\ 5c - 468 \end{bmatrix}$.

Operating $R_3 \to R_3 - 49R_2$, we get $\begin{bmatrix} 5 & 3 & 2 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ -2 \\ 5c - 370 \end{bmatrix}$.

If $5c - 370 \neq 0 \Rightarrow c \neq 74$. The equations are inconsistent (or incompatible).

If $\rho(A) = \rho(K)$, then equations are consistent.

This is possible only when c = 74. Thus the equations are incompatible unless c = 74.

IInd Part. Now when c = 74, then

$5x + 3y + 2z = 12$ and $2y + 3z = -2$.

Now putting $x = 2 + t$, $y = 2 - 3t$, $z = -2 + 2t$ and c = 74 in the given equations, we obtain

$5x + 3y + 2z = 12$

$\Rightarrow 5(2 + t) + 3(2 - 3t) + 2(-2 + 2t) = 12 \Rightarrow 10 + 5t + 6 - 9t - 24 + 4t = 12 \Rightarrow 12 = 12.$

Hence equation is satisfied.

On putting the given values of x, y, z in the equation, we get

$2x + 4y + 5z = 2$

$\Rightarrow 2(2+t)+4(2-3t)+5(-2+2t)=2 \Rightarrow 2=2$.

Hence equation is satisfied.

On putting the given values of x, y, z in the equation, we get

$39x+43y+45z=c$

$\Rightarrow 39(2+t)+43(2-3t)+4(-2+2t)=74 \Rightarrow 164-90=74 \Rightarrow 74=74$.

Q.No.21.: If $b\ell = am-n$, $cm=bn-\ell$, $an=c\ell-m$, prove that $1+a^2+b^2+c^2=0$.

Sol.: Given $am-n-b\ell=0$, $cm-bn+\ell=0$, $-m-an+c\ell=0$.

Putting them into determinant form, we get $\begin{vmatrix} a & -1 & -b \\ c & -b & 1 \\ -1 & -a & c \end{vmatrix}=0$.

$\Rightarrow a(-bc+a)+1(c^2+1)-b(-ac-b)$

$\Rightarrow -abc+a^2+c^2+1+bac+b^2=0$

$\Rightarrow 1+a^2+b^2+c^2=0$.

Hence, this completes the proof.

Q.No22.: Solve by calculating the inverse by elementary row operations

$x_1+x_2+x_3+x_4=0$, $x_1+x_2+x_3-x_4=4$,

$x_1+x_2-x_3+x_4=-4$, $x_1-x_2+x_3+x_4=2$.

Sol.: The system is written as AX = B, where

$$A=\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \end{bmatrix}, \quad X=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad B=\begin{bmatrix} 0 \\ 4 \\ -4 \\ 2 \end{bmatrix}$$

Inverse by elementary row operations

$$[A|I]=\left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 1 & -1 & 1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

Operating $R_{21(-1)}$, $R_{31(-1)}$, $R_{41(-1)}$ and $R_{2(-1)}$, $R_{3(-1)}$, $R_{4(-1)}$, we get

$$[A|I]=\left[\begin{array}{cccc|cccc}1&1&1&1&1&0&0&0\\0&0&0&2&1&-1&0&0\\0&0&2&0&1&0&-1&0\\0&2&0&0&1&0&0&-1\end{array}\right]$$

Operating R_{24}, $R_{2\left(\frac{1}{2}\right)}$, $R_{3\left(\frac{1}{2}\right)}$, $R_{4\left(\frac{1}{2}\right)}$, we get $[A|I]=\left[\begin{array}{cccc|cccc}1&1&1&1&1&0&0&0\\0&1&0&0&\frac{1}{2}&0&0&-\frac{1}{2}\\0&0&1&0&\frac{1}{2}&0&-\frac{1}{2}&0\\0&0&0&1&\frac{1}{2}&\frac{1}{2}&0&-1\end{array}\right]$

Operating $R_{14(-1)}$, $R_{13(-1)}$, $R_{12(-1)}$, we get $[A|I]=\left[\begin{array}{cccc|cccc}1&1&1&1&-\frac{1}{2}&\frac{1}{2}&\frac{1}{2}&\frac{1}{2}\\0&1&0&0&\frac{1}{2}&0&0&-\frac{1}{2}\\0&0&1&0&\frac{1}{2}&0&-\frac{1}{2}&0\\0&0&0&1&\frac{1}{2}&-\frac{1}{2}&0&0\end{array}\right]$.

Thus $A^{-1}=\frac{1}{2}\begin{bmatrix}-1&1&1&1\\1&0&0&-1\\1&0&-1&0\\1&-1&0&0\end{bmatrix}$.

The require solution is

$$X=A^{-1}B=\frac{1}{2}\begin{bmatrix}-1&1&1&1\\1&0&0&-1\\1&0&-1&0\\1&-1&0&0\end{bmatrix}\begin{bmatrix}0\\4\\-4\\2\end{bmatrix}=\begin{bmatrix}1\\-1\\2\\-2\end{bmatrix}$$

i.e., $X_1=1$, $X_2=-1$, $X_3=2$, $X_4=-2$.

Home Assignments:

System of homogenous equations

Q.No.1.: Solve the equations

$x+3y+2z=0$, $2x-y+3z=0$, $3x-5y+4z=0$, $x+17y+4z=0$.

Ans.: $x=11k$, $y=k$, $z=-7k$, where k is arbitrary.

Q.No.2.: Solve completely the system of equations

$3x + 4y - z - 6w = 0$, $2x + 3y + 2z - 3w = 0$,

$2x + y - 14z - 9w = 0$, $x + 3y + 13z + 3w = 0$.

Ans.: $x = 11k_2 + 6k_1$, $y = -8k_2 - 3k_1$, $z = k_2$, $w = k_1$,

where k_1, k_2 are arbitrary constants.

Q.No.3.: Using the loop current method on a circuit, the following equations were obtained: $7i_1 - 4i_2 = 12$, $-4i_1 + 12i_2 - 6i_3 = 0$, $-6i_2 + 14i_3 = 0$.

By matrix method, solve for i_1, i_2 and i_3.

Ans.: $i_1 = \dfrac{396}{175}$, $i_2 = \dfrac{24}{25}$, $i_3 = \dfrac{72}{175}$.

System of non-homogenous equations

Q.No.1.: Test for consistency and solve:

$5x + 3y + 7z = 4$, $3x + 26y + 2z = 9$, $7x + 2y + 10z = 5$.

Ans.: $x = \dfrac{(7 - 16k)}{11}$, $y = \dfrac{(3 + k)}{11}$, $z = k$, k arbitrary.

Q.No.2.: Test for consistency and solve:

$x_1 + x_2 - 2x_3 + x_4 + 3x_5 = 1$, $2x_1 - x_3 + 2x_3 + 2x_4 + 6x_5 = 2$,

$3x_1 + 2x_2 - 4x_3 - 3x_4 - 9x_5 = 3$.

Ans.: $x_1 = 1$, $x_2 = 2a$, $x_3 = a$, $x_4 = -3b$, $x_5 = b$, where a and b are arbitrary constants.

Q.No.3.: Test for consistency and solve:

$x_1 + x_2 + 2x_3 + x_4 = 5$, $2x_1 + 3x_2 - x_3 - 2x_4 = 2$, $4x_1 + 5x_2 + 3x_3 = 7$

Ans.: No solution, system inconsistent.

Q.No.4.: Test for consistency and solve:

$2x_1 + 3x_2 - x_3 = 1$, $3x_1 - 4x_2 + 3x_3 = -1$,

$2x_1 - x_2 + 2x_3 = -3$, $3x_1 + x_2 - 2x_3 = 4$.

Ans.: No solution, system inconsistent.

Q.No.5.: Test for consistency and solve:

$2x_1 + 2x_2 + x_3 = 3$, $2x_1 + x_2 + x_3 = 0$, $6x_1 + 2x_2 + 4x_3 = 6$.

Ans.: No solution, system inconsistent.

Q.No.6.: Test for consistency and solve:

$7x + 16y - 7z = 4$, $2x + 5y - 3z = -3$, $x + y + 2z = 4$.

Ans.: No solution, system inconsistent.

Q.No.7.: Test for consistency and solve:

$x + y + z = 4$, $2x + 5y - 2z = 3$, $x + 7y - 7z = 5$.

Ans.: No solution, system inconsistent.

Q.No.8.: Test for consistency and solve:

$-x_1 + x_2 + 2x_3 = 2$, $3x_1 - x_2 + x_3 = 6$, $-x_1 + 3x_2 + 4x_3 = 4$.

Ans.: $x_1 = 1$, $x_2 = -1$, $x_3 = 2$, Unique solution.

Q.No.9.: Test for consistency and solve:

$2x + y - z = 0$, $2x + 5y + 7z = 52$, $x + y + z = 9$.

Ans.: Unique solution, $x = 1$, $y = 3$, $z = 5$.

Q.No.10.: Test for consistency and solve:

$x + y + z = 6$, $2x - 3y + 4z = 8$, $x - y + 2z = 5$.

Ans.: $x_1 = 1$, $x_2 = 2$, $x_3 = 3$.

Q.No.11.: Show that the equations $x + 2y - z = 3$, $3x - y + 2z = 1$, $2x - 2y + 3z = 2$, $x - y + z = -1$ are consistent and solve them.

Ans.: $x = -1$, $y = 4$, $z = 4$.

Q.No.12.: Solve the following systems of equations by matrix method:

(i). $x + y + z = 8$, $x - y + 2z = 6$, $3x + 5y - 7z = 14$

(ii). $x + y + z = 6$, $x - y + 2z = 5$, $3x + y + z = 8$

(iii). $x + 2y + 3z = 1$, $2x + 3y + 2z = 2$, $3x + 3y + 4z = 1$.

Ans.: (i). $x = 5$, $y = \frac{5}{3}$, $z = \frac{4}{3}$ (ii). $x = 1$, $y = 2$, $z = 3$ (iii) $x = -\frac{3}{7}$, $y = \frac{8}{7}$, $= -\frac{2}{7}$.

Q.No.13.: For what values of a and b do the equations $x + 2y + 3z = 6$, $x + 3y + 5z = 9$, $2x + 5y + az = b$ have (i) have a no solution (ii) a unique solution (iii) more than one solution.

Ans.: (i). $a = 8$, $b \neq 15$ (ii). $a \neq 8$, b may have any value (ii) $a = 8$, $b = 15$.

Q.No.14.: Find the values of a and b for which the system has (i) no solution (ii) unique solution (iii) infinitely many solution for $2x+3y+5z=9$, $7x+3y-2z=8$, $2x+3y+az=b$.

Ans.: (i). No solution of $a=5,\ b\neq 9$.

(ii) Unique solution $a\neq 5$, b any value.

(iii) Infinitely many solutions a = 5, b = 9.

Q.No.15.: Find the values of a and b for which the system has (i) no solution (ii) unique solution (iii) infinitely many solution for $x+y+z=6$, $x+2y+3z=10$, $x+2y+az=b$.

Ans.: (i). $a=3,\ b\neq 10$ inconsistent

(ii) Unique solution $a\neq 3$, b any value.

(iii) Infinitely many solutions a = 3, b = 10.

Q.No.16.: Test for consistency $-2x+y+z=a$, $x-2y+z=b$, $x+y-2z=c$, where a, b, c are constants.

Ans.: (i) if $a+b+c\neq 0$**,** inconsistent.

(ii). $a+b+c=0$, infinite solution.

Q.No.17.: Find the value of k so that the equations $x+y+3z=0$**,** $4x+3y+kz=0$**,** $2x+y+2z=0$ have a non-trivial solution.

Ans.: $k=8$.

Q.No.18.: Show that if $\lambda\neq -5$, the system of equations $3x-y+4z=3$, $x+2y-3z=-2$, $6x+5y+\lambda z=-3$, have a unique solution. If $\lambda=-5$, show that the equations are consistent. Determine the solutions in each case.

Ans.: $\lambda\neq -5$**,** $x=\frac{4}{7}$**,** $y=-\frac{9}{7}$**,** $z=0$: $\lambda=-5$, $x=\frac{1}{7}(4-5)$, $y=\frac{1}{7}(13k-9)$, $z=k$ for all k.

Q.No.19.: Solve using A^{-1}(inverse of the coefficient matrix):

$2x_1+x_2+5x_3+x_4=5$, $x_1+x_2-3x_3-4x_4=-1$, $3x_1+6x_2-2x_3+x_4=8$, $2x_1+2x_2+2x_3-3x_4=2$.

Ans.: $x_1 = 2,\ x_2 = \frac{1}{5},\ x_3 = 0,\ x_4 = \frac{4}{5}$, unique solution.

Q.No.20.: Write the following equations in matrix form AX = B and solve for X by finding A^{-1}:

(i). $2x - 2y + z = 1,\ x + 2y + 2z = 2,\ 2x + y - 2z = 7$

(ii). $2x_1 - x_2 + x_3 = 4,\ x_1 + x_2 + x_3 = 1,\ x_1 - 3x_2 - 2x_3 = 2$.

Ans.: (i). $x = 2,\ y = 1,\ z = -1$ (ii). $x_1 = 1,\ x_2 = -1,\ x_3 = 1$.

Frequently asked questions and their replies:

Q.: What are the rank conditions for consistency of a linear algebraic system?

Ans.: Well, what is your definition of "rank"? The definition I would use is that the rank of a matrix is the number of non-zero rows left after you row-reduce the matrix. Obviously, that idea applies to non-square matrices. In fact, if you append a new column to a square matrix, to form the "augmented matrix", any non-zero row, after row-reduction, for the square matrix will still be non-zero for the augmented matrix- add values on the end can't destroy non-zero values already there. The only way the rank could be changed is if you have non-zero values in the new column on a row that is all zeroes except for that, so that the augmented rank has greater rank than the original matrix. That tells you that one of your matrices has reduced to 0x+ 0y+ 0z+ ...= a where a is non-zero and that is impossible. If there is no such case, you have at least one solution to each equation. Yes, the system is consistent if and only if the rank of the coefficient matrix is the same as the rank of the augmented matrix.

Q.: Whatdo you mean by row reduction, please elaborate?

Ans.: Row reduction refers to the algorithmic procedure of Gaussian elimination. There are three row-reduction techniques:

1. Swapping rows
2. Multiplying a row by a constant.
3. Adding a multiple of a row to another.

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6th Topic

Linear and Orthogonal Transformations

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Linear transformations:

Let (x, y) be co-ordinates of a point P referred to set of rectangular axes OX, OY. Then its co-ordinates (x', y') referred to OX', OY', obtained by rotating the former axes through an angle θ are given by

$$\left.\begin{array}{l} x' = x\cos\theta + y\sin\theta, \\ y' = -x\sin\theta + y\cos\theta \end{array}\right\}. \tag{i}$$

A more general transformation than (i) is

$$\left.\begin{array}{l} x' = a_1 x + b_1 y, \\ y' = a_2 x + b_2 y \end{array}\right\}, \tag{ii}$$

which in matrix notation is

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Such transformations as (i) and (ii), are called **linear transformations in two dimensions**.

$$\left.\begin{array}{l} x' = \ell_1 x + m_1 y + n_1 z \\ y' = \ell_2 x + m_2 y + n_2 z \\ z' = \ell_3 x + m_3 y + n_3 z \end{array}\right\} \tag{iii}$$

give a linear transformation from (x, y, z) to (x', y', z') in three dimensional problems.

In general, the relation Y=AX, where

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ .... \\ y_n \end{bmatrix}, \quad A = \begin{bmatrix} a_1 & b_1 & c_1 & .... & k_1 \\ a_2 & b_2 & c_2 & .... & k_2 \\ .... & .... & .... & .... & .... \\ a_n & b_n & c_n & .... & k_n \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ .... \\ x_n \end{bmatrix} \qquad \text{(iv)}$$

give linear transformation from n variables $x_1, x_2, ..., x_n$ to the variables $y_1, y_2, ..., y_n$, i.e., the transformation of the vector X to the vector Y

This transformation is called **linear** because the linear relations

(i) $A(X_1 + X_2) = AX_1 + AX_2$ and

(ii) $A(bX) = bAX$, hold for this transformation.

Singular and non-singular transformation:

If the transformation matrix A is singular, then the transformation is said to be singular, otherwise non-singular.

For a non-singular transformation Y = AX, we can also write the inverse transformation $X = A^{-1}Y$. A non-singular transformation is also called a regular transformation.

Remarks: If a transformation from (x_1, x_2, x_3) to (y_1, y_2, y_3) is given by Y = AX and another transformation of (y_1, y_2, y_3) to (z_1, z_2, z_3) is given by Z = BY, then the transformation from (x_1, x_2, x_3) to (z_1, z_2, z_3) is given by

Z = BY = B(AX) = (BA)X.

Orthogonal transformation:

The linear transformation Y = AX, where

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ .... \\ y_n \end{bmatrix}, \quad A = \begin{bmatrix} a_1 & b_1 & c_1 & .... & k_1 \\ a_2 & b_2 & c_2 & .... & k_2 \\ .... & .... & .... & .... & .... \\ a_n & b_n & c_n & .... & k_n \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ .... \\ x_n \end{bmatrix}$$

is said to be orthogonal if it transforms

$y_1^2 + y_2^2 + ... + y_n^2$ into $x_1^2 + x_2^2 + ... + x_n^2$.

The matrix A of this orthogonal transformation is called an **orthogonal matrix**.

Now $X'X = [x_1 x_2 .... x_n] \times \begin{bmatrix} x_1 \\ x_2 \\ .... \\ x_n \end{bmatrix} = x_1^2 + x_2^2 + ... + x_n^2$.

and similarly $Y'Y = y_1^2 + y_2^2 + ... + y_n^2$.

$\therefore$ If $Y = AX$ is an orthogonal transformation, then

$$X'X = x_1^2 + x_2^2 + ... + x_n^2 = y_1^2 + y_2^2 + ... + y_n^2 = Y'Y$$

$= (AX)'(AX) = (X'A')(AX) = X'(A'A)X$, which is possible only if $A'A = I$.

But $A^{-1}A = I$, therefore, $A' = A^{-1}$ for an orthogonal transformation.

Hence, a square matrix A is said to be orthogonal if $AA' = A'A = I$.

Result 1.: If A is orthogonal, then show that A' and A^{-1} are also orthogonal.

Proof: Since A is orthogonal $\Rightarrow AA' = I$.

Taking transpose on both sides, we get

$$(A'A)' = I' \Rightarrow A'(A')' = I$$

$\Rightarrow A'$ is orthogonal.

Again, Since A is orthogonal $\Rightarrow A'A = I$

Taking inverse on both sides, we get

$$(A'A)^{-1} = I^{-1}$$

$$\Rightarrow A^{-1}(A')^{-1} = I \Rightarrow A^{-1}(A^{-1})' = I \qquad \left[(A')^{-1} = (A^{-1})'\right]$$

$\Rightarrow A^{-1}$ is orthogonal.

Result 2.: If A and B are orthogonal matrices, then prove that AB is also orthogonal.

Proof: Let A and B are both n-rowed square matrices, therefore AB is also n-rowed square matrix.

Since $|AB| = |A||B|$ and $|A| \neq 0$, also $|B| \neq 0$.

$\therefore |AB| \neq 0$.

Hence, AB is non-singular matrix.

Now $(AB)' = B'A'$.

$\therefore (AB)'(AB) = (B'A')(AB) = B'(A'A)B$

$= B'IB \qquad [\because A'A = I]$

$= B'B = I \qquad [\because B'B = I]$

Hence, AB is also an orthogonal matrix.

Result 3.: If A is orthogonal, then show that $|A| = \pm 1$.

Proof: If A is orthogonal matrix, then $AA' = 1$

$\Rightarrow |A||A'| = |I| \Rightarrow |A|.|A'| = I \qquad [\because \det(AB) = (\det A).(\det B)]$

$\Rightarrow |A|.|A| = I, \qquad [\because |A'| = |A|]$

$\Rightarrow |A|^2 = 1 \Rightarrow |A| = \pm 1.$

Now, let us understand these transformations with the help of these problems:

Q.No.1.: Show that the transformation

$y_1 = 2x_1 + x_2 + x_3, \quad y_2 = x_1 + x_2 + 2x_3, \quad y_3 = x_1 - 2x_3$ is regular.

Also, write down the inverse transformation.

Sol.: In matrix notation, the given transformation is Y = AX, where

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \ Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}, \ A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & -2 \end{bmatrix}.$$

Now $$|A| = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & -2 \end{vmatrix} = 2(-2-0) - 1(-2-2) + 1(0-1) = -4 + 4 - 1 = -1 \neq 0.$$

Thus, the matrix A is non-singular and hence the given transformation is non-singular or regular.

$\therefore$ The inverse transformation is given by $X = A^{-1}Y$,

where $$A^{-1} = \frac{\operatorname{adj} A}{|A|} = \frac{1}{(-1)}\begin{bmatrix} -2 & 2 & 1 \\ 4 & -5 & -3 \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -2 & -1 \\ -4 & 5 & 3 \\ 1 & -1 & -1 \end{bmatrix}.$$

$$\therefore X = A^{-1}Y \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 & -2 & -1 \\ -4 & 5 & 3 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}.$$

$\Rightarrow x_1 = 2y_1 - 2y_2 - y_3; \ x_2 = -4y_1 + 5y_2 + 3y_3: \ x_3 = y_1 - y_2 - y_3,$

which is the required inverse transformation.

Q.No.2.: Represent each of the transformations

$x_1 = 3y_1 + 2y_2$, $y_1 = z_1 + 2z_2$, $x_2 = -y_1 + 4y_2$, $y_2 = 3z_1$,

by the use of matrices and find the composite transformation which expresses x_1, x_2 in terms of z_1, z_2.

Sol.: A transformation from the variable x_1, x_2 to y_1, y_2 can be represented by

$$X = A_1 Y, \text{ where } X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \; A_1 = \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix}, \; Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

Second transformation from the variable y_1, y_2 to x_1, x_2 can be represented by

$$Y = A_2 Z, \text{ where } Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \; A_2 = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}, \; Z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}.$$

Given $x_1 = 3y_1 + 2y_2 = 3(z_1 + 2z_2) + 2(3z_1) = 3z_1 + 6z_2 + 6z_1 = 9z_1 + 6z_2$.

and $x_2 = -y_1 + 4y_2 = -(z_1 + 2z_2) + 4(3z_1) = -z_1 + -2z_2 + 12z_1 = 11z_1 - 2z_2$.

The composite transformation, which expresses x_1, x_2 in terms of z_1, z_2 by the use of matrices is $X = AZ$.

$$\text{where } X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \; A = \begin{bmatrix} 9 & 6 \\ 11 & -2 \end{bmatrix}, \; z_2 = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}.$$

Q.No.3.: If $\xi = x\cos\alpha - y\sin\alpha$, $\eta = x\sin\alpha + y\cos\alpha$, write the matrix A of transformation and prove that $A^{-1} = A'$.

Hence write the inverse transformation.

Sol.: Let the transformed matrix of the equations

$\xi = x\cos\alpha - y\sin\alpha$ and $\eta = x\sin\alpha + y\cos\alpha$ is A.

$$\therefore A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}.$$

Thus, the given transformation can be written as $Y = AX$,

$$\text{where } Y = \begin{bmatrix} \xi \\ \eta \end{bmatrix}, \; A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}, \; X = \begin{bmatrix} x \\ y \end{bmatrix}.$$

Now $|A| = \cos^2\alpha + \sin^2\alpha = 1 \neq 0$.

Thus, the given transformation matrix A is non-singular and hence the transformation is non-singular or regular.

$\therefore$ The inverse transformation is given by $X = A^{-1}Y$

Now $A^{-1} = \dfrac{\text{adj } A}{|A|} = \dfrac{1}{1}\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} = A'$.

Thus, the inverse transformation is $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} \xi \\ \eta \end{bmatrix}$.

Thus $x = \xi\cos\alpha + \eta\sin\alpha$,

$y = \xi(-\sin\alpha) + \eta\cos\alpha$

is the inverse transformation of the given transformation.

Q.No.4.: A transformation from the variables x_1, x_2, x_3 to y_1, y_2, y_3 is given by $Y = AX$, and another transformation from y_1, y_2, y_3 to z_1, z_2, z_3 is given by $Z = BY$, where

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -2 \\ -1 & 2 & 1 \end{bmatrix},\quad B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{bmatrix}.$$

Obtain the transformation from x_1, x_2, x_3 to z_1, z_2, z_3.

Sol.: Given two transformation $Y = AX$ and $Z = BY$.

Now $Z = BY = B(AX) \Rightarrow Z = (BA)X$.

We have $B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{bmatrix}$ and $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -2 \\ -1 & 2 & 1 \end{bmatrix}$.

$$\text{Now } BA = \begin{bmatrix} 2+0-1 & 1+1+2 & 0-2+1 \\ 2+0-3 & 1+2+6 & 0-4+3 \\ 2+0-5 & 1+3+10 & 0-6+5 \end{bmatrix} = \begin{bmatrix} 1 & 4 & -1 \\ -1 & 9 & -1 \\ -3 & 14 & -1 \end{bmatrix}.$$

Now since $Z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

$$\therefore Z = (BA)X \quad \Rightarrow \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 & 4 & -1 \\ -1 & 9 & -1 \\ -3 & 14 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \Rightarrow \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} x_1 + 4x_2 - x_3 \\ -x_1 + 9x_2 - x_3 \\ -3x_1 + 14x_2 - x_3 \end{bmatrix}.$$

$\Rightarrow z_1 = x_1 + 4x_2 - x_3$,

$z_2 = -x_1 + 9x_2 - x_3$,

$z_3 = -3x_1 + 14x_2 - x_3$,

which is the required transformation.

Q.No.5.: Verify that the following matrix is orthogonal:

$$\textbf{(i)} \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}, \ \textbf{(ii)} \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}.$$

Sol.: Since, we know that a matrix is said to be orthogonal if $AA' = A, \ A = I$.

(i). Here $A = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$.

$$\therefore AA' = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} \times \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I.$$

Hence, the given matrix A is orthogonal matrix.

(ii). Here $A = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$.

$$\therefore AA' = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} . \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & 0 & -\cos\theta\sin\theta + \sin\theta\cos\theta \\ 0 & 1 & 0 \\ -\sin\theta\cos\theta + \cos\theta\sin\theta & 0 & \sin^2\theta + \cos^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I.$$

Hence the given matrix A is orthogonal matrix.

Q.No.6.: Prove that the following matrix is orthogonal:

$$\begin{bmatrix} -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{bmatrix}.$$

Sol.: Now, since we know that a matrix is said to be orthogonal if $AA' = A'A = I$.

Here $A = \frac{1}{3}\begin{bmatrix} -2 & 1 & 2 \\ 2 & 2 & 1 \\ 1 & -2 & 2 \end{bmatrix}$.

Now $AA' = \frac{1}{3}\begin{bmatrix} -2 & 1 & 2 \\ 2 & 2 & 1 \\ 1 & -2 & 2 \end{bmatrix} \times \frac{1}{3}\begin{bmatrix} -2 & 2 & 1 \\ 1 & 2 & -2 \\ 2 & 1 & 2 \end{bmatrix}$

$$= \frac{1}{9}\begin{bmatrix} 4+1+4 & -4+2+2 & -2-2+4 \\ -4+2+2 & 4+4+1 & 2-4+2 \\ -2-2+4 & 2-4+2 & 1+4+4 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I.$$

Hence, the given matrix is orthogonal.

Q.No.7.: Show that $A = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}$ is orthogonal.

Sol.: Here $A = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}$.

Now $AA^T = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix} \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}$.

$$= \frac{1}{4}\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} = I$$

i.e. $A^T = A^{-1}$ $\therefore$ A is orthogonal.

Q.No.8.: Is the matrix $\begin{bmatrix} 2 & -3 & 1 \\ 4 & 3 & 1 \\ -3 & 1 & 9 \end{bmatrix}$ orthogonal? If not, can it be converted into orthogonal matrix?

Sol.: Since, we know that a matrix is said to be orthogonal if $AA' = A'A = I$.

Here $A = \begin{bmatrix} 2 & -3 & 1 \\ 4 & 3 & 1 \\ -3 & 1 & 9 \end{bmatrix}$. $\therefore A' = \begin{bmatrix} 2 & 4 & -3 \\ -3 & 3 & 1 \\ 1 & 1 & 9 \end{bmatrix}$.

$$\therefore AA' = \begin{bmatrix} 4+9+1 & 8-9+1 & -6-3+9 \\ 8-9+1 & 16+9+1 & -12+3+9 \\ -6-3+9 & -12+3+9 & 9+1+81 \end{bmatrix} = \begin{bmatrix} 14 & 0 & 0 \\ 0 & 26 & 0 \\ 0 & 0 & 91 \end{bmatrix} \neq I.$$

Hence, the given matrix is not orthogonal.

It can be converted into orthogonal matrix.

It means first row is divided by $\sqrt{2^2 + (-3)^2 + (1)^2} = \sqrt{14}$,

Second row is divided by $\sqrt{4^2 + (3)^2 + (1)^2} = \sqrt{26}$,

Third row is divided by $\sqrt{(-3)^2 + (1)^2 + (9)^2} = \sqrt{91}$

Hence, the orthogonal matrix is $\begin{bmatrix} \frac{2}{\sqrt{14}} & \frac{-3}{\sqrt{14}} & \frac{1}{\sqrt{14}} \\ \frac{4}{\sqrt{26}} & \frac{3}{\sqrt{26}} & \frac{1}{\sqrt{26}} \\ \frac{-3}{\sqrt{91}} & \frac{1}{\sqrt{91}} & \frac{9}{\sqrt{91}} \end{bmatrix}$.

Q.No.9.: Prove that $\begin{bmatrix} \ell & m & n & 0 \\ 0 & 0 & 0 & -1 \\ n & \ell & -m & 0 \\ -m & n & -\ell & 0 \end{bmatrix}$ is orthogonal when $\ell = \frac{2}{7}$, $m = \frac{3}{7}$, $n = \frac{6}{7}$.

Sol.: Since, we know that a matrix is said to be orthogonal if $AA' = I$.

Now $AA' = \begin{bmatrix} \ell & m & n & 0 \\ 0 & 0 & 0 & -1 \\ n & \ell & -m & 0 \\ -m & n & -\ell & 0 \end{bmatrix} \begin{bmatrix} \ell & 0 & n & -m \\ m & 0 & \ell & n \\ n & 0 & -m & -\ell \\ 0 & -1 & 0 & 0 \end{bmatrix}$

$$= \begin{bmatrix} \ell^2 + m^2 + n^2 + 0 & 0 & \ell n + m\ell - nm & -\ell m + mn - n\ell \\ 0 & 1 & 0 & 0 \\ n\ell + \ell m - mn & 0 & n^2 + \ell^2 + m^2 & -nm + \ell n + m\ell \\ -m\ell + nm - \ell n & 0 & -mn + n\ell + \ell m & m^2 + n^2 + \ell^2 \end{bmatrix}.$$

Putting the values of ℓ, m and n, we get $AA' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I$.

Hence, the given matrix is orthogonal, if $\ell = \frac{2}{7}$, $m = \frac{3}{7}$, $n = \frac{6}{7}$.

Q.No.10.: Determine a, b, c so that A is orthogonal, where $A = \begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix}$.

Sol.: Here $A = \begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix}$.

For orthogonal matrix, we have $AA^T = I$. Therefore

$$AA^T = \begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix} \begin{bmatrix} 0 & a & a \\ 2b & b & -b \\ c & -c & c \end{bmatrix} = \begin{bmatrix} 4b^2+c^2 & 2b^2-c^2 & -2b^2+c^2 \\ 2b^2-c^2 & a^2+b^2+c^2 & a^2-b^2-c^2 \\ -2b^2+c^2 & a^2-b^2-c^2 & a^2+b^2+c^2 \end{bmatrix} = I$$

Solving $2b^2 - c^2 = 0$, $a^2 - b^2 - c^2 = 0$ (non-diagonal elements of I)

$c = \pm\sqrt{2}b$, $a^2 = b^2 + c^2 = b^2 + 2b^2 = 3b^2$, $a = \pm\sqrt{3}b$

From diagonal elements of I, we have

$4b^2 + c^2 = 1$, $4b^2 + 2b^2 = 1$.

$\therefore b = \pm\dfrac{1}{\sqrt{6}}$, $c = \pm\dfrac{1}{\sqrt{3}}$, $a = \pm\dfrac{1}{\sqrt{2}}$.

Q.No.11: Find the inverse transformation of $y_1 = x_1 + 2x_2 + 5x_3$, $y_2 = -x_2 + 2x_3$,

$y_3 = 2x_1 + 4x_2 + 11x_3$.

Sol.: Let $Y = [y_1 \quad y_2 \quad y_3]^T$ and $X = [x_1 \quad x_2 \quad x_3]^T$.

The coefficient matrix $A = \begin{bmatrix} 1 & 2 & 5 \\ 0 & -1 & 2 \\ 2 & 4 & 11 \end{bmatrix}$. Here $|A| = -1$.

$$\text{Adj } A = \begin{bmatrix} -19 & -2 & 9 \\ 4 & 1 & -2 \\ 2 & 0 & -1 \end{bmatrix}.$$

Thus, the inverse transformation is

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = A^{-1}Y = \frac{\text{adj}A}{|A|}Y = \begin{bmatrix} 19 & 2 & -9 \\ -4 & -1 & 2 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 19y_1 + 2y_2 - 9y_3 \\ -4y_1 - y_2 + 2y_3 \\ -2y_1 + y_3 \end{bmatrix}.$$

Home Assignments

Q.No.1.: Show that the transformation $y_1 = x_1 - x_2 + x_3$, $y_2 = 3x_1 - x_2 + 2x_3$,

$y_3 = 2x_1 - 2x_2 + 3x_3$ is non-singular.

Also find the inverse transformation.

Ans.: $x_1 = \frac{1}{2}(y_1 + y_2 - y_3), x_2 = \frac{1}{2}(-5y_1 + y_2 + y_3),\ x_3 = -2y_1 + y_3$.

Q.No.2.: Which of the following matrices is orthogonal?

$$\textbf{(i).}\ \frac{1}{9}\begin{bmatrix} -8 & 4 & 1 \\ 1 & 4 & -8 \\ 4 & 7 & 4 \end{bmatrix},\ \textbf{(ii).}\ \begin{bmatrix} 2 & -3 & 1 \\ 4 & 3 & 1 \\ -3 & 1 & 9 \end{bmatrix}.$$

Ans.: **(i).** Orthogonal, **(ii).** Not orthogonal.

Q.No.3.: Verify that the following matrix is orthogonal: $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Q.No.4.: Verify that the following matrix is orthogonal: $\frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$.

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7th Topic

Matrices

Characteristic Equations, Eigen Values and Eigen Vectors, Orthogonal vectors

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Characteristic matrix:

Let $A = [a_{ij}]_{n\times n}$ be any square matrix of order n and λ be a scalar. Then the matrix

$$A - \lambda I = \begin{bmatrix} a_{11}-\lambda & a_{12} & .... & a_{1n} \\ a_{21} & a_{22}-\lambda & .... & a_{2n} \\ .... & .... & .... & .... \\ a_{n1} & a_{n2} & .... & a_{nn}-\lambda \end{bmatrix}$$

is called the characteristic matrix of A, where I is the unit matrix of the order n.

Characteristic polynomial:

The determinant ofcharacteristic matrixis called the characteristic polynomial.

or

The determinant

$$|A - \lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & .... & a_{1n} \\ a_{21} & a_{22}-\lambda & .... & a_{2n} \\ .... & .... & .... & .... \\ a_{n1} & a_{n2} & .... & a_{nn}-\lambda \end{vmatrix},$$

which is an ordinary polynomial in λ of degree n, is called the characteristic polynomial of A.

Characteristic equation:

The equation $|A-\lambda I|=0$, is called the characteristic equation of A.

Characteristic roots:

The roots of characteristic equation, i.e. the roots of $|A-\lambda I|=0$, are called the characteristic roots or latent roots or characteristic values or eigen values or proper values of the matrix A.

Spectrum:

The set of all eigen values of A is called the spectrum of A.

Remarks:

If λ is a characteristic root of the matrix A, then $|A-\lambda I|=0$

$\Rightarrow$ The matrix $A-\lambda I$ is singular.

Therefore, $\exists$ a non-zero vector X (i.e. X$\neq$O),s.t.

$$(A-\lambda I)X=O \Rightarrow AX=\lambda X.$$

Characteristic vectors:

If λ is a characteristic root of an $n\times n$ matrix A, then a non-zero vector X (i.e. X$\neq$O),s.t. $AX=\lambda X$, is called a characteristic vector or eigen vector or latent vector of A corresponding to the characteristic root λ.

Relation between Characteristic roots and Characteristic vectors:

Theorem 1: Prove that, if λ is aneigenvalue of a matrix A if and only if there exists a non-zero vector X such that $AX=\lambda X$.

Proof: Suppose λ is aneigen value of the matrix A.

Then $|A-\lambda I|=0 \Rightarrow$ The matrix $A-\lambda I$ is singular.

Therefore, the matrix equation $(A-\lambda I)X=O$ possesses a non-zero solution,

i.e., $\exists$ a non-zero vector X s.t. $(A-\lambda I)X=O \Rightarrow AX=\lambda X$.

Converse Part:

Conversely, suppose there exists anon-zero vector X such that $AX=\lambda X$,

i.e., $(A-\lambda I)X=O$.

Since, the matrix equation $(A-\lambda I)X = O$ possesses a non-zero solution,

$\Rightarrow$ The coefficient matrix $A-\lambda I$ must be singular, i.e., $|A-\lambda I| = 0$.

Hence, λ is the eigenvalue of the matrix A.

This completes the proof.

Theorem 2.: Prove that, if X is aneigen vector of a matrix A, then X cannot correspond to more than one eigen values of A.

Proof: Let X be aneigen vector of a matrix A corresponding to two eigenvalues λ_1 and λ_2.

Then

$AX = \lambda_1 X$ and $AX = \lambda_2 X$.

Therefore $\lambda_1 X = \lambda_2 X$.

$\Rightarrow (\lambda_1 - \lambda_2)X = O \Rightarrow \lambda_1 - \lambda_2 = 0 \; [\because X \neq O]$

$\Rightarrow \lambda_1 = \lambda_2$.

This completes the proof.

Properties of eigen values:

Property No.(1): Show that the sum of eigen values of a matrix is the sum of the elements of the principal diagonal and the product of the eigen values of a matrix A is equal to its determinant.

Proof: Consider the square matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ of order 3.

$$\therefore |A-\lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{vmatrix} = -\lambda^3 + \lambda^2(a_{11}+a_{22}+a_{33}) + \lambda(........) + (.........). \quad \text{(i)}$$

Also, if λ_1, λ_2 and λ_3 be the eigen values of A, then

$$|A-\lambda I| = (\lambda_1-\lambda)(\lambda_2-\lambda)(\lambda_3-\lambda) = (-1)^3(\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3)$$

$$= -\lambda^3 + \lambda^2(\lambda_1+\lambda_2+\lambda_3) + \lambda(\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1) + (\lambda_1\lambda_2\lambda_3). \quad \text{(ii)}$$

(i). Equating R. H. S. of (i) and (ii) and comparing the coefficients of λ^2, we get

$\lambda_1 + \lambda_2 + \lambda_3 = a_{11} + a_{22} + a_{33}$.

(ii). Putting $\lambda = 0$ in (ii), we get $|A| = \lambda_1\lambda_2\lambda_3$. Hence, this proves the results.

Property No. (2): If λ is an eigen value of a matrix A, then show that $\frac{1}{\lambda}$ is the eigen value of A^{-1}.

Proof: Let λ be an eigen value of A and X be corresponding eigen vector.

Then $AX = \lambda X$.

Pre-multiplying by A^{-1}, we get

$$X = A^{-1}(\lambda X) = \lambda(A^{-1}X) \Rightarrow \frac{1}{\lambda}X = A^{-1}X$$

$$\Rightarrow A^{-1}X = \frac{1}{\lambda}X \qquad [\because A^{-1} \text{exist} \Rightarrow A \text{ is non-singular } \Rightarrow \lambda \neq 0]$$

$\Rightarrow \frac{1}{\lambda}$ is an eigen value of A^{-1} and X is the corresponding eigen vector.

Property No.(3): If λ is an eigen values of an orthogonal matrix , then show that $\frac{1}{\lambda}$ is also its eigen value.

Proof: Since we know that if λ is an eigen value of a matrix A, then

$\frac{1}{\lambda}$ is an eigen value of A^{-1}.

$\Rightarrow \frac{1}{\lambda}$ is an eigen value of A' $\qquad [\because$ A is orthogonal matrix, i.e., $AA' = I \Rightarrow A^{-1} = A'$]

But the matrices A and A' have same eigen values

$[\because$ the det. $|A - \lambda I|$ and $|A' - \lambda I|$ are the same]

Hence, $\frac{1}{\lambda}$ is also an eigen value of A.

Property No. (4): Show that if $\lambda_1, \lambda_2, \ldots\ldots\ldots, \lambda_n$ are the latent roots of a matrix A, then A^2 has the latent roots $\lambda_1^2, \lambda_2^2, \ldots\ldots\ldots, \lambda_n^2$.

Sol.: Let λ be a latent root of the matrix A

Then $\exists$ a non-zero vector X s.t. $AX = \lambda X$. (i)

Pre-multiplying both sides by A, we get

$$\Rightarrow A(AX) = A(\lambda X) \Rightarrow A^2X = \lambda(AX) \Rightarrow A^2X = \lambda(\lambda X) \Rightarrow A^2X = \lambda^2 X$$

Since X is a non-zero vector, therefore λ^2 is a latent root of the matrix A^2.

$\therefore$ If $\lambda_1, \lambda_2, \ldots\ldots\ldots, \lambda_n$ are the latent roots of a matrix A, then $\lambda_1^2, \lambda_2^2, \ldots\ldots\ldots, \lambda_n^2$ are the latent roots of the A^2.

Property No. (5): Show that if $\lambda_1, \lambda_2, \ldots\ldots\ldots, \lambda_n$ are the latent roots of a matrix A, then A^3 has the latent roots $\lambda_1^3, \lambda_2^3, \ldots\ldots\ldots, \lambda_n^3$.

Proof: Let λ be a latent root of the matrix A then $\exists$ a non-zero vector X s.t.

$$AX = \lambda X. \qquad \text{(i)}$$

Pre-multiplying both sides by A, we get

$$A(AX) = A(\lambda X) \Rightarrow A^2X = \lambda(AX) \Rightarrow A^2X = \lambda(\lambda X) \Rightarrow A^2X = \lambda^2 X.$$

Again pre-multiplying both sides by A, we get

$$A\left(A^2X\right) = A\left(\lambda^2 X\right) \Rightarrow A^3X = \lambda^2(AX) = \lambda^2(\lambda X) = \lambda^3 X.$$

Since X is a non-zero vector, therefore λ^3 is a latent root of the matrix A^3.

$\therefore$ If $\lambda_1, \lambda_2, \ldots\ldots\ldots, \lambda_n$ are the latent roots of a matrix A,

then $\lambda_1^3, \lambda_2^3, \ldots\ldots\ldots, \lambda_n^3$ are the latent roots of the A^3.

This completes the proof.

Property No. (6): If $\lambda_1, \lambda_2, \ldots\ldots\ldots\ldots, \lambda_n$ are the eigen values of a matrix A, then show that A^m has the eigen values $\lambda_1^m, \lambda_2^m, \ldots\ldots\ldots, \lambda_n^m$.

[m being positive integer]

Proof: Let λ_i be aneigen value of A and X_i be the corresponding eigen vector.

Then $AX_i = \lambda_i X_i$.

Pre-multiplying both sides by A, we get

$$A^2X_i = A(\lambda_i X_i) = \lambda i(AX_i) = \lambda_i(\lambda_i X_i) = \lambda_i^2 X_i.$$

Similarly, $A^3X_i = \lambda_i^3 X_i$.

In general, $A^mX_i = \lambda_i^m X_i$.

Thus, λ_i^m is an eigen value of A^m.

Hence $\lambda_1^m,\ \lambda_2^m, \ldots\ldots\ldots\ldots, \lambda_n^m$ are eigen values of A^m.

Property No. (7): If λ be aneigen value of a non-singular matrix A.

Show that $\frac{|A|}{\lambda}$ is an eigen value of matrix adj. A.

Proof: Since λ be an eigen value of a non-singular matrix A $\Rightarrow \lambda \neq 0$.

Also λ is an eigen value of A then $\exists$ a non-zero vector X. s. t. $AX = \lambda X$. (i)

Pre-multiplying both sides by Adj A, we get

$$(\text{Adj}\,A)(AX) = (\text{Adj}\,A)(\lambda X) \Rightarrow [(\text{Adj}\,A)A]X = \lambda[(\text{Adj}\,A)X]$$

$$\Rightarrow (|A|I)X = \lambda(\text{Adj }A)X \left[\because A^{-1} = \frac{\text{Adj}\,A}{|A|} \Rightarrow \text{Adj}\,A.A = |A|\,I\right]$$

$$\Rightarrow |A|X = \lambda(\text{Adj}\,A)X \Rightarrow \frac{|A|}{\lambda}X = (\text{Adj}\,A)X. \qquad [\because \lambda \neq 0]$$

$$\Rightarrow (\text{Adj}\,A)X = \frac{|A|}{\lambda}X.$$

Since X is a non-zero vector, therefore $\frac{|A|}{\lambda}$ is an eigen value of the matrix adj A.

Property No. (8): Show that the eigen values of a triangular matrix A are equal to the elements of the principal diagonal of A.

Proof: Let $A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ 0 & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & a_{nn} \end{bmatrix}$ be a triangular matrix of order n.

Then $|A - \lambda I| = \begin{vmatrix} (a_{11}-\lambda) & a_{12} & \ldots & a_{1n} \\ 0 & (a_{22}-\lambda) & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & (a_{nn}-\lambda) \end{vmatrix} = (a_{11}-\lambda)(a_{22}-\lambda)\ldots\ldots(a_{nn}-\lambda).$

$\therefore$ The roots of the equation $|A - \lambda I| = 0$ are $\lambda = a_{11}, a_{22}, \ldots\ldots\ldots, a_{nn}$.

Hence, the eigen valuesof A are $a_{11}, a_{22}, \ldots\ldots\ldots, a_{nn}$.

And as we define A,these are the diagonal elements of A.

This completes the proof.

Property No. (9).: Show that the eigen values of a unitary matrix have the absolute value 1.

or

Show that the eigen values of a unitary matrix are of unit modulus.

Proof: Suppose A is a unitary matrix $\Rightarrow A^{\theta}A = I$.

Let λ be an eigen value of A and X be corresponding eigen vector then $AX = \lambda X$. (i)

Taking conjugate transpose of both sides of (i), we get

$$(AX)^{\theta} = (\lambda X)^{\theta} \Rightarrow X^{\theta}A^{\theta} = \bar{\lambda}X^{\theta}. \quad \text{(ii)}$$

From (i) and (ii), we have

$$\left(X^{\theta}A^{\theta}\right)(AX) = \left(\bar{\lambda}X^{\theta}\right)(\lambda X)$$

$$\Rightarrow \left(X^{\theta}A^{\theta}\right)(AX) = \bar{\lambda}\lambda X^{\theta}X \Rightarrow X^{\theta}\left(A^{\theta}A\right)X = \bar{\lambda}\lambda X^{\theta}X \Rightarrow X^{\theta}IX = \bar{\lambda}\lambda X^{\theta}X$$

$$\Rightarrow X^{\theta}X = \bar{\lambda}\lambda X^{\theta}X \Rightarrow X^{\theta}X\left(\lambda\bar{\lambda} - 1\right) = O. \quad \text{(iii)}$$

Since $X^{\theta}X \neq O$, (since $X \neq O$),

$\therefore$ (iii) gives $\lambda\bar{\lambda} - 1 = 0 \Rightarrow \lambda\bar{\lambda} = 1 \Rightarrow |\lambda|^2 = 1$.

Thus $|\lambda| = 1 \Rightarrow$ The eigen values of a unitary matrix have the absolute value 1.

This completes the proof.

Property No. (10): Show that the characteristic roots of Hermitian matrix are real.

Proof: Let λ be an eigen value of a Hermitian matrix A and X be the corresponding eigen vector.

Then $AX = \lambda X$. (i)

Pre-multiplying both sides of (i) by X^{θ}, we get

$$X^{\theta}(AX) = X^{\theta}(\lambda X) \Rightarrow X^{\theta}AX = \lambda X^{\theta}X. \quad \text{(ii)}$$

Taking transpose conjugate of both sides of (ii), we get

$$\left(X^{\theta}AX\right)^{\theta} = \left(\lambda X^{\theta}X\right)^{\theta} \Rightarrow X^{\theta}A^{\theta}\left(X^{\theta}\right)^{\theta} = \bar{\lambda}X^{\theta}\left(X^{\theta}\right)^{\theta}$$

$$\Rightarrow X^{\theta}AX = \bar{\lambda}X^{\theta}X. \quad \text{(iii)}$$

$$\left[\because \left(X^{\theta}\right)^{\theta} = X \text{ and } A^{\theta} = A, A \text{ being Hermitian}\right]$$

From (ii) and (iii), we have

$\lambda X^{\theta}X = \bar{\lambda}X^{\theta}X \Rightarrow (\lambda - \bar{\lambda})X^{\theta}X = O$.

But X is not a zero vector. $\therefore X^{\theta}X \neq 0$.

Hence $\lambda - \bar{\lambda} = 0 \Rightarrow \lambda = \bar{\lambda} \Rightarrow \lambda$ is real.

This completes the proof.

Property No. (11): Show that the characteristic roots of a Skew-Hermitian matrix are either pure imaginary or zero.

Proof: Suppose A is a Skew-Hermitian matrix. Then iA is Hermitian.

Let λ be a characteristic root of A and X be corresponding eigen vector. Then

$AX = \lambda X$.

Pre-multiplying both sides by i, we get $(iA)X = (i\lambda)X$

$\Rightarrow (i\lambda)$ is a characteristic root of iA, which is Hermitian.

Hence $(i\lambda)$ is real.

Therefore, either λ must be zero or pure imaginary.

Now let us solve some more important results:

Result No.1.: Show that the matrices A and A' have the same eigen values.

Sol.: We have $(A - \lambda I)' = A' - \lambda I' = A' - \lambda I$.

$\therefore |(A - \lambda I)'| = |A' - \lambda I|$

$\Rightarrow |(A - \lambda I)| = |A' - \lambda I|$ $[\because |B'| = |B|]$

$\therefore |(A - \lambda I)| = 0$ if and only if $|A' - \lambda I| = 0$

i.e., λ is an eigen value of A if and only if λ is an eigen value of A'.

This completes the proof.

Result No.2.: Show that the characteristic roots of A^{θ} are the conjugates of the characteristic roots of A.

Sol.: We have $|A^{\theta} - \bar{\lambda}I| = |(A - \lambda I)^{\theta}| = \overline{|A - \lambda I|}$ [Note that $|B^{\theta}| = |\overline{(B')}| = \overline{|B'|} = \overline{|B|}$]

$\therefore |A^{\theta} - \lambda\bar{I}| = 0$ iff $\overline{|A - \lambda I|} = 0$

$\Rightarrow |A^{\theta} - \bar{\lambda}I| = 0$ iff $|A - \lambda I| = 0$ [$\because$ if z is a complex number, then $z = 0$ iff $\bar{z} = 0$]

$\Rightarrow \overline{\lambda}$ is an eigen values of A^{θ} if and only if λ is an eigen value of A.

Result No.3.: Show that 0 is a characteristic root of a matrix if and only if the matrix is singular.

Sol.: We have given 0 is an eigen value of $A \Rightarrow \lambda = 0$ satisfies the equation $|A - \lambda I| = 0$

$\Rightarrow |A| = 0 \Rightarrow$ A is singular.

Conversely, if A is singular $\Rightarrow |A| = 0 \Rightarrow \lambda = 0$ satisfy the equation $|A - \lambda I| = 0$

$\Rightarrow 0$ is an eigen value of A.

This completes the proof.

The process of finding the eigen values and eigen vectors of a matrix:

Let $A = [a_{ij}]_{n \times n}$ be a square matrix of order n.

First we should write the characteristic equation of the matrix A, i.e., the equation $|A - \lambda I| = 0$. This equation will be of degree n in λ. So it will have n roots. These n roots will give us the eigen values of the matrix A. If λ_1 is an eigen value of A, then the corresponding eigenvectors of A will be given by the non-zero vectors

$$X = [x_1, x_2, \ldots\ldots\ldots\ldots, x_n]'$$

satisfy the equation .

$$AX = \lambda_1 X \Rightarrow (A - \lambda_1 I)X = O.$$

Orthogonal Vectors:

Let X and Y be two real-n-vectors, then X is said to be orthogonal to Y if

$$X'Y = O.$$

Let X and Y be two complex-n-vectors, then X is said to be orthogonal to Y if

$$X^{\theta}Y = O.$$

Now let us solve some problems by using the properties of eigen values and eigen vectors:

Q.No.1.: Find the **sum and product** of the eigen values of $\begin{bmatrix} 2 & 3 & -2 \\ -2 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$.

Sol.: Since, we know that the sum of the eigen values of a matrix is the sum of the elements of the principal diagonal and the product of the eigen values of a matrix is equal to its determinant.

Here $A = \begin{bmatrix} 2 & 3 & -2 \\ -2 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$.

If $\lambda_1, \lambda_2, \lambda_3$ be its eigen values of A, then $\lambda_1 + \lambda_2 + \lambda_3 = 2 + 1 + 2 = 5$. Ans.

and $\lambda_1 \lambda_2 \lambda_3 = |A| = \begin{vmatrix} 2 & 3 & -2 \\ -2 & 1 & 1 \\ 1 & 0 & 2 \end{vmatrix} = 2\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} - 3\begin{vmatrix} -2 & 1 \\ 1 & 2 \end{vmatrix} + (-1)\begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix}$

$= 2(2-0) - 3(-4-1) + (-2)(0-1) = 4 + 15 + 2 = 21$. Ans.

Q.No.2.: Find the **product** of the eigen values of $\begin{bmatrix} 7 & 2 & 2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$.

Sol.: Since, we know that the product of the eigen values of a matrix is equal to its determinant.

Here $A = \begin{bmatrix} 7 & 2 & 2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$.

If $\lambda_1, \lambda_2, \lambda_3$ be its eigen values of A, then

$$\lambda_1 \lambda_2 \lambda_3 = |A| = \begin{vmatrix} 7 & 2 & 2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{vmatrix} = 7\begin{vmatrix} -1 & 2 \\ 2 & -1 \end{vmatrix} - 2\begin{vmatrix} -6 & 2 \\ 6 & -1 \end{vmatrix} + 2\begin{vmatrix} -6 & -1 \\ 6 & 2 \end{vmatrix}$$

$= 7(1-4) - 2(6-12) + 2(-12+6) = -21 + 12 - 12 = -21$. Ans.

Now let us solve some problems of evaluation of eigen values and eigen vectors:

Q.No.1.: Find the **eigen values and eigen vectors** of the matrix $\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$.

Sol.: The characteristic equation is of A is $|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0$.

$\Rightarrow \lambda^2 - 7\lambda + 6 = 0 \Rightarrow (\lambda - 6)(\lambda - 1) = 0 \Rightarrow \lambda = 6,\ 1$.

Thus, the roots of this equation are $\lambda_1 = 6$, $\lambda_2 = 1$.

Therefore, the eigen values are 6 and 1.

The eigen vectors $X_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ of A corresponding to the eigen value 6 are given by the non-zero solution of the equation $(A - 6I)X_1 = O$

$$\Rightarrow \begin{bmatrix} 5-6 & 4 \\ 1 & 2-6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 4 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Operating $R_2 \to R_2 + R_1$, we get $\begin{bmatrix} -1 & 4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

The coefficient matrix of these equations is of rank 1. Therefore, these equations have $2-1$, i.e., 1 linearly independent solution. These equations reduced to the single equation $-x_1 + 4x_2 = 0$.

Obviously, $x_1 = 4$, and $x_2 = 1$ is a solution of this equation.

Therefore, $X_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ is an eigen vector of A corresponding to the eigen values 6. The set of all eigen vectors of A corresponding to the eigen values 6 is given by c_1X_1 where c_1 is any non-zero scalar.

The eigen vectors X_2 of A corresponding to the eigen value 1 is given by the non-zero solutions of the equation

$$(A - 1I)X_2 = O \Rightarrow \begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow 4x_1 + 4x_2 = 0,\quad x_1 + x_2 = 0.$$

From these $x_1 = -x_2$. Let us take $x_1 = 1$, $x_2 = -1$.

Then $X_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigen vector of A corresponding to the eigen value 1.

Every non-zero multiple of the vector X_2 is an eigen vector of A corresponding to the eigen value 1.

Q.No.2.: Find the **eigen values and eigen vectors** of the matrices:

(a) $\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$, **(b)** $\begin{bmatrix} 1 & -2 \\ -5 & 4 \end{bmatrix}$.

Sol.: (a). Let $A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$. The characteristic equation of A is $|A - \lambda I| = \begin{vmatrix} 1-\lambda & 4 \\ 3 & 2-\lambda \end{vmatrix} = 0$

$\Rightarrow (1-\lambda)(2-\lambda) - 12 = 0 \Rightarrow \lambda^2 - 3\lambda - 10 = 0 \Rightarrow \lambda^2 - 5\lambda + 2\lambda - 10 = 0$

$\Rightarrow \lambda(\lambda - 5) + 2(\lambda - 5) = 0 \Rightarrow (\lambda - 5)(\lambda + 2) = 0 \Rightarrow \lambda = 5, \ -2$.

If x, y, z be the components of eigen vector corresponding to eigenvalue λ.

Then $[A - \lambda I][X] = 0 \Rightarrow \begin{bmatrix} 1-\lambda & 4 \\ 3 & 2-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$.

Put $\lambda = 5$, we get $\begin{bmatrix} -4 & 4 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$.

Operating $R_2 \to 4R_2 - 3R_1$, we get

$\begin{bmatrix} -4 & 4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow -4x + 4y = 0 \Rightarrow x - y = 0 \Rightarrow x = y = k$.

When $k = 1$, then $x = y = 1$.

Now putting $\lambda = -2$, we get

$\begin{bmatrix} 3 & 4 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0 \Rightarrow 3x + 4y = 0 \Rightarrow 3x + 4y = 0$.

Solving $x = 4, \quad y = -3$.

So eigen vectors are $(1, 1), \ (4, -3)$. Ans.

(b). Let $A = \begin{bmatrix} 1 & -2 \\ -5 & 4 \end{bmatrix}$. The characteristic equation A is $|A - \lambda I| = 0$

$\Rightarrow |A - \lambda I| = \begin{vmatrix} 1-\lambda & -2 \\ -5 & 4-\lambda \end{vmatrix} = (1-\lambda)(4-\lambda) - 10 = 0 \Rightarrow \lambda^2 - 5\lambda - 6 = 0$

$\Rightarrow \lambda = -1, \ 6$.

If x, y, be the components of eigen vector corresponding to eigen value λ.

Then $[A-\lambda I][X]=0 \Rightarrow \begin{bmatrix} 1-\lambda & -2 \\ -5 & 4-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=0$. (i)

Putting $\lambda=-1$ in (i), we get $\begin{bmatrix} 2 & -2 \\ -5 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=0$.

Operating $R_2 \to 2R_2-5R_1$, we get

$\begin{bmatrix} 2 & -2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow 2x-2y=0 \Rightarrow x=y=k$.

When $k=1$, then $x=y=1$.

Now putting $\lambda=6$ in (i), we get $\begin{bmatrix} -5 & -2 \\ -5 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Operating $R_2 \to R_2-R_1$, we get

$\begin{bmatrix} -5 & -2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow -5x-2y=0 \Rightarrow 5x+2y=0$.

Solving, we get $x=2$, $y=-5$

Hence, the eigen vectors of A are $(1, 1)$ and $(2, -5)$. Ans.

Q.No.3.: (i) Find the eigen values and eigen vectors of $A=\begin{pmatrix} 8 & -4 \\ 2 & 2 \end{pmatrix}$.

(ii) Also find the eigen values and eigen vectors of $A^T=\begin{pmatrix} 8 & 2 \\ -4 & 2 \end{pmatrix}$.

(iii) Find the eigen values and eigen vectors of $A^{-1}=\frac{1}{24}\begin{bmatrix} 2 & 4 \\ -2 & 8 \end{bmatrix}$.

(iv) Find the eigen values and eigen vectors of $B=kA$ where $k=-\frac{1}{2}$.

(v) Find the eigen values and eigen vectors of $A^2=\begin{pmatrix} 56 & -40 \\ 20 & -4 \end{pmatrix}$.

(vi) Find the eigen values and eigen vectors of

$B=A \pm kI=\begin{pmatrix} 8 & -4 \\ 2 & 2 \end{pmatrix} \pm k\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}=\begin{pmatrix} 8 \pm k & -4 \\ 2 & 2 \pm k \end{pmatrix}$.

(vii) Find the eigen values and eigen vectors of $D=2A^2-\frac{1}{2}A+3I$.

(viii) Find the sum and product of eigen values of A.

Sol.: 1st Part: Find the eigen values and eigen vectors of $A = \begin{pmatrix} 8 & -4 \\ 2 & 2 \end{pmatrix}$.

The eigen values are the roots of the characteristic equation

$$\begin{vmatrix} 8-\lambda & -4 \\ 2 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (8-\lambda)(2-\lambda)+8=0 \Rightarrow \lambda^2 - 10\lambda + 24 = 0 \Rightarrow (\lambda-4)(\lambda-6)=0.$$

The two distinct eigen values are $\lambda = 4, 6$.

Eigen vector corresponding to eigen value $\lambda = 4$

$$(A-\lambda I)X = 0 \Rightarrow \begin{pmatrix} 18-4 & -4 \\ 2 & 2-4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4 & -4 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$4x_1 - 4x_2 = 0$

$2x_1 - 2x_2 = 0$

$\therefore x_1 = x_2 \ \overline{X}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

X_2corresponding $\lambda = 6$: $\begin{pmatrix} 8-6 & -4 \\ 2 & 2-6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 & -4 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

$2x_1 - 4x_2 = 0 \therefore x_1 = 2x_2$. $\overline{X}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

2nd Part: Find the eigen values and eigen vectors of $A^T = \begin{pmatrix} 8 & 2 \\ -4 & 2 \end{pmatrix}$.

Characteristic equation $\begin{vmatrix} 8-\lambda & 2 \\ -4 & 2-\lambda \end{vmatrix} = 0$

Characteristic equation is $\lambda^2 - 10\lambda + 24 = 0$ same as the characteristic equation of A. Thus, the eigen values of A and A^T are same. However, the eigen vectors are not the same.

For $\lambda = 4$: $(A-\lambda I)X = 0 \Rightarrow \begin{pmatrix} 8-4 & -2 \\ -4 & 2-4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

$4x_1 -+2x_2 = 0$. $\quad \therefore x_2 = -2x_1$.

$X_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

X_2corresponding $\lambda = 6: \begin{pmatrix} 8-6 & 2 \\ -4 & 2-6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

$2x_1 + 2x_2 = 0$

$X_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$

3rd Part: Find the eigen values and eigen vectors of $A^{-1} = \frac{1}{24}\begin{bmatrix} 2 & 4 \\ -2 & 8 \end{bmatrix}.$

Characteristic equation is $\left|A^{-1} - \lambda I\right| = 0$

$$\begin{vmatrix} \frac{1}{12} - \lambda & \frac{1}{6} \\ -\frac{1}{12} & \frac{1}{3} - \lambda \end{vmatrix} = \left(\frac{1}{12} - \lambda\right)\left(\frac{1}{3} - \lambda\right) + \frac{1}{12}.\frac{1}{6} = 0$$

$24\lambda^2 - 10\lambda + 1 = 0$, $\left(\lambda - \frac{1}{4}\right)\left(\lambda - \frac{1}{6}\right) = 0$.

The eigen values of A^{-1} are $\frac{1}{4}, \frac{1}{6}$ which are the reciprocal of 4, 6 of A.

Also the given vectors of A^{-1} and A are same

For $\lambda = \frac{1}{4}: \begin{pmatrix} \frac{1}{12} - \frac{1}{4} & \frac{1}{6} \\ -\frac{1}{12} & \frac{1}{3} - \frac{1}{4} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

$-2x_1 + x_2 = 0 \therefore x_1 = x_2$. $X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$

For $\lambda = \frac{1}{6}: \begin{pmatrix} \frac{1}{12} - \frac{1}{6} & \frac{1}{6} \\ -\frac{1}{12} & \frac{1}{3} - \frac{1}{6} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

$-x_1 + 2x_2 = 0 \therefore x_1 = 2x_2$. $X_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$

4th Part: Find the eigen values and eigen vectors of $B = kA$ where $k = -\frac{1}{2}$.

$B = -\frac{1}{2}A = \begin{pmatrix} -4 & +2 \\ -1 & -1 \end{pmatrix}$.

Characteristic equation of B is $|B - \lambda I| = \begin{vmatrix} -4-\lambda & 2 \\ -1 & -1-\lambda \end{vmatrix} = 0$

$(4+\lambda)(1+\lambda) + 2 = 0 \Rightarrow \lambda^2 + 5\lambda + 6 = 0$

So the eigen values of B are -2, -3, which are $-\frac{1}{2}$ times of eigen values 4, 6 of A. Also the eigen vectors of B and A are same.

For $\lambda = -2$: $\begin{bmatrix} -4+2 & 2 \\ -1 & -1+2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$. $\therefore x_1 = x_2$. $X_1 = C_1\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

For $\lambda = -3$: $\begin{bmatrix} -4+3 & 2 \\ -1 & -1+3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$. $-x_1 + 2x_2 = 0$. $X_2 = C_2\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

5th Part: Find the eigen values and eigen vectors of $A^2 = \begin{pmatrix} 56 & -40 \\ 20 & -4 \end{pmatrix}$.

Characteristic equation of A^2 is $\begin{vmatrix} 56-\lambda & -40 \\ 20 & -4-\lambda \end{vmatrix} = 0$

$\Rightarrow \lambda^2 - 52\lambda + 576 = (\lambda - 16)(\lambda - 36) = 0$

So eigen values of A^2 are 16, 36 which are square of the eigen values 4, 6 of A. Also the eigen vectors of A and A^2 are same.

For $\lambda = 16$: $\begin{bmatrix} 56-16 & -40 \\ 20 & -4-16 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$. $\therefore x_1 = x_2$. $X_1 = C_1\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

For $\lambda = 36$: $\begin{bmatrix} 56-36 & -40 \\ 20 & -4-36 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$. $x_1 - 2x_2 = 0 \therefore x_1 = 2x_2$. $X_2 = C_2\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

6th Part: Find the eigen values and eigen vectors of

$B = A \pm kI = \begin{pmatrix} 8 & -4 \\ 2 & 2 \end{pmatrix} \pm k\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 8\pm k & -4 \\ 2 & 2\pm k \end{pmatrix}$.

Characteristic equation of B is $|B - \lambda I| = 0$

$\begin{vmatrix} 8\pm k-\lambda & -4 \\ 2 & 2\pm k-\lambda \end{vmatrix} = 0 \Rightarrow (8\pm k-\lambda)(2\pm k-\lambda) + 8 = 0$

$\Rightarrow \lambda^2 - (10 \pm 2k)\lambda + (k^2 \pm 10k + 24) = 0$

Roots are $\frac{10+2}{2} \pm k$. i.e., $4 \pm k$ and $6 \pm k$ which are 4, 6 of A with $\pm k$.

Eigen vectors of B and A are same

For $\lambda = 4 \pm k$: $\begin{bmatrix} 8 \pm k - (4 \pm k) & -4 \\ 2 & 2 \pm k - (4 \pm k) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$4x_1 - 4x_2 = 0 \Rightarrow x_1 = x_2$ etc.

7th Part: Find the eigen values and eigen vectors of $D = 2A^2 - \frac{1}{2}A + 3I$.

$$D = 2\begin{pmatrix} 56 & -40 \\ 20 & -4 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 8 & -4 \\ 2 & 2 \end{pmatrix} + 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 111 & -78 \\ 39 & -6 \end{pmatrix}$$

Characteristic equation of D is $\begin{bmatrix} 111-\lambda & -78 \\ 39 & -6-\lambda \end{bmatrix} = 0$

$\Rightarrow \lambda^2 - 105\lambda + 2376 = (\lambda - 33)(\lambda - 72) = 0$.

Thus, the eigen values of D are 33, 72.

Note that $33 = 2.16 - \frac{1}{2}.4 + 3$ and $72 = 2.36 - \frac{1}{2}.6 + 3$ i.e., eigen value of D is $2\lambda^2 - \frac{1}{2}\lambda + 3$ where λ is the eigen value of A.

The eigen vectors of A and D are same.

For $\lambda = 33$: $\begin{bmatrix} 111-33 & -78 \\ 39 & -6-33 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow 78x_1 - 78x_2 = 0 \Rightarrow x_1 = x_2$ etc.

8th Part: Find the sum and product of eigen values of A.

Sum of eigen values of A = $4 + 6 = 10 =$ trace of A $= a_{11} + a_{22} = 8 + 10$.

Product of eigen values of $A = 4.6 = 24 = |A| = 16 + 8 = 24$.

Q.No.4.: Find the **characteristic roots and characteristic vectors** of the matrices:

(a) $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$, **(b)** $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$.

Sol.: (a). The characteristic equation of the matrix A is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (8-\lambda)\{(7-\lambda)(3-\lambda)-16\}+6\{-6(3-\lambda)+8\}+2\{24-2(7-\lambda)\}=0$$

$$\Rightarrow \lambda^3 - 18\lambda^2 + 45\lambda = 0 \Rightarrow \lambda(\lambda-3)(\lambda-15)=0$$

Hence, the characteristic roots of A are 0, 3 and 15.

The eigen vectors $X = [x_1, x_2, x_3]'$ of A corresponding to the eigen value 0 are given by the non-zero solutions of the equation $(A - 0I)X = O$

$$\Rightarrow \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 8 & -4 & 3 \\ -6 & -5 & 5 \\ 2 & 10 & -10 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} (\text{ by } R_1 \to R_3)$$

$$\Rightarrow \begin{bmatrix} 2 & -4 & 3 \\ 0 & -5 & 5 \\ 0 & 10 & -10 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad (\text{by } R_2 \to +3R_1,\ R_3 \to R_3 - 4R_1)$$

$$\Rightarrow \begin{bmatrix} 2 & -4 & 3 \\ 0 & -5 & 5 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad (\text{by } R_3 \to R_3 + 2R_2)$$

The coefficient matrix of these equations is of rank 2. Therefore these equations have $3-2=1$ linearly independent solution. Thus, there is only one linearly independent eigen vector corresponding to the eigen value 0. These equations can be written as

$2x_1 - 4x_2 + 3x_3 = 0,\ -5x_2 + 5x_3 = 0.$

From the last equation, we get $x_2 = x_3$.

Let us take $x_2 = 1,\ x_3 = 1$. Then, the first equation gives $x_1 = \frac{1}{2}$.

Therefore $X_1 = \begin{bmatrix} \frac{1}{2} & 1 & 1 \end{bmatrix}'$ is an eigen vector of A corresponding to the eigen vector 0.

If c_1 is any non-zero scalar, then e_1X_1 is also an eigen vector of A corresponding to the eigen value 0.

The eigen vector of A corresponding to the eigen value 3 are given by the non-zero solution of the equation

$$(A-3\ I)X = O \Rightarrow \begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -1 & -2 & -2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad (\text{by } R_1 \to R_1 + R_3)$$

$$\Rightarrow \begin{bmatrix} -1 & -2 & -2 \\ 0 & 16 & 8 \\ 0 & -8 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad (\text{by } R_2 \to R_2 - 6R_1,\ \ R_3 \to R_3 + 2R_1)$$

$$\Rightarrow \begin{bmatrix} -1 & -2 & -2 \\ 0 & 16 & 8 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad (\text{by } R_2 \to R_3 + \frac{1}{2}R_2)$$

The coefficient matrix of these equations is of rank 2.

Therefore, these equations have $3-2=1$ linearly independent solution.

These equations can be written as

$-x_1 - 2x_2 - 2x_3 = 0,\ 16x_2 + 8x_3 = 0$.

From the second equation we get $x_2 = -\frac{1}{2}x_3$.

Let us take $x_3 = 4$, $x_2 - 2$, then the first equation gives $x_1 = -4$.

Therefore, $X_2 = [-4 \ \ -2 \ \ 4]'$ is an eigen vector of A corresponding to eigen value 3. Every non-zero multiple of X_2 is an eigen vector of A corresponding to the eigen value 3.

The eigen vectors of A corresponding to the eigen value 15 are given by the non-zero solutions of the equation $A - 15\ I = O$.

$$\Rightarrow \begin{bmatrix} 8-15 & -6 & -2 \\ -6 & 7-15 & -4 \\ 2 & -4 & 3-15 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -7 & -6 & -2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -1 & 2 & 6 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad (\text{by } R_1 \to R_1 - R_2)$$

$$\Rightarrow \begin{bmatrix} -1 & 2 & 6 \\ 0 & -20 & -40 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad (\text{by } R_2 \to R_2 - 6R_1, \quad R_3 \to R3 + 2R_1)$$

The coefficient matrix of these equations is of rank 2.

Therefore, these equations have $3-2=1$ linearly independent solution.

These equations can be written as

$-x_1 + 2x_2 + 6x_3 = 0$, $20x_2 - 40x_3 = 0$.

The last equation gives $x_2 = -2x_3$.

Let us take $x_3 = 1$, $x_2 = -2$, then the first equation gives $x_1 = 2$.

Therefore $X_3 = [2 \quad -2 \quad 1]'$ is an eigen vector of A corresponding to the eigen value 15, if k is any non-zero scalar, then kX_3 is also an eigen vector of A corresponding to the eigen value 15.

(b). $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$.

Let λ be the eigen value of A, then characteristic equation is $|A - \lambda I| = 0$.

$$\Rightarrow |A - \lambda I| = \begin{vmatrix} 2-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (2-\lambda)(2-\lambda)(2-\lambda) - 1(2-\lambda) = 0$$

$$\Rightarrow -\lambda^3 + 6\lambda^2 - 11\lambda + 6 = 0 \Rightarrow \lambda = 1,\ 2,\ 3$$

When $\lambda = 1$, we get $(A - \lambda I)X = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$x + z = 0$, $y = 0$, $x + z = 0$

By solving these equations, we get $x = 1$, $y = 0$, $z = -1$.

When $\lambda = 2$, we get $(A - \lambda I)X = 0 \Rightarrow \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow x = 0,\ y = k,\ z = 0.$

By solving these equations, we get $x = 0,\ y = 1,\ z = 0$.

When $\lambda = 3$, we get $(A - \lambda I)X = 0 \Rightarrow \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow -x + z = 0,\ y = 0.$

By solving these equations, we get $x = 1,\ y = 0,\ z = 1$.

Hence, eigen vectors are $(1, 0, -1)$, $(0, 1, 0)$, $(1, 0, 1)$.

Q.No.5.: Find the **characteristic roots and characteristic vectors** of the matrices:

$$\textbf{(a)} \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}, \textbf{(b)} \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}.$$

(a). Let $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$.

The characteristic equation of A is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{vmatrix} = (2-\lambda)\left(-\lambda + \lambda^2 - 12\right) = 0 \Rightarrow \lambda^3 + \lambda^2 - 14\lambda - 24 = 0$$

$\Rightarrow \lambda = 5, -3, -3$.

If x, y, z be the components of eigen vector corresponding to the eigen value λ. Then

$$(A - \lambda I)X = \begin{bmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

When $\lambda = 5$, we get $\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -1 \\ -1 & -2 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow -7x + 2y - 32 = 0 \Rightarrow 2x - 4y - 2 = 0 \Rightarrow -x - 2y = 52 = 0$

$\therefore x = 1,\ y = 2,\ \ z = -1$

When $\lambda = -3$, we get $\begin{bmatrix} 1 & 2 & -3 \\ 9 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow x + 2y - 32 = 0$,

$9x + 4y - 62 = 0$,

$-x - 2y + 32 = 0$.

Solving these equations, we get $x = -2,\ \ y = -1,\ \ z = 0$

Hence, the vectors are $(-2, -1, 0)$ and $(1, 2, -1)$.

(b). $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$.

The characteristic equation of A is $|A - \lambda I| = 0 \Rightarrow \begin{bmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Operating $C_3 \to C_3 + C_2$, we get

$$\begin{bmatrix} 6-\lambda & -2 & 0 \\ -2 & 3-\lambda & 2-\lambda \\ 2 & -1 & 2-\lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow (2-\lambda)\begin{bmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & 1 \\ 2 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Operating $R_2 \to R_2 - R_3$, we get

$$(2-\lambda)\begin{bmatrix} 6-\lambda & -2 & 0 \\ -4 & 4-\lambda & 0 \\ 2 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow (2-\lambda)[(6-\lambda)(4-\lambda) - 8] = 0$$

$$\Rightarrow (2-\lambda)(\lambda^2 - 10\lambda + 16) = 0 \Rightarrow (2-\lambda)(\lambda - 2)(\lambda - 8) = 0$$

Therefore, the characteristic roots of A are given by $\lambda = 2,\ 2,\ 8$.

The characteristic vectors of A corresponding to the characteristic root 8 are given by the non-zero solutions of the equation $(A - 8I)X = O$

$$\Rightarrow\begin{bmatrix}6-8 & -2 & 2\\ -2 & 3-8 & -1\\ 2 & -1 & 3-8\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}\Rightarrow\begin{bmatrix}-2 & -2 & 2\\ -2 & -5 & -1\\ 2 & -1 & -5\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}.$$

Operating $R_2 \to R_2 - R_1$, $R_3 \to R_3 + R_1$, we get $\begin{bmatrix}-2 & -2 & 2\\ 0 & -3 & -3\\ 0 & -3 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$.

Operating $R_3 \to R_3 - R_2$, we get $\begin{bmatrix}-2 & -2 & 2\\ 0 & -3 & -3\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$.

The coefficient matrix of these equations is of rank 2. Therefore, these equations possess $3-2=1$ linearly independent solution.

These equations can be written as

$-2x_1-2x_2+2x_3=0$, $-3x_2-3x_3=0$.

From the last equation, we get $x_2=-x_3$. Let us take $x_3=1$, $x_2=-1$. Then the first equation gives $x_1=2$.

Therefore, $X_1=\begin{bmatrix}2\\-1\\1\end{bmatrix}$ is an eigen vector of A corresponding to the eigen value 8.

Every non-zero multiple of X_1 is also an eigen vector of A corresponding to the eigen value 8.

The eigen vectors of A corresponding to the eigen value 2 are given by the non-zero solution of the equation

$(A-32\,I)X=O \Rightarrow \begin{bmatrix}4 & -2 & 2\\ -2 & 1 & -1\\ 2 & -1 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$.

Operating $R_1 \leftrightarrow R_2$, we get $\begin{bmatrix}-2 & 1 & -1\\ 4 & -2 & 2\\ 2 & -1 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$.

Operating $R_2 \to R_2 + 2R_1$, $R_3 \to R_3 + R_1$, we get $\begin{bmatrix} -2 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

The coefficient matrix of these equations is of rank 1. Therefore, these equations possess $3-1=2$ linearly independent solution. We see that these equations reduce to the single equation

$2x_1 - x_2 - x_3 = 0$.

Obviously $X_2 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$, $X_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ are two linearly independent solutions of this equation.

Therefore, X_2 and X_3 are two linearly independent eigen vectors of A corresponding to the eigen value 2.

If c_1, c_2 are scalars not both equal to zero, then $c_1X_2 + c_2X_3$ gives all the eigen vectors of A corresponding to the eigen value 2.

Q.No.6.: Find the **eigen values and eigen vectors** of the matrix $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$.

Sol.: The characteristic equation is $|A - \lambda I| = 0 = \begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0$

$\Rightarrow \lambda^3 - 7\lambda^2 + 36 = 0$.

Since $\lambda = -2$ satisfies it, we can write this equation as

$(\lambda+2)(\lambda^2 - 9\lambda + 18) = 0 \Rightarrow (\lambda+2)(\lambda-3)(\lambda-6) = 0$.

Thus, the roots of this equation are $\lambda_1 = 2$, $\lambda_2 = 3$, $\lambda_3 = 6$.

Therefore, the eigen values of A are $\lambda = -2,\ 3,\ 6$.

If x, y, z be the components of an eigen vector corresponding to the eigenvalue λ, we have

$$[A - \lambda I]X = \begin{bmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0. \qquad \text{(i)}$$

Putting $\lambda = -2$, we have $3x + y + 3z = 0$, $x + 7y + z = 0$, $3x + y + 3z = 0$.

The first and third equations being the same, we have from first two

$$\frac{x}{-20} = \frac{y}{0} = \frac{z}{20} \Rightarrow \frac{x}{-1} = \frac{y}{0} = \frac{z}{1}.$$

Hence, the eigen vectors are $(-1, 0, 1)$. Also every non-zero multiple of this vector is an eigen vector corresponding to $\lambda = -2$.

Similarly, the eigen vectors corresponding to $\lambda = 3$ and $\lambda = 6$ are the arbitrary non-zero multiples of the vectors $(1, -1, 1)$ and $(1, 2, 1)$ which are obtained from (i).

Hence, the three eigen vectors may be taken as $(-1, 0, 1)$, $(1, -1, 1)$, $(1, 2, 1)$.

Q.No.7.: Find the eigen values and eigen vectors of $A = \begin{pmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{pmatrix}$.

Sol.: For upper triangular, lower triangular and diagonal matrices, the eigen values are given by the diagonal elements.

Characteristic equation is $|A - \lambda I| = \begin{vmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 5-\lambda \end{vmatrix} = 0$

$\Rightarrow (3-\lambda)(2-\lambda)(5-\lambda) = 0$.

So eigen values of A are 3, 2, 5 which are the diagonal elements of A.

Eigen vector $X_1$for $\lambda = 3$: $\begin{bmatrix} 0 & 1 & 4 \\ 0 & -1 & 6 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$\Rightarrow x_2 + 4x_3 = 0$, $-x_2 + 6x_3 = 0$, $2x_3 = 0$

$\Rightarrow x_2 = 0$, $x_3 = 0$, $x_1 =$ arbitrary. $X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

Eigen vector $X_2$for $\lambda = 2$: $x_1 + x_2 + 4x_3 = 0$, $6x_3 = 0$, $3x_3 = 0$

$\Rightarrow x_3 = 0$, $x_1 = -x_2$. $X_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$.

Eigen vector X_3 for $\lambda = 5: -2x_1 + x_2 + 4x_3 = 0$, $-3x_2 + 6x_3 = 0$,

$$\Rightarrow x_1 = 3x_3,\ \ x_2 = 2x_3.\ \ X_3 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}.$$

Q.No.8.: Find the eigen values and eigen vectors of $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$.

Determine whether the eigen vectors are orthogonal.

Sol.: Characteristic equation is $\begin{vmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{vmatrix} = 0$

$$\Rightarrow \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0 \Rightarrow (\lambda-1)(\lambda-2)(\lambda-3) = 0$$

So $\lambda =$ 1, 2, 3 are three distinct eigen values of A

For $\lambda = 1$: $\begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $\quad x_3 = 0,\ \ x_1 + x_2 + x_3 = 0 \Rightarrow x_1 + x_2 = 0 \Rightarrow x_2 = -x_1$.

Let $x_1 = 1 \Rightarrow x_2 = -1$. Also $x_3 = 0$. Thus $X_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$.

For $\lambda = 2$: $\begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$,

$x_1 + x_3 = 0 \Rightarrow x_3 = -x_1$. And $2x_1 + 2x_2 + x_3 = 0 \Rightarrow x_2 = \frac{1}{2}x_3$.

Let $x_1 = 2 \Rightarrow x_3 = -2$ and $x_2 = -1$. Thus $X_2 = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$.

For $\lambda = 3$: $\begin{bmatrix} -2 & 0 & -1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $\quad x_1 = -x_2,\ \ x_1 - x_2 + x_3 = 0 \Rightarrow x_1 = -\frac{1}{2}x_3$.

Let $x_1 = 1 \Rightarrow x_2 = -1$. Also $x_3 = -2$. Thus $X_3 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$.

Thus, there are three linearly independent eigen vectors X_1, X_2, X_3 corresponding to the three distinct eigen values.

Since $X_1^T X_2 = 3 \neq O$, $X_2^T X_3 = 5 \neq O$, $X_3^T X_1 = 2 \neq O$.

Therefore, no pair of eigen vectors are orthogonal.

Q.No.9.: Find the eigen values and eigen vectors of $A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$.

Determine the algebraic and geometric multiplicity.

Sol.: Characteristic equation is $\begin{vmatrix} 1-\lambda & 2 & 2 \\ 0 & 2-\lambda & 1 \\ -1 & 2 & 2-\lambda \end{vmatrix} = 0$

$\Rightarrow \lambda^3 - 5\lambda^2 + 8\lambda - 4 = (\lambda - 1)(\lambda - 2)^2 = 0$.

So $\lambda =$ 1, 2, 2 are eigen values with $\lambda = 2$ repeated twice (double root) of multiplicity 2.

The algebraic multiplicity of the eigen values $\lambda = 2$ is 2.

For $\lambda = 1$: $\begin{bmatrix} 0 & 2 & 2 \\ 0 & 1 & 1 \\ -1 & 2 & 1 \end{bmatrix}$, $x_2 = -x_3$ $x_1 = -x_3$. $X_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$.

For $\lambda = 2$: $\begin{bmatrix} -1 & 2 & 2 \\ 0 & 0 & 1 \\ -1 & 2 & 0 \end{bmatrix}$, $x_3 = 0$, $x_1 = 2x_2$. $X_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$.

Thus, only one eigen vector X_2 corresponds to the repeated eigenvalue $\lambda = 2$.

The geometric multiplicity of eigen value $\lambda = 2$ is one.

Q.No.10.: Find the eigen values and eigen vectors of $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{bmatrix}$.

Determine the algebraic and geometric multiplicity.

Sol.: Characteristic equation is $\lambda^3 - 3\lambda^2 + 3\lambda - 1 = (\lambda - 1)^3 = 0$.

$\lambda = 1, 1, 1$ is an eigen value of algebraic multiplicity 3.

For $\lambda = 1$:

$-x_1 + x_2 = 0, \quad \therefore x_1 = x_2$

$-x_2 + x_3 = 0, \quad x_2 = x_3$

$x_1 - 3x_2 + 2x_3 = 0$

$$X = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Thus, only one eigen value X Corresponds to the thrice repeated eigenvalues $\lambda = 1$, so geometric multiplicity is one.

Q.No.11.: Find the eigen values and eigen vectors of $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

Determine the algebraic and geometric multiplicity.

Sol.: Characteristic equation is $\begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)(\lambda-1)(\lambda-3) = 0$.

Thus $\lambda = 1, 1, 3$ is an eigen values of A.

So the algebraic multiplicity of eigenvalue $\lambda = 1$. Is two.

For $\lambda = 3$: $\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix} \sim x_3 = 0,\ x_1 = x_2$. $X_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

For $\lambda = 1$: $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad n = 3, \quad r = 1$

$n - r = 3 - 1 = 2$ = arbitrary

$x_1 + x_2 + x_3 = 0 \Rightarrow x_1 = -x_2 - x_3$

where x_2 and x_3 are arbitrary.

For a choice $x_2 = 0,\ x_3 =$ arbitrary.

$$X_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}.$$

For a choice of $x_2 \neq 0$, $x_3 = 0$

$$X_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

Thus, for the repeated eigenvalue $\lambda = 1$, there corresponds two linearly independent eigen vectors X_2 and X_3. So the geometric multiplicity of eigen value $\lambda = 1$ is 2.

Q.No.12.: Find the eigen values of orthogonal matrix $B = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$.

Sol.: Characteristic equation of $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$ is

$$\begin{vmatrix} 1-\lambda & 2 & 2 \\ 2 & 1-\lambda & -2 \\ 2 & -2 & 1-\lambda \end{vmatrix} = \lambda^3 - 3\lambda^2 - 9\lambda + 27 = 0 \Rightarrow (\lambda - 3)^2(\lambda + 3) = 0.$$

The eigen values of A are 3, 3, -3, so the eigen values of $B = \frac{1}{3}A$ are 1, 1, -1.

Note that $\lambda = 1$ is an eigen value of B then its reciprocal $\frac{1}{\lambda} = \frac{1}{1} = 1$ is also an eigen values of B.

Q.No13.: Show that $A = \begin{bmatrix} 2 & 3+4i \\ 3-4i & 2 \end{bmatrix}$ is Hermitian.

Find its eigen values and eigen vectors.

Sol.: Since here $A = \begin{bmatrix} 2 & 3+4i \\ 3-4i & 2 \end{bmatrix}$.

Therefore $\overline{A} = \begin{bmatrix} 2 & 3-4i \\ 3+4i & 2 \end{bmatrix}$, $\overline{A}^T = \begin{bmatrix} 2 & 3+4i \\ 3-4i & 2 \end{bmatrix} = A$.

Thus A is Hermitian. (Note that the diagonal elements of A are real).

The characteristic equation for A is $|A-\lambda I| = \begin{vmatrix} 2-\lambda & 3+4i \\ 3-4i & 2-\lambda \end{vmatrix} = 0$

$\Rightarrow (2-\lambda)^2 - (3+4i)(3-4i) = 4+\lambda^2-4\lambda-[9+16] = 0$

$\Rightarrow \lambda^4 - 4\lambda - 21 = (\lambda+3)(\lambda-7) = 0$.

Eigen values of A, Hermitian matrix are real $-3,\ 7$.

For $\lambda = -3$: $\begin{bmatrix} 5 & 3+4i \\ 3-4i & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$.

$x_1 = -\left(\frac{3+4i}{5}\right)x_2$.

The eigen vector corresponding to $\lambda = -3$ is $X_1 = \begin{bmatrix} -3-4i \\ 5 \end{bmatrix}$.

For $\lambda = 7$: $\begin{bmatrix} -5 & 3+4i \\ 3-4i & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$.

$x_1 = \left(\frac{3+4i}{5}\right)x_2$.

The eigen vector corresponding to $\lambda = 7$ is $X_1 = \begin{bmatrix} 3+4i \\ 5 \end{bmatrix}$.

Q.No.14.: Show that $A = \begin{bmatrix} i & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix}$ is Skew-Hermitian and also unitary. Find the eigen values and eigen vectors.

Sol.: $\overline{A} = \begin{bmatrix} -i & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{bmatrix}$, $\overline{A}^T = \begin{bmatrix} -i & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{bmatrix} = -A$.

Thus, A is Skew-Hermitian.

Consider $A\overline{A}^T = \begin{bmatrix} i & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix}\begin{bmatrix} -i & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & ! \end{bmatrix} = I$.

Thus $\overline{A}^T = A^{-1}$,

i.e., A is unitary matrix also.

The characteristic equation of A is $|A-\lambda I| = \begin{vmatrix} i-\lambda & 0 & 0 \\ 0 & 0-\lambda & i \\ 0 & i & 0-\lambda \end{vmatrix} = 0$

$\Rightarrow (i-\lambda)(\lambda^2+1) = \lambda^3 - i\lambda^2 + \lambda - i = 0 \Rightarrow (\lambda+i)(\lambda-i)^2 = 0$.

The eigen values of A are $\lambda = -i,\ i,\ i$ which are purely imaginary (for Skew-Hermitian) and are of absolute value unity (i.e. $|-i| = |i| = 1$)

For $\lambda = -i$: $\begin{bmatrix} 2i & 0 & 0 \\ 0 & i & i \\ 0 & i & i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$.

Solving $x_1 = 0,\ \ x_2 = -x_3$.

Thus the eigen vector corresponding to $\lambda = -i$ is $X_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$.

For $\lambda = i$: $\begin{bmatrix} 0 & 0 & 0 \\ 0 & -i & i \\ 0 & i & -i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$.

Solving $x_1 =$ arbitrary, $\ x_2 = x_3$.

Choose x_1, so that two linearly independent eigen vectors are obtained (with $x_1 = 0$, $x_2 = 1$ and $x_1 = 1$, $x_2 = 0$)

$X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ and $X_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

Q.No.15.: Find the Hermitian form H for

$A = \begin{bmatrix} 0 & i & 0 \\ -i & 1 & -2i \\ 0 & 2i & 2 \end{bmatrix}$ with $X = \begin{bmatrix} i \\ 1 \\ -i \end{bmatrix}$.

Sol.: Since $H = \overline{X}^T AX = \begin{bmatrix} -i & 1 & i \end{bmatrix}\begin{bmatrix} 0 & i & 0 \\ -i & 1 & -2i \\ 0 & 2i & 2 \end{bmatrix}\begin{bmatrix} i \\ 1 \\ -i \end{bmatrix}$

$$=\begin{bmatrix}-i & 1+1-2 & 0\end{bmatrix}\begin{bmatrix}i\\1\\-i\end{bmatrix}=1\text{, real.}$$

Q.No.16.: Determine the Skew-Hermitian form S for

$$A=\begin{pmatrix}2i & 3i\\3i & 0\end{pmatrix}\text{with }\ X=\begin{bmatrix}4i\\-5\end{bmatrix}.$$

Sol.: Since $S=\overline{X}^{T}AX=\begin{bmatrix}-4i & -5\end{bmatrix}\begin{bmatrix}2i & 3i\\3i & 0\end{bmatrix}\begin{bmatrix}4i\\-5\end{bmatrix}$

$$=\begin{pmatrix}8-15i & 12\end{pmatrix}\begin{pmatrix}4i\\-5\end{pmatrix}=32i+60-60=32i\text{, purely imaginary.}$$

Orthogonal Vectors:

Let X and Y be two real-n-vectors, then X is said to be orthogonal to Y if

$$X'Y=O.$$

Let X and Y be two complex-n-vectors, then X is said to be orthogonal to Y if

$$X^{\theta}Y=O.$$

Q.No.1.: For a symmetrical square matrix, show that the eigen vectors corresponding to two unequal eigen values are orthogonal.

Proof: Let X_1 and X_2 be two eigen vectors corresponding to two unequal eigen values λ_1 and λ_2 of a symmetrical square matrix A. Then, by definition

$$AX_1=\lambda_1X_1 \qquad \text{(i)}$$

and $$AX_2=\lambda_2X_2 \qquad \text{(ii)}$$

Since A is symmetrical square matrix therefore $A'=A$.

Also $\lambda_1\neq\lambda_2$.

To show: X_1 and X_2 are orthogonal vectors, i.e., $X_2'X_1=O$.

Now $\lambda_1X_2'X_1=X_2'(\lambda_1X_1)=X_2'(AX_1)=(X_2'A)X_1=(X_2'A')X_1$

$$=(AX_2)'X_1=(\lambda_2X_2)'X_1=\lambda_2X_2'X_1$$

$$\Rightarrow\lambda_1X_2'X_1=\lambda_2X_2'X_1\Rightarrow(\lambda_1-\lambda_2)X_2'X_1=O.$$

But $\lambda_1 \neq \lambda_2 \Rightarrow (\lambda_1 - \lambda_2) \neq 0$.

Thus $X_2' X_1 = O$.

Hence X_1 and X_2 are orthogonal vectors.

Q.No.2.: Show that any eigen vectors corresponding to two distinct eigen values of a Hermitian matrix are orthogonal.

or

Show that the eigen vectors X_i, X_j corresponding to two distinct eigen values λ_i, λ_j of a Hermitian matrix H are orthogonal, i.e. $\overline{X}_i^T X_j = 0$.

Proof: Let X_1 and X_2 be two eigen vectors corresponding to two distinct eigen values λ_1 and λ_2 of a Hermitian matrix A. Then by definition

$$AX_1 = \lambda_1 X_1 \qquad \text{(i)}$$

and $$AX_2 = \lambda_2 X_2. \qquad \text{(ii)}$$

Since A is Hermitian matrix, then both the eigen values are real $\Rightarrow \lambda_1, \lambda_2$ are real.

Also $A^\theta = A$.

To show: X_1 and X_2 are orthogonal vectors, i.e., $X_2^\theta X_1 = O$.

Now $\lambda_1 X_2^\theta X_1 = X_2^\theta(\lambda_1 X_1) = X_2^\theta(AX_1) = (X_2^\theta A)X_1 = (X_2^\theta A^\theta)X_1$

$$= (AX_2)^\theta X_1 = (\lambda_2 X_2)^\theta X_1 = \overline{\lambda}_2 X_2^\theta X_1 = \lambda_2 X_2^\theta X_1 \qquad [\because \lambda_2 \text{ is real}]$$

$\Rightarrow \lambda_1 X_2^\theta X_1 = \lambda_2 X_2^\theta X_1 \Rightarrow (\lambda_1 - \lambda_2) X_2^\theta X_1 = O$.

But $\lambda_1 \neq \lambda_2 \Rightarrow (\lambda_1 - \lambda_2) \neq 0$.

Thus $X_2^\theta X_1 = O$.

Hence, X_1 and X_2 are orthogonal vectors.

Q.No.3.: Show that any eigen vectors corresponding to two distinct eigen values of a unitary matrix are orthogonal.

Proof: Let X_1 and X_2 be two eigen vectors corresponding to two distinct eigen values λ_1 and λ_2 of a unitary matrix A. Then by definition

$$AX_1 = \lambda_1 X_1 \qquad \text{(i)}$$

and $AX_2 = \lambda_2 X_2$. (ii)

Since A is unitary matrix, then the eigen values have the absolute value 1.

i.e. $\therefore |\lambda_1| = 1 \Rightarrow |\lambda_1|^2 = 1 \Rightarrow \lambda_1\bar{\lambda}_1 = 1 \Rightarrow \bar{\lambda}_1 = \dfrac{1}{\lambda_1}$

$|\lambda_2| = 1 \Rightarrow |\lambda_2|^2 = 1 \Rightarrow \lambda_2\bar{\lambda}_2 = 1 \Rightarrow \bar{\lambda}_2 = \dfrac{1}{\lambda_2}$

Also $AA^\theta = I$.

To show: X_1 and X_2 are orthogonal vectors, i.e., $X_2^\theta X_1 = O$.

Taking conjugate transpose of (ii), we get

$$(AX_2)^\theta = (\lambda_2 X_2)^\theta \Rightarrow X_2{}^\theta A^\theta = \bar{\lambda}_2 X_2{}^\theta\,. \qquad \text{(iii)}$$

From (i) and (iii), we get

$$\left(X_2{}^\theta A^\theta\right)(AX_1) = \left(\bar{\lambda}_2 X_2{}^\theta\right)(\lambda_1 X_1)$$

$$\Rightarrow X_2{}^\theta\left(A^\theta A\right)X_1 = \bar{\lambda}_2\lambda_1 X_2{}^\theta X_1$$

$$\Rightarrow \left(1 - \bar{\lambda}_2\lambda_1\right)X_2{}^\theta X_1 = O\,. \qquad \text{(iv)}$$

Also $\bar{\lambda}_2 = \dfrac{1}{\lambda_2}$. (iv)

Thus, from (iv), we get

$$\left(1 - \frac{\lambda_1}{\lambda_2}\right)X_2{}^\theta X_1 = O \Rightarrow \left(\frac{\lambda_2 - \lambda_1}{\lambda_2}\right)X_2{}^\theta X_1 = O\,.$$

But $\lambda_2 \neq \lambda_1 \Rightarrow \lambda_2 - \lambda_1 \neq 0$.

Thus $X_2^\theta X_1 = O$.

Hence, X_1 and X_2 are orthogonal vectors.

Home Assignments:

Use of properties:

Q.No.1.: Show that, if λ is a characteristic root of the matrix A, then $\lambda + k$ is a characteristic root of the $A + kI$.

Q.No.2.: If $\lambda_1, \lambda_2, \ldots\ldots\lambda_n$ are eigen values of a matrix A, then A^m has the eigen values $\lambda_1{}^m, \lambda_2{}^m, \ldots\ldots\lambda_n{}^m$ (m being a positive integer).

Q.No.3.: Find the sum and product of the eigen value of

$$A = \begin{bmatrix} 2 & 3 & -2 \\ -2 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}.$$

Ans.: Sum = trace = 2 +1 +2 = 5, Product = $|A| = 21$.

Find the eigen values and eigen vectors of 2×2 matrices:

Q.No.1.: Find the eigen values and eigen vectors of the matrix: $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$.

Ans.: $5, -2$, $(1, 1)$, $4, -3$.

Q.No.2.: Find the eigen value and eigen vector of $\begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$.

Ans.: $\lambda^2 + 7\lambda + 6 = 0$, $\lambda = -1, -6$, $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

Q.No.3.: Find the eigen value and eigen vector of $\begin{bmatrix} 6 & 8 \\ 8 & -6 \end{bmatrix}$.

Ans.: $10, -10$, $\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

Q.No.4.: Find the eigen value and eigen vector of $\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$.

Ans.: $2, -1$, $\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Q.No.5.: Find the eigen value and eigen vector of $\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$.

Ans.: $4, \ -1, \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Find the eigen values and eigen vectors of 3×3 matrices:

Q.No.1.: Find the eigen values and eigen vectors of the matrices:

(i). $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ **(ii).** $\begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$.

Ans.: (i). $1, 1, 3; \ (1,-2,1), \ (1,-1,0), \ (1,1,0)$ **(ii).** $2, 3, 5; \ (1,-1,0), \ (1,0,0), \ (2,0,1)$.

Q.No.2.: Find the eigen value and eigen vector of $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$.

Ans.: $5, \ -3, \ -3, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$.

Q.No.3.: Find the eigen value and eigen vector of $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$.

Ans.: $\lambda^3 - 7\lambda^2 + 36 = 0, \ \ \lambda = -2, \ 3, \ 6, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

Q.No.4.: Find the eigen value and eigen vector of $\begin{bmatrix} -3 & -7 & -5 \\ 2 & 4 & 3 \\ 1 & 2 & 2 \end{bmatrix}$.

Ans.: $(\lambda - 1)^3 = 0, \ \ \lambda = 1, \ 1, \ 1, \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$.

Q.No.5.: Find the eigen value and eigen vector of $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$.

Ans.: $\lambda^3 - 18\lambda^2 + 45\lambda = 0,\ \ \lambda = 0,\ 3,\ 15$, $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$.

Q.No.6.: Find the eigen value and eigen vector of $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$.

Ans.: $\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0,\ \ \lambda = 5,\ 1,\ 1$, $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$.

Q.No.7.: Find the eigen value and eigen vector of $\begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$

Ans.: $\lambda^3 - 7\lambda^2 + 16\lambda - 12 = 0,\ \ \lambda = 2,\ 2,\ 3$, For $\lambda = 2, \begin{bmatrix} 5 \\ 2 \\ -3 \end{bmatrix}$, For $\lambda = 3, \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$.

Q.No.8.: Find the eigen value and eigen vector of $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$.

Ans.: $\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0,\ \ \lambda = 2,\ 2,\ 8$, $\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, For $\lambda = 8, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$.

Q.No.9.: Find the eigen value and eigen vector of $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$.

Ans.: $(\lambda - 2)^3 = 0,\ \ \lambda = 2,\ 2,\ 2$, $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

Q.No.10.: Find the eigen value and eigen vector of $\begin{bmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$.

Ans.: $\lambda^3 - 2\lambda^2 - 4\lambda + 8 = 0$, $\lambda = 2, 2, -2$, For $\lambda = 2$, $[0 \quad 1 \quad 1]^T$

For $\lambda = -2$, $[-4 \quad -1 \quad 7]^T$.

Q.No.11.: Find the eigen value and eigen vector of $\begin{bmatrix} 3 & -2 & -5 \\ 4 & -1 & -5 \\ -2 & -1 & -3 \end{bmatrix}$.

Ans.: $(\lambda + 5)(\lambda - 2)^2 = 0$, $\lambda = 5, 2, 2$, For $\lambda = 5$, $X_1 = [3 \quad 2 \quad 4]^T$

For $\lambda = 2$, $X_2 = [1 \quad 3 \quad -1]^T$.

Q.No.12.: Two eigen values of the matrix $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ are = 1 each.

Find the eigen values of A^{-1}.

Ans.: 1, 1, $\frac{1}{5}$.

Find the eigen values and eigen vectors of 4×4 matrices:

Q.No.1.: Find the eigen value and eigen vector of $\begin{bmatrix} 1 & -4 & -1 & -4 \\ 2 & 0 & 5 & -4 \\ -1 & 1 & -2 & 3 \\ -1 & 4 & -1 & 6 \end{bmatrix}$.

Ans.: $\lambda^4 - 5\lambda^3 + 9\lambda^2 - 7\lambda + 2 = 0$, $\lambda = 2, 1, 1, 1$, For $\lambda = 2$, $\begin{bmatrix} 2 \\ 3 \\ -2 \\ -3 \end{bmatrix}$, For $\lambda = 1$, $\begin{bmatrix} 3 \\ 6 \\ -4 \\ -5 \end{bmatrix}$.

Find the eigen values and eigen vectors of SPECIAL matrices:

Q.No.1.: Show that eigen values of the skew-symmetric matrix

$$A = \begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix}$$ are purely imaginary or zero.

Ans.: Eigen values are $0, \ -25i, \ 25i$.

Q.No.2.: Prove that $A = \begin{bmatrix} 4 & 1-3i \\ 1+3i & 7 \end{bmatrix}$ is Hermitian matrix. Find its eigen values.

Ans.: Characteristic equation: $\lambda^2 - 11\lambda + 18 = 0$, eigen values 9, 2.

Q.No.3.: Find the eigen vectors of the Hermitian matrix $A = \begin{pmatrix} a & b+ic \\ b-ic & k \end{pmatrix}$.

Ans.: $\lambda_{1,2} = \dfrac{\left[(a+k) \pm (a-k)^2 + 4\left(b^2 + c^2\right)\right]}{2}$

Eigen vectors: $\begin{bmatrix} \dfrac{-\left(b^2 + c^2\right)}{(a-\lambda)(b-ic)} & 1 \end{bmatrix}^T_{\text{at } \lambda=\lambda_1\lambda_2}$.

Q.No.4.: Find the Hermitian form of $A = \begin{bmatrix} 3 & 2-i \\ 2+i & 4 \end{bmatrix}$ with $X = \begin{bmatrix} 1+i \\ 2i \end{bmatrix}$.

Ans.: 34.

Q.No.5.: Find the Hermitian form of $A = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$, $X = \begin{pmatrix} 1 \\ i \end{pmatrix}$.

Ans.: -2.

Q.No.6.: Show that $B = \begin{bmatrix} 3i & 2+i \\ -2+i & -i \end{bmatrix}$ is Skew-Hermitian. Find its eigen values.

Ans.: Characteristic equation: $\lambda^2 - 2i\lambda + 8 = 0$, eigen values $4i, \ -2i$.

Q.No.7.: Find the eigen vectors of the Skew Hermitian matrix $A = \begin{bmatrix} 2i & 3i \\ 3i & 0 \end{bmatrix}$.

Ans.: $\lambda_{1,2} = \left(1 \pm \sqrt{10}\right)i$, eigen vectors: $\begin{pmatrix} 1 \pm \dfrac{\sqrt{10}-1}{3} \end{pmatrix}^T$.

Q.No.8.: Find the Skew-Hermitian form for

(a) $A = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ with $X = \begin{pmatrix} 1 \\ i \end{pmatrix}$,

(b). $A = \begin{pmatrix} 2i & 4 \\ -4 & 0 \end{pmatrix}$ with $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

Ans.: (a). 0, (b). $2i|x_1|^2 + 8i \ \text{Im}(\bar{x}_1 x_2)$.

Q.No.9.: Find the Skew- Hermitian form for $A = \begin{bmatrix} -i & 1 & 2+i \\ -1 & 0 & 3i \\ -2+i & 3i & i \end{bmatrix}$ with $X = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$.

Ans.: 16i.

Q.No.10.: $C = \begin{bmatrix} \frac{i}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{i}{2} \end{bmatrix}$ is unitary matrix. Find its eigen values.

Ans.: $\lambda^2 - i\lambda - 1 = 0$, $\lambda = (\sqrt{3} + i)/2$, $(-\sqrt{3} + i)/2$.

Q.No.11.: Show that the column (and also row) vectors of the unitary matrix

$$A = \frac{1}{2}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix} \text{ form an orthogonal system.}$$

Q.No.12.: Determine the eigen values and eigen vectors of the unitary matrix $\frac{1}{2}\begin{bmatrix} 1 & i \\ -i & -1 \end{bmatrix}$.

Ans.: Eigen values 1, -1, eigen vectors $\left[1i \pm i\sqrt{2}\right]^T$.

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8th Topic

Matrices

Cayley-Hamilton Theorem

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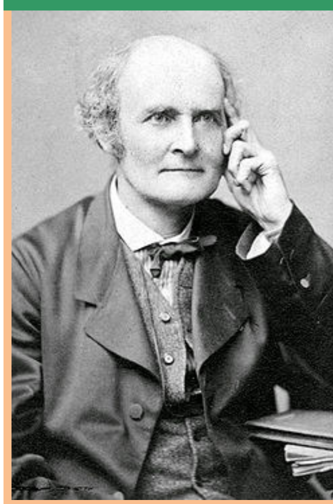
CAYLEY-HAMILTON THEOREM



Arthur Cayley

(16 August 1821 – 26 January 1895)

Sir William Rowan Hamilton

(4 August 1805 – 2 September 1865)

Cayley learned about matrices while attending one of Hamilton's lectures in Dublin, and later they both created their Cayley-Hamilton Theorem.

Statement: "Every square matrix over the real or complex field satisfies its own characteristic equation".

i.e., if the characteristic equation for the nth order square matrix A is $|A - \lambda I| = 0$

$$\Rightarrow (-1)^n \lambda^n + \left[(-1)^{n-1} b_1\right]\lambda^{n-1} + \left[(-1)^{n-2} b_2\right]\lambda^{n-2} + \ldots\ldots\ldots + \left[(-1)^{n-n} b_n\right] = 0$$

$$\Rightarrow (-1)^n \left[\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \ldots\ldots\ldots + a_n\right] = 0$$

$$\Rightarrow \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \ldots\ldots\ldots\ldots + a_n = 0.$$

Then $\boxed{\mathbf{A^n + a_1A^{n-1} + a_2A^{n-2} + \ldots\ldots\ldots\ldots + a_nI = O}}$.

Proof: As we know, matrix $A - \lambda I$ is characteristic matrix of A.

This matrix can be written as

$$A - \lambda I = \begin{bmatrix} a_{11} - \lambda & a_{12} & \ldots. & a_{1n} \\ a_{21} & a_{22} - \lambda & \ldots. & a_{2n} \\ \ldots. & \ldots. & \ldots. & \ldots. \\ a_{n1} & a_{n2} & \ldots. & a_{nn} - \lambda \end{bmatrix}$$

This matrix shows that the elements of $A - \lambda I$ are at most of the 1^{st} degree in λ.

$\therefore$ The elements of Adj $(A - \lambda I)$ are ordinary polynomials in λ of degree $(n-1)$ or less.

$$\begin{bmatrix} \lambda^3 + 2\lambda + 1 & 4\lambda^3 + 5\lambda + 3 & 2\lambda + 9 \\ \lambda^3 + 5 & \lambda^3 + 2\lambda & 2\lambda^3 + 5\lambda + 7 \\ 3\lambda^3 + 2\lambda + 6 & \lambda^3 + 4 & \lambda^3 + 8\lambda + 3 \end{bmatrix} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}\lambda^3 + \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}\lambda^2 + \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}\lambda + \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

Now Adj $(A - \lambda I)$ can be written as matrix polynomials in λ, and is given by

$$\text{Adj}(A - \lambda I) = B_0\lambda^{n-1} + B_1\lambda^{n-2} + \ldots\ldots\ldots\ldots + B_{n-2}\lambda + B_{n-1},$$

where $B_0,\ B_1, \ldots\ldots\ldots\ldots, B_{n-1}$ are matrices of the type $n \times n$, whose elements are functions of a_{ij}'s. [the elements of A].

Now, since $A\ \text{adj}A = |A|I_n$

Replacing A by $A - \lambda I$, we obtain

$$(A - \lambda I)\ \text{Adj.}\ (A - \lambda I) = |A - \lambda I|\,I_n$$

$$\Rightarrow (A - \lambda I)\left[B_0\lambda^{n-1} + B_1\lambda^{n-2} + \ldots\ldots\ldots + B_{n-2}\lambda + B_{n-1}\right] = (-1)^n\left[\lambda^n + a_1\lambda^{n-1} + \ldots\ldots\ldots + a_n\right]I_n$$

Comparing coefficients of the like powers of λ on both sides, we get

$$-IB_0 = (-1)^n I$$

$$AB_0 - IB_1 = (-1)^n a_1 I$$

$$AB_1 - IB_2 = (-1)^n a_2 I$$

....

$$AB_{n-1} = (-1)^n a_n I$$

Pre-multiplying these successively by $A^n,\ A^{n-1},\ldots\ldots\ldots, A,\ I$ and adding, we get

$$O = (-1)^n\left[A^n + a_1A^{n-1} + \ldots\ldots\ldots\ldots + a_n I\right]$$

$$\Rightarrow A^n + a_1A^{n-1} + a_2A^{n-2} + \ldots\ldots\ldots\ldots + a_n I = O. \qquad \text{(i)}$$

i.e., Every square matrix satisfies its own characteristic equation.

This completes the proof.

Another method of finding the inverse:

If A be a non-singular matrix $\Rightarrow |A| \neq 0$.

Since $|A - \lambda I| = (-1)^n\left[\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \ldots\ldots\ldots + a_n\right]$

$$\Rightarrow |A| = (-1)^n a_n \Rightarrow a_n \neq 0.$$

Pre-multiplying (i) by A^{-1}, we get

$$A^{n-1} + a_1A^{n-2} + a_2A^{n-3} + \ldots\ldots\ldots\ldots\ldots + a_n A^{-1} = O.$$

$$\Rightarrow \boxed{A^{-1} = -\frac{1}{a_n}\left[A^{n-1} + a_1A^{n-2} + \ldots\ldots\ldots\ldots\ldots\ldots + a_{n-1}I\right]}. \qquad \text{(since } a_n \neq 0\text{)}.$$

Now let us understand this important theorem by the following problems:

Q.No.1.: Find the characteristic roots of the matrix $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ and verify Cayley-Hamilton theorem for this matrix. Find the inverse of the matrix A and also express $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10\ I$ as a linear polynomial in A.

Sol.: **Find: Characteristic roots**

The characteristic equation of the matrix A is $|A - \lambda I| = 0$.

$$\Rightarrow \begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(3-\lambda) = 0 \Rightarrow \lambda^2 - 4\lambda - 5 = 0 \ \Rightarrow (\lambda-5)(\lambda+1) = 0. \qquad \text{(i)}$$

The roots of this equation are $\lambda = 5, -1$ and these are the characteristic roots of A.

By Cayley-Hamilton theorem, the matrix A must satisfy its characteristic equation (i) so we must have

$$A^2 - 4A - 5I = O. \qquad \text{(ii)}$$

Verification of Cayley-Hamilton theorem:

Since $A^2 = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix}$.

Therefore $A^2 - 4A - 5I = \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix} - \begin{bmatrix} 4 & 16 \\ 8 & 12 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$.

This verifies the theorem.

Find: Inverse of A

Now multiplying (ii) by A^{-1}, we get

$$A^2A^{-1} - 4AA^{-1} - 5IA^{-1} = OA^{-1} \Rightarrow A - 4I - 5A^{-1} = O \Rightarrow A^{-1} = \frac{1}{5}(A - 4I).$$

Now $A - 4I = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \begin{bmatrix} -3 & 4 \\ 2 & -1 \end{bmatrix}$

$\therefore A^{-1} = \frac{1}{5}(A - 4I) = \frac{1}{5}\begin{bmatrix} -3 & 4 \\ 2 & -1 \end{bmatrix}$.

Express $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10\ I$ as a linear polynomial in A:

Now (ii) $\Rightarrow A^2 = 4A + 5I$.

Multiplying by A^3, we get $A^5 = 4A^4 + 5A^3$.

Multiplying by A^2, we get $A^4 = 4A^3 + 5A^2$.

Multiplying by A, we get $A^3 = 4A^2 + 5A$.

Now $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10\ I$

$= (4A^4 + 5A^3) - 4A^4 - 7(4A^2 + 5A) + 11A^2 - A - 10\ I$

$= 5A^3 - 17A^2 - 36A - 10\ I = 5(4A^2 + 5A) - 17A^2 - 36A - 10\ I$

$= 3A^2 - 11A - 10\ I = 3A^2 - 12A + A - 15\ I + 5\,I = 3\left(A^2 - 4A - 5I\right) + A + 5I = A + 5\,I$,

which is a linear polynomial in A.

Q.No.2.: Find the characteristic equation of the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ and, hence find the matrix represented by $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$.

Sol.: Here $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$.

Let λ be the eigen value of the matrix A, then $|A - \lambda I| = \begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{vmatrix}$.

Operating $C_3 \to C_3 + C_2$, we get

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 1-\lambda & \lambda-1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 1-\lambda & -1 \\ 1 & 1 & 1 \end{vmatrix}.$$

Operating $R_2 \to R_2 + R_3$, we get

$$|A - \lambda I| = (1-\lambda)\begin{vmatrix} 2-\lambda & 1 & 0 \\ 1 & 2-\lambda & 0 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= (1-\lambda)\left[(2-\lambda)^2 - 1\right] = (1-\lambda)\left[4 + \lambda^2 - 4\lambda - 1\right] = -\lambda^3 + 5\lambda^2 - 7\lambda + 3.$$

$\therefore |A - \lambda I| = \lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$ is the characteristic equation of the matrix A. (i)

Then by Cayley-Hamilton theorem, the matrix A must satisfy (i), we have

$A^3 - 5A^2 + 7A - 3I = O$. (ii)

From (ii), we get

$A^3 = 5A^2 - 7A + 3I$, $\therefore$ $A^4 = 5A^3 - 7A^2 + 3A$ and $A^8 = 5A^7 - 7A^6 + 3A^5$ and

Now $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$

$= \left(5A^7 - 7A^6 + 3A^5\right) - 5A^7 + 7A^6 - 3A^5 + \left(5A^3 - 7A^2 + 3A\right) - 5A^3 + 8A^2 - 2A + I$

$$= A^2 + A + I = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}. \text{ Ans.}$$

Finding inverse by Cayley-Hamilton Theorem

Q.No.3.: Find the characteristic equation of the matrix $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$, and

hence find its inverse.

Sol.: **Find: Characteristic Equation**

The characteristic equation is $A = \begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 3-\lambda & -3 \\ -2 & -4 & -4-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^3 - 20\lambda + 8 = 0$.

which is the required characteristic equation of A.

Find: Inverse of A

Now since by Cayley-Hamilton theorem, we have $A^3 - 20A + 8I = O$

$\Rightarrow A^2 - 20I + 8A^{-1} = O$

$$\Rightarrow A^{-1} = \frac{5}{2}I - \frac{1}{8}A^2 = \frac{5}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{8}\begin{bmatrix} -4 & -8 & -12 \\ 10 & 22 & 6 \\ 2 & 2 & 22 \end{bmatrix} = \begin{bmatrix} 3 & 1 & \frac{3}{2} \\ -\frac{5}{4} & -\frac{1}{4} & -\frac{3}{4} \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}. \text{ Ans.}$$

Q.No.4.: Using Cayley-Hamilton theorem, find the inverse of

(i) $\begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$, **(ii)** $\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$, **(iii)** $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ 2 & -4 & -4 \end{bmatrix}$.

Sol.: (i). Let $A = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$.

If λ be the eigen value of the matrix A, then characteristic equation of A is $|A-\lambda I|=0$.

$$\Rightarrow \begin{bmatrix} 5-\lambda & 3 \\ 3 & 2-\lambda \end{bmatrix} = 0 \Rightarrow (5-\lambda)(2-\lambda)-9 \Rightarrow 10-5\lambda-2\lambda+\lambda^2-9=0$$

$$\Rightarrow \lambda^2-7\lambda+1=0.$$

Then, by Cayley-Hamilton theorem, we have $A^2-7A+1=0$.

Pre-multiplying both sides by A^{-1}, we get $A-7\ I+A^{-1}=O$

$$\Rightarrow A^{-1}=-A+7\ I=-\begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}+7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}=\begin{bmatrix} -5+7 & -3+0 \\ -3+0 & -2+7 \end{bmatrix}=\begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}. \text{ Ans.}$$

(ii). Let $A=\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$.

If λ be the eigen value of the matrix A, then characteristic equation of A is $|A-\lambda I|=0$.

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 & 3 \\ 2 & 1-\lambda & -1 \\ 1 & -1 & 1-\lambda \end{vmatrix}=0 \qquad \Rightarrow (1-\lambda)^3-(1-\lambda)+3(-2-1+\lambda)=0$$

$$\Rightarrow (1-\lambda)^3-(1-\lambda)+3(\lambda-3)=0$$

$$\Rightarrow (1-\lambda)^3+4\lambda-10=0 \Rightarrow -\lambda^3+3\lambda^2+\lambda-9=0.$$

Then, by Cayley Hamilton theorem, we have $-A^3+3A^2+A-9\ I=O$.

Pre-multiplying both sides by A^{-1}, we get $-A^2+3A+I-9A^{-1}=O$

$$\Rightarrow 9A^{-1}=-A^2+3A+I \ \Rightarrow A^{-1}=\frac{1}{9}\left(-A^2+3A+I\right).$$

Now $A^2=\begin{bmatrix} 1+0+3 & 0+0-3 & 3+0+3 \\ 2+2-1 & 0+1+1 & 6-1-1 \\ 1-2+1 & 0-1-1 & 3+1+1 \end{bmatrix} \Rightarrow \begin{bmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{bmatrix}$

$$\therefore A^{-1}=\frac{1}{9}\left[-A^2+3A+1\right]=\frac{1}{9}\begin{bmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{bmatrix}+\frac{3}{9}\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}+\frac{1}{9}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} -4+3+1 & 3+0+0 & -6+9+0 \\ -3+6+0 & -2+3+1 & -4-3+0 \\ 0+3+0 & 2-3+0 & -5+3+1 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 0 & 3 & 3 \\ 3 & 2 & -7 \\ 3 & -1 & -1 \end{bmatrix}. \text{ Ans.}$$

(iii). Let $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ 2 & -4 & -4 \end{bmatrix}$.

If λ be the eigen value of matrix A, then characteristic equation of A is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 3-\lambda & -3 \\ 2 & -4 & -4-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)\left[(3-\lambda)(-4-\lambda)-12\right]-(-4-\lambda+6)+3\left[-4-2(3-\lambda)\right]=0$$

$$\Rightarrow (1-\lambda)\left[-12-3\lambda+4\lambda+\lambda^2-12\right]+(4+\lambda-6)-12-18+6\lambda=0$$

$$\Rightarrow (1-\lambda)\left[\lambda^2+\lambda-24\right]+(\lambda-2)-30+6\lambda=0$$

$$\Rightarrow \lambda^2+\lambda-24-\lambda^3-\lambda^2+24\lambda+\lambda-2-30+6\lambda=0$$

$$\Rightarrow -\lambda^3+32\lambda-56=0$$

Then, by Cayley-Hamilton theorem, we have $-A^3+32A-56I=O$.

Pre-multiplication both sides by A^{-1}, we have $-A^2+32I-56A^{-1}=O$

$$\Rightarrow -A^2+32I=56A^{-1} \Rightarrow A^{-1}=\frac{1}{56}\left[-A^2+32I\right]$$

Now $A^2 = A.A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ 2 & -4 & -4 \end{bmatrix}.\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ 2 & -4 & -4 \end{bmatrix}$

$$\Rightarrow A^2 = \begin{bmatrix} 1+1+6 & 1+3-12 & 3-3-12 \\ 1+3-6 & 1+9+12 & 3-9+12 \\ 2-4-8 & 2-12+16 & 6+12+16 \end{bmatrix} = \begin{bmatrix} 8 & -8 & -12 \\ -2 & 22 & 6 \\ -10 & 6 & 34 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{56}\left[-A^2+32I\right] = \frac{1}{56}\begin{bmatrix} -8+32 & 8 & 12 \\ 2 & -22+32 & -6 \\ 10 & -6 & -34+32 \end{bmatrix} = \frac{1}{56}\begin{bmatrix} 24 & 8 & 12 \\ 2 & 10 & -6 \\ 10 & -6 & -2 \end{bmatrix}.$$

Q.No.5.: Verify Cayley-Hamilton theorem for the matrix A and find its inverse.

(i) $\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$, **(ii)** $\begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$.

Sol.: (i). Let $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$.

If λ be the eigen value of matrix A, then characteristic equation of A is $|A - \lambda I| = 0$.

$$\Rightarrow \begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)\{(2-\lambda)^2 - 1\} + 1\{-1(2-\lambda)+1\} + 1\{1-(2-\lambda)\} = 0$$

$$\Rightarrow (2-\lambda)(3 - 4\lambda + \lambda^2) + (\lambda - 1) + (\lambda - 1) = 0 \quad \Rightarrow -\lambda^3 + 6\lambda^2 - 9\lambda + 4 = 0.$$

Then, by Cayley-Hamilton's theorem, we get $-A^3 + 6A^2 - 9A + 4\ I = O$

Multiplying both sides by A^{-1}, we get $\quad -A^2 + 6A - 9\ I + 4A^{-1} = O \qquad$ (i)

$$\Rightarrow A^{-1} = \frac{1}{4}\left[A^2 - 6A + 9\ I\right].$$

First verify result (i):

Now $A^2 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$,

$$A^3 = A^2.A = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

$\therefore -A^3 + 6A^2 - 9A + 4I$

$$= -\begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} + \begin{bmatrix} 36 & -30 & 30 \\ -30 & 36 & -30 \\ 30 & -30 & 36 \end{bmatrix} - \begin{bmatrix} 18 & -9 & 9 \\ -9 & 18 & -9 \\ 9 & -9 & 0 \end{bmatrix} + \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O.$$

Hence Cayley-Hamilton theorem verified.

IInd: Find the inverse of A.

$$\text{Now } A^{-1} = \frac{1}{4}\left(A^2 - 6A + 9I\right) = \frac{1}{4}\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - \frac{3}{2}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + \frac{9}{4}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} \frac{6-12+9}{4} & \frac{-5+6+0}{4} & \frac{5-6+0}{4} \\ \frac{-5+6+0}{4} & \frac{6-12+9}{4} & \frac{-5+6+0}{4} \\ \frac{5-6+0}{4} & \frac{-5+6+0}{4} & \frac{6+2++9}{4} \end{bmatrix} = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} & -\frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} & \frac{3}{4} \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}. \text{ Ans.}$$

(ii). Given $A = \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$.

If λ is the eigen value of A then characteristic equation of A is $|A - \lambda I| = 0$.

$$\Rightarrow \begin{vmatrix} 7-\lambda & 2 & -2 \\ -6 & -1-\lambda & 2 \\ 6 & 2 & -1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)^2\begin{bmatrix} 7-\lambda & 2 & -2 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \Rightarrow (1-\lambda)^2[(1-\lambda) - 2(1) - 2] = 0$$

$$\Rightarrow -\lambda^3 + 2\lambda^2 - \lambda + 3 - 6\lambda + 3\lambda^2 = 0 \quad \Rightarrow \lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0.$$

Now by Cayley-Hamilton theorem, we get $A^3 - 5A^2 + 7A - 3I = O$.

Now $A^2 = \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}\begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 25 & 8 & -8 \\ -24 & -7 & 8 \\ 24 & 8 & -7 \end{bmatrix}$

$A^3 = A^2.A = \begin{bmatrix} 25 & 8 & -8 \\ -24 & -7 & 8 \\ 24 & 8 & -7 \end{bmatrix}\begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 79 & 26 & -26 \\ -78 & -25 & 26 \\ 78 & 26 & -25 \end{bmatrix}$

$A^3 - 5A^2 + 7A - 3I$

$= \begin{bmatrix} 79 & 26 & -26 \\ -78 & -25 & 26 \\ 78 & 26 & -25 \end{bmatrix} - 5\begin{bmatrix} 25 & 8 & -8 \\ -24 & -7 & 8 \\ 24 & 8 & -7 \end{bmatrix} + 7\begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$

Hence this proves the result.

Now $A^{-1} = \frac{1}{3}\left[A^2 - 5A + 7I\right] = \frac{1}{3}\left\{\begin{bmatrix} 25 & 8 & -8 \\ -24 & -7 & 8 \\ 24 & 8 & -7 \end{bmatrix} - \begin{bmatrix} 35 & 10 & -10 \\ -30 & -5 & 10 \\ 30 & 10 & -5 \end{bmatrix} + \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}\right\}$

$= \frac{1}{3}\begin{bmatrix} 25+7-35 & 8-10+0 & -8+10+0 \\ -24+35+0 & -7+5+7 & 8-10+0 \\ 24-30+0 & 8-10+0 & -7+5+7 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -3 & -2 & 2 \\ 6 & 5 & -2 \\ -6 & -2 & 5 \end{bmatrix}$. Ans.

Q.No.6.: Find the characteristic equation of the matrix $A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$. Show that the equation is satisfied by A and hence obtains the inverse of the given matrix.

Sol.: **Find: Characteristic Equation**

Given $A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$.

If λ be an eigen value of matrix A, then the characteristic equation of A is $|A - \lambda I| = 0$.

$$\Rightarrow \begin{vmatrix} 1-\lambda & 3 & 7 \\ 4 & 2-\lambda & 3 \\ 1 & 2 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)[(2-\lambda)(1-\lambda)-6]-3[4(1-\lambda)-3]+7[8-(2-\lambda)]=0$$

$$\Rightarrow (1-\lambda)[2-2\lambda-\lambda+\lambda^2-6]-3(1-4\lambda)+7(6+\lambda)=0$$

$$\Rightarrow (1-\lambda)(\lambda^2-3\lambda-4)-3+12\lambda+42+7\lambda=0$$

$$\Rightarrow \lambda^2-3\lambda-4-\lambda^3+3\lambda^2+4\lambda-3+12\lambda+42+7\lambda=0$$

$$\Rightarrow -\lambda^3+4\lambda^2+20\lambda+35=0 \qquad \Rightarrow \lambda^3-4\lambda^2-20\lambda-35=0,$$

which is the required characteristic equation.

To shows the above characteristic equation is satisfied by A.

i.e., $A^3-4A^2-20A-35\ I=O$.

$$\text{Now } A^2 = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1+12+7 & 3+6+14 & 7+9+7 \\ 4+8+3 & 12+4+6 & 28+6+3 \\ 1+8+1 & 3+4+2 & 7+6+1 \end{bmatrix} = \begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix}$$

$$A^3 = A^2.A = \begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix}\begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 20+92+23 & 60+46+46 & 140+69+23 \\ 15+88+37 & 45+44+74 & 105+66+37 \\ 10+36+14 & 30+18+28 & 70+27+14 \end{bmatrix}$$

$$= \begin{bmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{bmatrix}$$

$$\therefore A^3-4A^2-20A-35\ I$$

$$= \begin{bmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{bmatrix} - 4\begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix} - 20\begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} - 35\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0.$$

Find: Inverse of A

Since $A^3-4A^2-20A-35\ I=O$

Multiplying both sides by A^{-1}, we get $A^2-4A-20\ I-35\ A^1=O$

$$\Rightarrow A^{-1}=\frac{1}{35}\left[A^2-4A-20\ I\right]=\frac{1}{35}\left\{\begin{bmatrix}20&23&23\\15&22&37\\10&9&14\end{bmatrix}-4\begin{bmatrix}1&3&7\\4&2&3\\1&2&1\end{bmatrix}-20\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\right\}$$

$$=\frac{1}{35}\begin{bmatrix}20-4-20&23-12+0&23-28+0\\15-16+0&22-8-20&37-12+0\\10-4+0&9-8+0&14-4-20\end{bmatrix}=\frac{1}{35}\begin{bmatrix}-4&11&-5\\-1&-6&25\\6&1&-10\end{bmatrix}. \text{ Ans.}$$

Q.No.7.: Verify Cayley Hamilton theorem for the matrix $A=\begin{bmatrix}1&2\\2&-1\end{bmatrix}$, find A^{-1}**.**

Determine A^8.

Sol.: The characteristic equation is $|A-\lambda I|=\begin{vmatrix}1-\lambda&2\\2&-1-\lambda\end{vmatrix}=0$

$\Rightarrow(\lambda-1)(1+\lambda)-4=0\Rightarrow\lambda^2-5=0$.

$A^2=A.A=\begin{bmatrix}1&2\\2&-1\end{bmatrix}\begin{bmatrix}1&2\\2&-1\end{bmatrix}=\begin{vmatrix}5&0\\0&5\end{vmatrix}=5I\ \Rightarrow A^2-5I=O$

Thus A satisfies the characteristic equation.

To find A^{-1}, multiply $A^2-5I=O$ by A^{-1}, we get

$A^{-1}.A^2-5A^{-1}I=O\quad\Rightarrow A-5A^{-1}=O$

So $A^{-1}=\frac{1}{5}A=\frac{1}{5}\begin{bmatrix}1&2\\2&-1\end{bmatrix}$.

To find A^8, multiply $A^2-5I=0$ by A^6, we get

$A^6.A^2-5I.A^6=O$

$A^8=5A^6=5.A^2.A^2.A^2=5.(5I)(5I)(5I)$

$A^8=625$ I.

Q.No.8.: Verify Cayley-Hamilton theorem for the matrix $A=\begin{bmatrix}1&2&3\\2&4&5\\3&5&6\end{bmatrix}$ and hence find

the inverse of A. Find A^4.

Express $B=A^8-11A^7-4A^6+A^5+A^4-11A^3-3A^2+2A+I$ as a quadratic polynomial in A. Find B.

Sol.: The characteristic equation of A is

$$|A-\lambda I| = \begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & 4-\lambda & 5 \\ 3 & 5 & 6-\lambda \end{vmatrix}$$

$$= (1-\lambda)[(4-\lambda)(6-\lambda)-25]-2[2(6-\lambda)-15]+3[10-3(4-\lambda)] = 0,$$

$$\Rightarrow \lambda^3 - 11\lambda^2 - 4\lambda + 1 = 0.$$

Cayley Hamilton theorem is verified if A satisfies the above characteristic equation,

i.e., $A^3 - 11A^2 - 4A + I = 0$.

Now $A^2 = A.A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{bmatrix}$.

$$A^3 = A.A^2 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}\begin{bmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{bmatrix} = \begin{bmatrix} 157 & 283 & 353 \\ 283 & 510 & 636 \\ 353 & 636 & 793 \end{bmatrix}.$$

Verification:

$A^3 - 11A^2 - 4A + I$

$$= \begin{bmatrix} 157 & 283 & 353 \\ 283 & 510 & 636 \\ 353 & 636 & 793 \end{bmatrix} - 11\begin{bmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{bmatrix} - 4\begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 3 & 5 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

To find A^{-1}:

From characteristic equation $A^{-1} = -A^2 + 11A + 4I$.

So $A^{-1} = -\begin{bmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{bmatrix} + 11\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} + 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix}$.

To find A^4:

From Cayley-Hamilton theorem

$$A^3 - 11A^2 - 4A + I = 0 \Rightarrow A^3 = 11A^2 + 4A - I.$$

Multiplying both sides by A

$$A^4 = A.A^3 = A\left(11A^2 + 4A - I\right) = 11A^3 + 4A^2 - A$$

$$=11\begin{bmatrix}157 & 283 & 353\\ 283 & 510 & 636\\ 353 & 636 & 793\end{bmatrix}+\begin{bmatrix}14 & 25 & 31\\ 25 & 45 & 56\\ 31 & 56 & 70\end{bmatrix}-\begin{bmatrix}1 & 2 & 3\\ 2 & 4 & 5\\ 3 & 5 & 6\end{bmatrix}=\begin{bmatrix}1782 & 3211 & 4004\\ 3211 & 5786 & 7215\\ 4004 & 7215 & 8997\end{bmatrix}$$

To find B:

Rewrite $B = A^8 - 11A^7 - 4A^6 + A^5 + A^4 - 11A^3 - 3A^2 + 2A + I$

$$= A^5\left(A^3 - 11A^2 - 4A + I\right) + A\left(A^3 - 11A^2 - 4A + 1\right) + A^2 + A + I$$

$$= A^5(0) + A(0) + A^2 + A + I.$$

Thus, the quadratic polynomial in A of B is $A^2 + A + I$.

Now $B = A^2 + A + I = \begin{bmatrix}14 & 25 & 31\\ 25 & 45 & 56\\ 31 & 56 & 70\end{bmatrix}+\begin{bmatrix}1 & 2 & 3\\ 2 & 4 & 5\\ 3 & 5 & 6\end{bmatrix}+\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}.$

$$B = \begin{bmatrix}16 & 27 & 34\\ 27 & 50 & 61\\ 34 & 61 & 77\end{bmatrix}.$$

Q.No.9.: Determine A^{-1}, A^{-2}, A^{-3} if $A = \begin{bmatrix}4 & 6 & 6\\ 1 & 3 & 2\\ -1 & -4 & -3\end{bmatrix}$.

Sol.: The characteristic equation of A is

$$|A - \lambda I| = \begin{vmatrix}4-\lambda & 6 & 6\\ 1 & 3-\lambda & 2\\ -1 & -4 & -3-\lambda\end{vmatrix} = \lambda^3 - 4\lambda^2 - \lambda + 4 = 0$$

It follows from Cayley-Hamilton theorem that

$$A^3 - 4A^2 - A + 4I = 0$$

Multiplying by A^{-1},

$$A^{-1}A^3 - 4A^{-1}A^2 - A^{-1}.A + A^{-1}4I = 0$$

Solving $A^{-1} = \dfrac{1}{4}\left(I + 4A - A^2\right)$

$$A^2 = A.A = \begin{bmatrix}16 & 18 & 18\\ 5 & 7 & 6\\ -5 & -6 & -5\end{bmatrix}$$

$$A^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix} - \frac{1}{4}\begin{bmatrix} 16 & 18 & 18 \\ 5 & 7 & 6 \\ -5 & -6 & -5 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & 6 & 6 \\ -1 & 6 & 2 \\ 1 & -10 & -6 \end{bmatrix}.$$

Multiplying A^{-1} by A^{-1}, we have

$$A^{-2} = A^{-1}A^{-1} = A^{-1}\frac{1}{4}\left[I + 4A - A^2\right] = \frac{1}{4}\left[A^{-1} + 4I - A\right] = \frac{1}{4}\begin{bmatrix} \frac{1}{4} & -\frac{9}{2} & -\frac{9}{2} \\ -\frac{5}{4} & \frac{5}{2} & -\frac{3}{2} \\ \frac{5}{4} & \frac{3}{2} & \frac{11}{2} \end{bmatrix}.$$

$$A^{-3} = A^{-1}A^{-2} = A^{-1}\left[A^{-1} + 4I - A\right]\frac{1}{4} = \frac{1}{4}\left[A^{-2} + 4A^{-1} - I\right] = \frac{1}{64}\begin{bmatrix} 1 & 78 & 78 \\ -21 & 90 & 26 \\ 21 & -154 & -90 \end{bmatrix}$$

Home Assignments

Problems on verification of Cayley-Hamilton theorem

Q.No.1.: Verify Cayley-Hamilton theorem for the matrix $\begin{bmatrix} 2 & 5 \\ 1 & -3 \end{bmatrix}$.

Ans.: Characteristic polynomial: $\lambda^2 + \lambda - 11$.

Q.No.2.: Verify Cayley-Hamilton theorem for the matrix $\begin{bmatrix} 2 & -3 \\ 7 & -4 \end{bmatrix}$.

Ans.: Characteristic polynomial: $\lambda^2 + 2\lambda + 13$.

Q.No.3.: Verify Cayley-Hamilton theorem for the matrix $\begin{bmatrix} 1 & 4 & -3 \\ 0 & 3 & 1 \\ 0 & 2 & -1 \end{bmatrix}$.

Ans.: Characteristic equation: $\lambda^3 - 3\lambda^2 - 3\lambda + 5 = 0$.

Q.No.4.: Find the characteristic equation of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}$. Show that the equation is satisfied by A.

Ans.: $\lambda^3 + \lambda^2 - 18\lambda - 40 = 0$.

Problems on finding the inverse by using Cayley-Hamilton theorem

Q.No.5.: Using Cayley-Hamilton theorem, find the inverse of

(i) $\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$, **(ii).** $\begin{bmatrix} 7 & -1 & 3 \\ 6 & 1 & 4 \\ 2 & 4 & 8 \end{bmatrix}$.

Ans.: **(i).** $\begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix}$ **(ii).** $\frac{1}{50}\begin{bmatrix} 8 & 20 & -7 \\ 40 & 50 & -10 \\ 22 & -30 & 13 \end{bmatrix}$.

Q.No.6.: Using Cayley-Hamilton theorem, find the inverse of $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$.

Ans.: Characteristic equation: $\lambda^3 - 6\lambda^2 - 9\lambda - 4 = 0$, $A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$.

Q.No.7.: Using Cayley-Hamilton theorem, find the inverse of $A = \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$.

Ans.: Characteristic equation: $\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$, $A^{-1} = \frac{1}{3}\begin{bmatrix} -3 & -2 & 2 \\ 6 & 5 & -2 \\ -6 & -2 & 5 \end{bmatrix}$.

Problems on verifications and finding the inverse by using Cayley-Hamilton theorem

Q.No.8.: Find the characteristic equation of the matrix $A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$. Show that the equation is satisfied by A and hence obtain the inverse of the given matrix.

Ans.: $\lambda^3 - 4\lambda^2 - 20\lambda - 35 = 0$, $A^{-1} = \dfrac{1}{35}\begin{bmatrix} -4 & 11 & -5 \\ -1 & -6 & 25 \\ 6 & 1 & 10 \end{bmatrix}$.

Q.No.9.: Verify Cayley-Hamilton theorem to find A^{-1} if $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$.

Ans.: Characteristic equation: $\lambda^3 - 20\lambda + 8 = 0$, $A^{-1} = \dfrac{1}{4}\begin{bmatrix} 12 & 4 & 6 \\ -5 & -1 & -3 \\ -1 & -1 & -1 \end{bmatrix}$.

Q.No.10.: Verify Cayley-Hamilton theorem and hence find A^{-1} for $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$.

Ans.: Characteristic equation: $\lambda^4 - \lambda^3 - \lambda + 1 = 0$, $A^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$.

Problems on finding the matrix polynomials

Q.No.11.: Verify Cayley-Hamilton theorem for the matrix $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$. Find A^{-1}.

Find $B = A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10\ I$.

Ans.: Characteristic equation: $\lambda^2 - 4\lambda - 5 = 0$, $A^{-1} = \dfrac{1}{5}\begin{bmatrix} -3 & 4 \\ 2 & -1 \end{bmatrix}$,

$$B = A + 5\ I = \begin{bmatrix} 6 & 4 \\ 2 & 8 \end{bmatrix}$$

Q.No.12.: If $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$, find A^{-1}.

Find $B = A^8 - 5A^7 + 7A^6 - 3A^5 - 5A^3 + 8A^2 - 2A + I$

Ans.: Characteristic equation: $\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$,

$$A^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ 0 & 3 & 0 \\ -1 & -1 & 2 \end{bmatrix},\ B = \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}.$$

Q.No.13.: Find $B = A^6 - 4A^5 + 8A^4 - 12A^3 + 14A^2$ if $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$.

Ans.: Characteristic equation : $\lambda^2 - 4\lambda + 5 = 0$, $B = 5\ I - 4A = \begin{bmatrix} 1 & -8 \\ 4 & -7 \end{bmatrix}$.

Q.No.14.: Find A^{-1} and A^4 if $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$.

Ans.: Characteristic equation: $\lambda^3 - 3\lambda^2 - \lambda + 9 = 0$,

$$A^{-1} = \frac{1}{9}\begin{bmatrix} 0 & 3 & 3 \\ 3 & 2 & -7 \\ 3 & -1 & -1 \end{bmatrix},\ A^4 = \begin{bmatrix} 7 & -30 & 42 \\ 18 & -13 & 46 \\ -6 & -14 & 17 \end{bmatrix}.$$

Q.No.15.: Compute A^{-1}, A^{-2}, A^3 and A^4 if $A = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 1 \\ 2 & 3 & 1 \end{bmatrix}$.

Ans.: Characteristic equation: $\lambda^3 - 3\lambda^2 - 7\lambda - 11 = 0$, $A^{-1} = \frac{1}{11}\begin{bmatrix} -2 & 5 & -1 \\ -1 & -3 & 5 \\ 7 & -1 & -2 \end{bmatrix}$,

$$A^{-2} = \frac{1}{121}\begin{bmatrix} -8 & -24 & 29 \\ 40 & -1 & -24 \\ -27 & 40 & -8 \end{bmatrix},\ A^3 = \begin{bmatrix} 42 & 31 & 29 \\ 45 & 39 & 31 \\ 53 & 45 & 42 \end{bmatrix},\ A^4 = \begin{bmatrix} 193 & 160 & 144 \\ 224 & 177 & 160 \\ 272 & 224 & 193 \end{bmatrix}.$$

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9^{th} Topic

Matrices

Reduction to Diagonal form (Modal Matrix)

Reduction to Quadratic form to Canonical form

Nature. Index, signature of Quadratic form

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Reduction to Diagonal form:

Theorem: If a square matrix A of order n has n linearly independent eigen vectors, then a matrix P can be found such that $P^{-1}AP$ is a diagonal matrix.

This result will be proved for a square matrix of order 3 but the method will be capable of easy extension to matrices of any order.

Proof:

Let A be a square matrix of order 3.

Let $\lambda_1, \lambda_2, \lambda_3$ be its eigen values

and $X_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$, $X_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$, $X_3 = \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix}$ be the corresponding eigen vectors.

Denoting the square matrix $[X_1X_2X_3] = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$ by P, we have

$$AP = A[X_1X_2X_3] = [AX_1\ AX_2\ AX_3] = [\lambda_1X_1\ \lambda_2X_2\ \lambda_3X_3]$$

$$=\begin{bmatrix}\lambda_1x_1 & \lambda_2x_2 & \lambda_3x_3\\ \lambda_1y_1 & \lambda_2y_2 & \lambda_3y_3\\ \lambda_1z_1 & \lambda_2z_2 & \lambda_3z_3\end{bmatrix}=\begin{bmatrix}x_1 & x_2 & x_3\\ y_1 & y_2 & y_3\\ z_1 & z_2 & z_3\end{bmatrix}\begin{bmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{bmatrix}=PD,$$

where D is the diagonal matrix.

$\therefore P^{-1}AP = P^{-1}PD = D$,

which proves the theorem.

Remarks:

1. The matrix P, which diagonalises A is called the **modal matrix** of A and the resulting diagonal matrix D is known as a **spectral matrix** of A.
2. The diagonal matrix has the eigen values of A as its diagonal elements.
3. The matrix P is found by grouping the eigen vectors of A into a square matrix.

Similarities of matrices:

A square matrix $\hat{A}$ of order n is called **similar** to a square matrix A of order n if $\hat{A} = P^{-1}AP$ for some non-singular $n \times n$ matrix P.

Similarity Transformation: This transformation of a matrix A by a non-singular matrix P to $\hat{A}$ is called a **similarity transformation.**

Remarks:

1. If the matrix $\hat{A}$ is similar to the matrix A, then $\hat{A}$ has the same eigen values of A.
2. If **X** is an eigen vector of A, then $Y = P^{-1}X$ is an eigen vector of $\hat{A}$ corresponding to the same eigen value.

Powers of a matrix:

Result: Diagonalisation of a matrix is quite useful for obtaining powers of a matrix.

Proof: Let A be the square matrix.

Then, a non-singular matrix P can be found such that $D = P^{-1}AP$.

$\therefore D^2 = (P^{-1}AP)(P^{-1}AP) = P^{-1}A^2P$. $\qquad [\because PP^{-1} = I]$

Similarly, $D^3 = P^{-1}A^3P$ and in general $D^n = P^{-1}A^nP$. (i)

To obtain A^n:

Pre-multiply (i) by P and post-multiply by P^{-1}, we get

$PD^nP^{-1} = PP^{-1}A^nPP^{-1} = A^n$ which gives A^n.

Thus, $A^n = PD^nP^{-1}$, where $D^n = \begin{bmatrix} \lambda_1^n & 0 & 0 \\ 0 & \lambda_2^n & 0 \\ 0 & 0 & \lambda_3^n \end{bmatrix}$.

Working procedure:

1. Find the eigen values of the square matrix A.
2. Find the corresponding eigen vectors and write the normal matrix A.
3. Find the diagonal matrix D from $D = P^{-1}DP$.
4. Obtain A^n from $A^n = PDP^{-1}$.

Quadratic Forms:

Definition: A homogeneous polynomial of second degree in any number of variables is called a quadratic form.

For examples:

(i) $ax^2 + 2hxy + by^2$

(ii) $ax^2 + by^2 + cz^2 + 2hxy + 2gyz + 2fzx$ and

(iii) $ax^2 + by^2 + cz^2 + dw^2 + 2hxy + 2gyz + 2fzx + 2\ell xw + 2myw + 2nzw$

are quadratic forms in two, three and four variables.

Theorem: Every quadratic form can be written as $\sum_{j=1}^{n}\sum_{i=1}^{n} a_{ij}x_ix_j = X'AX$, so that the matrix A is always symmetric, where $A = \lfloor a_{ij} \rfloor$ and $X' = [x_1,\ x_2,\dots, x_n]$.

Proof: In n-variables $x_1, x_2, \dots\dots\dots\dots x_n$, the general quadratic form is $\sum_{j=1}^{n}\sum_{i=1}^{n} b_{ij}x_ix_j$.

In the expansion, the co-efficient of $x_ix_j = (b_{ij} + b_{ji})$.

Suppose $2a_{ij} = b_{ij} + b_{ij}$ where $a_{ij} = a_{ji}$ and $a_{ii} = b_{ii}$

$$\therefore \sum_{j=1}^{n}\sum_{i=1}^{n} b_{ij}x_ix_j = \sum_{j=1}^{n}\sum_{i=1}^{n} a_{ij}x_ix_j, \text{ where } a_{ij} = \frac{1}{2}(b_{ij} + b_{ji}).$$

Hence, every quadratic form can be written as $\sum_{j=1}^{n}\sum_{i=1}^{n} a_{ij}x_i x_j = X'AX$, so that the matrix A is always symmetric, where $A = \left[a_{ij}\right]$ and $X' = \left[x_1,\ \ x_2,\dots,x_n\right]$.

Now writing the above said examples of quadratic forms in matrix form, we get

(i). $ax^2 + 2hxy + by^2 = \begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} a & h \\ h & b \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$.

(ii). $ax^2 + by^2 + cz^2 + 2hxy + 2gyz + 2fzx = \begin{bmatrix} x & y & z \end{bmatrix}\begin{bmatrix} a & h & f \\ h & b & g \\ f & g & c \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and

(iii). $ax^2 + by^2 + cz^2 + dw^2 + 2hxy + 2gyz + 2fzx + 2\ell xw + 2myw + 2nzw$

$$= \begin{bmatrix} x & y & z & w \end{bmatrix}\begin{bmatrix} a & h & f & \ell \\ h & b & g & m \\ f & g & c & n \\ \ell & m & n & d \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}.$$

Linear Transformation of a Quadratic form:

Let $X'AX$ be a quadratic form in n-variables and let $X = PY$, (i)

where P is a non-singular matrix, be the non-singular transformation.

From (i), we get $X' = (PY)' = Y'P'$.

Thus $X'AX = Y'P'APY = Y'(P'AP)Y = Y'BY$, (ii)

where $B = P'AP$.

Therefore, $Y'BY$ is also a quadratic form in n-variables.

Hence, it is a linear transformation of the quadratic form $X'AX$ under the linear transformation $X = PY$ and $B = P'AP$.

Note:

(i) Here $B' = (P'AP)' = P'AP = B$.

(ii) $\rho(B) = \rho(A)$.

$\therefore$ A and B are **congruent matrices**.

Canonical Form:

If a **real quadratic form** be expressed as a **sum or difference of the square of new variables** by means of any real non-singular linear transformation, then the later quadratic expression is called a canonical form (or sum of squares form or Principal axes form) of the given quadratic form.

i.e., if the quadratic form $X'AX = \sum_{j=1}^{n} \sum_{i=1}^{n} a_{ij} x_i x_j$ can be reduced to the quadratic form

$Y'BY = \sum_{i=1}^{n} \lambda_i y_1^2$ by a non-singular linear transformation X = PY, then $Y'BY$ is called the **canonical form** of the given one.

$\therefore$ If $B = P'AP = \text{diag.}(\lambda_1, \lambda_2, \ldots\ldots\ldots, \lambda_n)$,

then $X'AX = Y'BY = \sum_{i=1}^{n} \lambda_i y_1^2$.

Remarks:

1. Here some of λ_1 (eigen values) may be positive or negative or zero.
2. A quadratic form is said to be real if the elements of the symmetric matrix are real.
3. If $\rho(A) = r$, then the quadratic form $X'AX$ will contain only r terms.

Index and Signature of the quadratic form:

Index:

The number p of positive terms in the canonical form is called the index of the quadratic form.

Signature:

(The number of terms) – (The number of negative terms)

i.e., $p-(r-p) = 2p - r$ is called signature of the quadratic form, where $\rho(A) = r$.

Definite, Semi-definite and Indefinite Real Quadratic form:

Let $X'AX$ be real quadratic form in n-variables x_1, x_2, ………… x_n with rank r and index p.

Then, we say that the quadratic form is

(i) positive definite if r = n, p = r

(ii) negative definite if r = n, p = 0

(iii) positive semi-definite if $r < n$, $p = r$ and

(iv) negative semi-definite if $r < n$, $p = 0$.

If the canonical form has both positive and negative terms, the quadratic form is said to be indefinite.

Remarks: If $X'AX$ is positive definite then $|A| > 0$.

OR

Nature of Quadratic Form:

A real quadratic form $X'AX$ in a variable is said to be

(i) Positive definite if all the eigen values of $A > 0$.

(ii) Negative definite if all the eigen values of $A < 0$.

(iii) Positive semi-definite if all the eigen values of $A \geq 0$ and at least one eigen value = 0.

(iv) Negative semi-definite if all the eigen values of $A \leq 0$ and at least one eigen value = 0.

(v) Indefinite if some of the eigen values of A are positive and others negative.

Law-of-Inertia of Quadratic form:

Statement:

"The index of real quadratic form is invariant under real non-singular transformation".

Reduction to Canonical form by Orthogonal Transformation:

Let $X'AX$ be a given quadratic form. The modal matrix B of A is that matrix whose columns are characteristic vectors of A. If B represent the orthogonal matrix of A (the normalized modal matrix of A whose column vectors are pair-wise orthogonal),

then $X = BY$ will reduce $X'AX$ to $Y'DY$,

where $D = \text{diag}\left(\lambda_1, \lambda_2, ..., \lambda_n\right)$ and $\lambda_1, \lambda_2, ..., \lambda_n$ are characteristic roots of A.

Remarks: This method works successfully if the characteristic vectors A are linearly dependent which are pairwise orthogonal.

Determination of real symmetric matrix C of the quadratic form:

Q.No.1.: Find a real symmetric matrix C of the **quadratic form**

$$Q = x_1^2 + 3x_2^2 + 2x_3^2 + 2x_1x_2 + 6x_2x_3.$$

Sol.: The coefficient matrix of Q is $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 2 \end{bmatrix}$,

Thus C = symmetric matrix = $\frac{1}{2}\left[A + A^T\right]$.

$$C = \frac{1}{2}\left\{\begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 0 & 6 & 2 \end{bmatrix}\right\} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 3 \\ 0 & 3 & 2 \end{bmatrix}.$$

Remarks: The simplest way writing C is

1. Put coefficients of square terms as the diagonal elements.

2. Place $\frac{1}{2}$ of a_{ij}, the coefficients of x_i, x_j, x_{ij} and the remaining $\frac{1}{2}$ of a_{ij}, at c_{ji}, i.e.,

$c_{ij} = c_{ji} = \frac{1}{2}a_{ij}$ such that $c_{ij} + c_{ji} = \frac{1}{2}\left(a_{ij} + a_{ij}\right) = a_{ij}$.

Determine the nature, index and signature

Q.No.1.: Determine the nature, index and signature of the quadratic form

$$2x_1^2 + 2x_2^2 + 3x_3^2 + 2x_1x_2 - 4x_1x_3 - 4x_2x_3.$$

Sol.: The real symmetric matrix A associated with the quadratic form is

$$A = \begin{bmatrix} 2 & 1 & -2 \\ 1 & 2 & -2 \\ -2 & -2 & 3 \end{bmatrix}.$$

Its characteristic equation is $\begin{vmatrix} 2-\lambda & 1 & -2 \\ 1 & 2-\lambda & -2 \\ -2 & -2 & 3-\lambda \end{vmatrix} = 0$

$$\Rightarrow \lambda^3 - 7\lambda^2 + 7\lambda - 1 = 0 \Rightarrow (\lambda - 1)\left(\lambda - \left(3 + \sqrt{8}\right)\right)\left(\lambda - \left(3 - \sqrt{8}\right)\right) = 0.$$

The eigen values are $\lambda =$ 1, 0.1715, 3.1715, which are all positive.

Since, we know that if all the eigen values of A > 0, then the quadratic form is positive definite

So, here quadratic form is positive define.

Index: 3, Signature: $3-0=3$.

Q.No.2.: Find the nature, index and signature of quadratic form $2x_1x_2+2x_1x_3+2x_2x_3$.

Sol.: The real symmetric matrix A associated with the quadratic form is $A=\begin{bmatrix}0 & 1 & 1\\ 1 & 0 & 1\\ 1 & 1 & 0\end{bmatrix}$.

Its characteristic equation is $\begin{vmatrix}-\lambda & 1 & 1\\ 1 & -\lambda & 1\\ 1 & 1 & -\lambda\end{vmatrix}=0$.

$\Rightarrow \lambda^3-3\lambda-2=0 \Rightarrow (\lambda+1)^2(\lambda-2)=0$.

The eigen values are $2,\ -1,\ -1$, some are positive and some are negative.

So quadratic form is indefinite.

Index: 1, Signature: $1-2=-1$.

Q.No.3.: Identify the nature, index and signature of the quadratic form $x_1^2+4x_2^2+x_3^2-4x_1x_2+2x_3x_1-4x_2x_3$.

Sol.: The real symmetric matrix A associated with the quadratic form is

$$A=\begin{bmatrix}1 & -2 & 1\\ -2 & 4 & -2\\ 1 & -2 & 1\end{bmatrix}.$$

Its characteristic equation is $|A-\lambda I|=\begin{vmatrix}1-\lambda & -2 & 1\\ -2 & 4-\lambda & -2\\ 1 & -2 & 1-\lambda\end{vmatrix}=\lambda^2(\lambda-6)=0$.

Eigen values are $\lambda=0,\ 0,\ 6$.

So quadratic form is positive semi definite.

Index: 3, Signature: 3.

Q.No.4.: Classify the quadratic form and find the index and signature of $-3x_1^2-3x_2^2-3x_3^2-2x_1x_2-2x_1x_3+2x_2x_3$.

Sol.: The real symmetric matrix A associated with the quadratic form is

$$A=\begin{bmatrix}-3 & -1 & -1\\ -1 & -3 & 1\\ -1 & 1 & -3\end{bmatrix}.$$

Its characteristic equation is $\begin{vmatrix}-3-\lambda & -1 & -1\\ -1 & -3-\lambda & 1\\ -1 & 1 & -3-\lambda\end{vmatrix}=0$

$\Rightarrow \lambda^3+9\lambda^2+24\lambda+16=(\lambda+1)(\lambda+4)^2=0$.

All the eigen values $-1,\ -4,\ -4$, are negative.

So quadratic form is negative definite.

Index: 0, Signature: $0-3=-3$

Note: $Q=3x_1^2+3x_2^2-3x_3^2+2x_1x_2+2x_1x_3-2x_2x_3$ is positive definite.

Reduction to diagonal form

Q.No.1.: Find the matrix P which **diagonalises** the matrix $A=\begin{bmatrix}4 & 1\\ 2 & 3\end{bmatrix}$, verify that

$P^{-1}AP=D$, where D is diagonal matrix , hence find A^6.

Sol.: *Since we know, if a square matrix A of or order n has n linearly independent eigen vectors, then a matrix P can be found such that* $P^{-1}AP$ *is a diagonal matrix.*

A is diagonalizable by P whose columns are the linearly independent eigen vectors of A.

The characteristic equation of A is $|A-\lambda I|=\begin{vmatrix}4-\lambda & 1\\ 2 & 3-\lambda\end{vmatrix}=0$

$\Rightarrow (4-\lambda)(3-\lambda)-2=\lambda^2-7\lambda+10=(\lambda-2)(\lambda-5)=0$.

So $\lambda=2,\ 5$ are two distinct eigen values of A.

For $\lambda=2$: $2x_1+x_2=0$, $x_2=-2x_1$, $X_1=\begin{bmatrix}1\\ -2\end{bmatrix}$.

For $\lambda=5$: $-x_1+x_2=0$, $x_2=x_1$, $X_2=\begin{bmatrix}1\\ 1\end{bmatrix}$.

Thus, the matrix P which diagonalises A is $P=[X_1,\ X_2]=\begin{bmatrix}1 & 1\\ -2 & 1\end{bmatrix}$.

Verification: Since $P^{-1} = \frac{1}{3}\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$.

Therefore $P^{-1}AP = \frac{1}{3}\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 & -2 \\ 10 & 5 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$

$$P^{-1}AP = \frac{1}{3}\begin{bmatrix} 6 & 0 \\ 0 & 15 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} = D = \text{diagonal matrix}$$

D contain eigen values 2, 5 as diagonal elements.

To find A^6:

$$A^6 = PD^6P^{-1} = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 2^6 & 0 \\ 0 & 5^6 \end{bmatrix}\frac{1}{3}\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$$

$$A^6 = \frac{1}{3}\begin{bmatrix} 64 & 15625 \\ 128 & 15625 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 31314 & 15561 \\ 31122 & 15753 \end{bmatrix}$$

$$\therefore A^6 = \begin{bmatrix} 10438 & 5187 \\ 10374 & 5251 \end{bmatrix}. \text{ Ans.}$$

Q.No.2.: Define modal matrix & spectral matrix of a matrix.

Reduce the matrix $A = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$ into a diagonal matrix, by finding its modal matrix P, and hence write its spectral matrix.

Sol.: 1st Part: We know that if a square matrix A of order n has n linearly independent eigen vectors, then a matrix P can be found such that $P^{-1}AP$ is a diagonal matrix.

Modal matrix: The matrix P, which diagonalises A is called the **modal matrix** of A.

Spectral matrix: The resulting diagonal matrix D is known as a **spectral matrix** of A.

2nd Part:

The characteristic equation of A is $\begin{vmatrix} 1-\lambda & 0 \\ 2 & -1-\lambda \end{vmatrix} = 0$

$$\Rightarrow (1-\lambda)(-1-\lambda) = 0 \Rightarrow (1-\lambda)(1+\lambda) = 0 \Rightarrow \lambda = 1, -1.$$

So eigen values of A are $\lambda = 1, -1$.

For $\lambda = -1$, we have $2x_1 = 0 \Rightarrow x_1 = 0$

Thus $X_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

For $\lambda = 1$, we have $2x_1 - 2x_2 = 0 \Rightarrow x_1 = x_2$

Thus $X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Thus, the **modal matrix** is $P = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$

Spectral matrix is $D = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

Also $P^{-1} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$

Verification: $A = PDP^{-1}$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$$

Q.No.3.: Diagonalise $A = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ and hence find A^8. Find the modal matrix.

Sol.: The non-singular square matrix P containing eigen vectors of A as columns, diagonalises A.

The characteristic equation of A is $\begin{vmatrix} 1-\lambda & 6 & 1 \\ 1 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0 \Rightarrow (\lambda+1)(\lambda-3)(\lambda-4) = 0$.

So eigen values of A are $\lambda = -1,\ 3,\ 4$.

For $\lambda = -1$, we have

$$2x_1 + 6x_2 + x_3 = 0$$
$$x_1 + 3x_2 + 0 = 0$$
$$4x_3 = 0$$

$\therefore x_3 = 0\ x_1 = -3x_2$. Thus $X_1 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$.

For $\lambda = 3$, we have $-2x_1 + 6x_2 + x_3 = 0$

$$x_1 - x_2 = 0$$

$\therefore x_3 = x_2,\ x_3 = -4x_2$. Thus $X_2 = \begin{bmatrix} 1 \\ 1 \\ -4 \end{bmatrix}$.

For $\lambda = 4$, we have $-3x_1 + 6x_2 + x_3 = 0$

$$x_2 - 2x_2 = 0$$

$$-x_3 = 0$$

$\therefore x_3 = 0,\ x_2 = 2x_2$. Thus $X_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$.

Thus $P = \begin{bmatrix} -3 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & -4 & 0 \end{bmatrix}$ is the modal matrix.

To find P^{-1}: Now $\begin{bmatrix} -3 & 1 & 2 & : & 1 & 0 & 0 \\ 1 & 1 & 1 & : & 0 & 1 & 0 \\ 0 & -4 & 0 & : & 0 & 0 & 1 \end{bmatrix}$

Operating $R_{12},\ R_{21(3)}$, we get $\sim \begin{bmatrix} 1 & 1 & 1 & : & 0 & 1 & 0 \\ 0 & 4 & 5 & : & 1 & 3 & 0 \\ 0 & -4 & 0 & : & 0 & 0 & 1 \end{bmatrix}$

Operating $R_{32(1)}$, we get $\sim \begin{bmatrix} 1 & 1 & 1 & : & 0 & 1 & 0 \\ 0 & 4 & 5 & : & 1 & 3 & 0 \\ 0 & 0 & 5 & : & 1 & 3 & 1 \end{bmatrix}$

Operating $R_{2\left(\frac{1}{4}\right)},\ R_{3\left(\frac{1}{5}\right)}$, we get $\sim \begin{bmatrix} 1 & 1 & 1 & : & 0 & 1 & 0 \\ 0 & 1 & \frac{5}{4} & : & \frac{1}{4} & \frac{3}{4} & 0 \\ 0 & 0 & 1 & : & \frac{1}{5} & \frac{3}{5} & \frac{1}{5} \end{bmatrix}$

Operating $R_{23\left(-\frac{5}{4}\right)}$, $R_{3(-1)}$, we get $\sim \begin{bmatrix} 1 & 1 & 0 & : & -\frac{1}{5} & \frac{2}{5} & -\frac{1}{5} \\ 0 & 1 & 0 & : & 0 & 0 & -\frac{1}{4} \\ 0 & 0 & 1 & : & \frac{1}{5} & \frac{3}{5} & \frac{1}{5} \end{bmatrix}$

Operating $R_{12(-1)}$, we get $\sim \begin{bmatrix} 1 & 1 & 0 & : & -\frac{1}{5} & \frac{2}{5} & \frac{1}{20} \\ 0 & 1 & 0 & : & 0 & 0 & -\frac{1}{4} \\ 0 & 0 & 1 & : & \frac{1}{5} & \frac{3}{5} & \frac{1}{5} \end{bmatrix}$

Thus $P^{-1} = \frac{1}{20}\begin{bmatrix} -4 & 8 & 1 \\ 0 & 0 & -5 \\ 4 & 12 & 4 \end{bmatrix}$.

Diagonalisation:

$$D = P^{-1}AP = \frac{1}{20}\begin{bmatrix} -4 & 8 & 1 \\ 0 & 0 & -5 \\ 4 & 12 & 4 \end{bmatrix}\begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} -3 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & -4 & 0 \end{bmatrix}$$

$$= \frac{1}{20}\begin{bmatrix} 4 & -8 & -1 \\ 0 & 0 & -15 \\ 16 & 48 & 16 \end{bmatrix}\begin{bmatrix} -3 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & -4 & 0 \end{bmatrix}$$

$$= \frac{1}{20}\begin{bmatrix} -20 & 0 & 0 \\ 0 & 60 & 0 \\ 0 & 0 & 80 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

To find A^8 :

Now $A^8 = PDP^{-1} = \begin{bmatrix} -3 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & -4 & 0 \end{bmatrix}\begin{bmatrix} (-1)^8 & 0 & 0 \\ 0 & 3^8 & 0 \\ 0 & 0 & 4^8 \end{bmatrix}\frac{1}{20}\begin{bmatrix} -4 & 8 & 1 \\ 0 & 0 & -5 \\ 4 & 12 & 4 \end{bmatrix}$

$$= \begin{bmatrix} -3 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & -4 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 6561 & 0 \\ 0 & 0 & 65536 \end{bmatrix}\frac{1}{20}\begin{bmatrix} -4 & 8 & 1 \\ 0 & 0 & -5 \\ 4 & 12 & 4 \end{bmatrix}$$

$$=\frac{1}{20}\begin{bmatrix}-3 & 6561 & 131072\\ 1 & 6561 & 65536\\ 0 & -26244 & 0\end{bmatrix}\begin{bmatrix}\begin{bmatrix}-4 & 8 & 1\\ 0 & 0 & 5\\ 4 & 12 & 4\end{bmatrix}\end{bmatrix}=\frac{1}{20}\begin{bmatrix}524300 & 1572840 & 491480\\ 262140 & 786440 & 229340\\ 0 & 0 & 131220\end{bmatrix}$$

$$A^8=\begin{bmatrix}26215 & 78642 & 24574\\ 13107 & 39322 & 11467\\ 0 & 0 & 6561\end{bmatrix}.\text{ Ans}$$

Q.No.4.: Find a matrix P, which transforms the matrix $A=\begin{bmatrix}1 & 1 & 3\\ 1 & 5 & 1\\ 3 & 1 & 1\end{bmatrix}$ to **diagonal form**.

Hence, calculate A^4.

Sol.: *Since we know, if a square matrix A of or order n has n linearly independent eigen vectors, then a matrix P can be found such that* $P^{-1}AP$ *is a diagonal matrix.*

The eigen values of A are $-2,\ 3,\ 6$ and

the eigen vectors are $(-1,\ 0,\ 0),\ (1,\ -1,\ 1),\ (1, 2, 1)$.

Writing these eigen vectors as the three columns, the required transformation matrix (modal matrix) is

$$P=\begin{bmatrix}-1 & 1 & 1\\ 0 & -1 & 2\\ 1 & 1 & 1\end{bmatrix}.$$

To find P^{-1}: $|P|=\begin{vmatrix}-1 & 1 & 1\\ 0 & -1 & 2\\ 1 & 1 & 1\end{vmatrix}=\begin{vmatrix}a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{vmatrix}$ (say).

$A_1-3,\ B_1=2,\ C_1=1,\ A_2=0,\ B_2=-2,\ C_2=2,\ A_3=3,\ B_2=3,\ C_3=1.$

Also $|P|=a_1A_1+b_1B_1+c_1C_1=6.$

$$\therefore P^{-1}=\frac{1}{|P|}\begin{bmatrix}A_1 & A_2 & A_3\\ B_1 & B_2 & B_3\\ C_1 & C_2 & C_3\end{bmatrix}=\frac{1}{6}\begin{bmatrix}-3 & 0 & 3\\ 2 & -2 & 2\\ 1 & 2 & 1\end{bmatrix}.$$

Thus $D = P^{-1}AP = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$.

$$\therefore D^4 = \begin{bmatrix} (-2)^4 & 0 & 0 \\ 0 & 3^4 & 0 \\ 0 & 0 & 6^4 \end{bmatrix} = \begin{bmatrix} 16 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 1296 \end{bmatrix}.$$

$$\text{Hence, } A^4 = PD^4P^{-1} = \frac{1}{6}\begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 16 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 1296 \end{bmatrix}\begin{bmatrix} -3 & 0 & 3 \\ 2 & -2 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} -8 & 0 & 8 \\ 27 & -27 & 27 \\ 216 & 512 & 216 \end{bmatrix} = \begin{bmatrix} 251 & 485 & 235 \\ 485 & 1051 & 485 \\ 235 & 485 & 251 \end{bmatrix}.$$

Reduction of quadratic form to Canonical form by linear transformation

Q.No.1.: Reduce $3x^2 + 3z^2 + 4xy + 8xz + 8yz$ into **canonical form**.

or

Diagonalise the **quadratic form** $3x^2 + 3z^2 + 4xy + 8xz + 8yz$ by linear transformation and write the linear transformation.

or

Reduce the **quadratic form** $3x^2 + 3z^2 + 4xy + 8xz + 8yz$ into "sum of squares".

Sol.: The given quadratic form can be written as $X'AX$,

where $X' = [x, y, z]$ and the symmetric matrix $A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 4 \\ 4 & 4 & 3 \end{bmatrix}$.

Let us reduce A into diagonal matrix.

We know that $A = I_3 A I_3$, i.e., $\begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 4 \\ 4 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Operating $R_2 \to R_2 - \frac{2}{3}R_1,\ \ R_3 \to R_3 - \frac{4}{3}R_1$, we get

$$\begin{bmatrix} 3 & 2 & 4 \\ 0 & -\frac{4}{3} & \frac{4}{3} \\ 0 & \frac{4}{3} & -\frac{7}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{2}{3} & 1 & 0 \\ -\frac{4}{3} & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Operating $C_2 \to C_2 - \frac{2}{3}C_1,\ \ C_3 \to C_3 - \frac{4}{3}C_1$, we get

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & -\frac{4}{3} & \frac{4}{3} \\ 0 & \frac{4}{3} & -\frac{7}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{2}{3} & 1 & 0 \\ -\frac{4}{3} & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -\frac{2}{3} & -\frac{4}{3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Operating $R_3 \to R_3 + R_2$, we get $\begin{bmatrix} 3 & 0 & 0 \\ 0 & -\frac{4}{3} & \frac{4}{3} \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{2}{3} & 1 & 0 \\ -2 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -\frac{2}{3} & -\frac{4}{3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Operating $C_3 \to C_3 + C_2$, we get $\begin{bmatrix} 3 & 0 & 0 \\ 0 & -\frac{4}{3} & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{2}{3} & 1 & 0 \\ -2 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -\frac{2}{3} & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

$\Rightarrow \text{Diag.}\left(3,\ -\frac{4}{3},\ -1\right) = P'AP$.

$\therefore$ The canonical form of the given quadratic form is

$$Y'(P'AP)Y = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & -\frac{4}{3} & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = 3y_1^2 - \frac{4}{3}y_2^2 - y_3^2.$$

Here Rank of A = 3, Index = 1, Signature = $1-2=-1$.

Remarks: In this problem the non-singular transformation which reduces the given quadratic form into the canonical form is X = PY

i.e., $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -\frac{2}{3} & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$,

i.e., $x = y_1 - \frac{2}{3}y_2 - 2y_3$, $y = y_2 + y_3$, $z = y_3$.

Q.No.2.: Reduce the **quadratic form** $x^2 - 4y^2 + 6z^2 + 2xy - 4xz + 2w^2 - 6zw$ into the **"sum of squares".**

Sol.: The matrix form of the given quadratic form is $X'AX$,

where $X' = [x \quad y \quad z \quad w]$ and $A = \begin{bmatrix} 1 & 1 & -2 & 0 \\ 1 & -4 & 0 & 0 \\ -2 & 0 & 6 & -3 \\ 0 & 0 & -3 & 2 \end{bmatrix}$.

Let us reduce A to the diagonal matrix.

We know that $A = I_4 A I_4 \Rightarrow \begin{bmatrix} 1 & 1 & -2 & 0 \\ 1 & -4 & 0 & 0 \\ -2 & 0 & 6 & -3 \\ 0 & 0 & -3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Operating $R_{21} - R_1$, $R_3 + 2R_1$, we get $\begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & -5 & 2 & 0 \\ 0 & 2 & 2 & -3 \\ 0 & 0 & -3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Operating $C_2 - C_1$, $C_3 + 2C_1$, we get $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -5 & 2 & 0 \\ 0 & 2 & 2 & -3 \\ 0 & 0 & -3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Operating, $R_3 + \frac{2}{5}R_2$, we get $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -5 & 2 & 0 \\ 0 & 2 & \frac{14}{5} & -3 \\ 0 & 0 & -3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ \frac{8}{5} & \frac{2}{5} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Operating, $C_3 + \frac{2}{5}C_2$ we get $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 2 & \frac{14}{5} & -3 \\ 0 & 0 & -3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ \frac{8}{5} & \frac{2}{5} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & \frac{8}{5} & 0 \\ 0 & 1 & \frac{2}{5} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Operating, $R_4 + \frac{15}{14}C_2$, we get $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & \frac{14}{5} & -3 \\ 0 & 0 & -3 & -\frac{17}{14} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ \frac{8}{5} & \frac{2}{5} & 1 & 0 \\ \frac{12}{7} & \frac{3}{7} & \frac{15}{14} & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & \frac{8}{5} & 0 \\ 0 & 1 & \frac{2}{5} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Operating, $C_4 + \frac{15}{14}C_3$, we get $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & \frac{14}{5} & 0 \\ 0 & 0 & 0 & -\frac{17}{14} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ \frac{8}{5} & \frac{2}{5} & 1 & 0 \\ \frac{12}{7} & \frac{3}{7} & \frac{15}{14} & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & \frac{8}{5} & \frac{12}{7} \\ 0 & 1 & \frac{2}{5} & \frac{3}{7} \\ 0 & 0 & 1 & \frac{15}{14} \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

i.e., $\text{diag.}\left(1, -5, \frac{14}{5}, -\frac{17}{14}\right) = P'AP$.

$\therefore$ The canonical form of the given quadratic form is

$$Y'(P'AP)Y = Y'\,\text{diag.}\left(1, -5, \frac{14}{5}, -\frac{17}{14}\right)Y = \begin{bmatrix} y_1 & y_2 & y_3 & y_4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & \frac{14}{5} & 0 \\ 0 & 0 & 0 & -\frac{17}{14} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

$$= y_1^2 - 5y_2^2 + \frac{14}{5}y_3^2 - \frac{17}{14}y_4^2,$$

which is the sum of squares.

Remarks: Here rank of A = 4

Index = 2

Signature = 2 – 2 = 0.

Reduction of quadratic form to Canonical form by Orthogonal Transformation:

Q.No.1.: Reduce $8x^2 + 7y^2 + 3z^2 - 12xy + 4xz - 8yz$ into **canonical form** by **orthogonal transformation.**

Sol.: The matrix of the quadratic form is $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$.

The characteristic of A are given by $|A - \lambda I| = 0$

i.e., $\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0 \Rightarrow \lambda(\lambda - 15) = 0$.

$\therefore \lambda = 0,\ 3,\ 15$.

Characteristic vector for $\lambda = 0$ is given by $[A - (0)I]X = O$.

i.e., $8x_1 - 6x_2 + 2x_3 = 0$

$-6x_1 + 7x_2 - 4x_3 = 0$

$2x_1 - 4x_2 + 3x_3 = 0$.

Solving first two, we get $\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{3}$ giving the eigen vector $X_1 = (1, 2, 2)'$.

When $\lambda = 3$, the corresponding characteristic vector is given by $[A - (3)I]X = O$.

i.e., $5x_1 - 6x_2 + 2x_3 = 0$

$-6x_1 + 4x_2 - 4x_3 = 0$

$2x_1 - 4x_2 = 0$

Solving any two equations, we get $X_2 = (2, 1, -2)'$.

Similarly, characteristic vector corresponding to $\lambda = 15$ is $X_3 = (2, -2, 1)'$.

Now X_1, X_2, X_3 are pairwise orthogonal, i.e., $X_1 . X_2 = X_2 . X_3 = X_3 . X_1 = 0$.

$\therefore$ The normalized modal matrix is $B = \left[\frac{X_1}{\|X_1\|}, \frac{X_2}{\|X_2\|}, \frac{X_3}{\|X_3\|}\right] = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} \end{bmatrix}$.

Now B is the orthogonal matrix i.e., $B^{-1} = B^T$ and $|B| = 1$.

Now $B^{-1}AB = D$ = diag (0, 3, 15)

$$\Rightarrow \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} \end{bmatrix} A \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}$$

Now $X'AX = Y'\left(B^{-1}AB\right)Y = Y'DY = [y_1, \ y_2, \ y_3]\begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = 0y_1^2 + 3y_2^2 + 15y_3^2$,

which is the required canonical form

Note: Here the orthogonal transformation is X = BY

Rank of quadratic form = 2

Index = 2

Signature = 2, it is a positive semi-definite.

Q.No.2.: Reduce $6x_1^2 + 3x_2^2 + 3x_3^2 - 4x_1x_2 - 2x_2x_3 + 4x_3x_1$ into **canonical form** by **orthogonal transformation.**

Sol.: The matrix of the quadratic form is $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$.

The characteristic roots are given by $|A - \lambda I| = 0$

$$\Rightarrow \begin{bmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{bmatrix} = 0$$

$\Rightarrow \lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$, which on solving gives $\lambda = 8, \ 2, \ 2$.

The vector corresponding to $\lambda = 8$ is given by $[A - 8I]X = O$

$$\Rightarrow \begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -2x_1 - 2x_2 + 2x_3 = 0$$

$$-2x_1 - 5x_2 - x_3 = 0$$

$$2x_1 - x_2 - 5x_3 = 0$$

Solving any two of the equations, we get the vectors as $[2, -1, 1]'$.

The characteristic vector for $\lambda = 2$ is given by $[A - 2I]X = O$, which reduces to single equation

$2x_1 - x_2 + x_3 = 0$.

Putting $x_1 = 0$, we get $\frac{x_2}{1} = \frac{x_3}{1}$ or vectors is $[0, 1, 1]'$.

Again by putting $x_2 = 0$, we get $\frac{x_1}{1} = \frac{x_3}{-2}$ or the vectors $[2, 0, -2]'$

Now $X_1 = [2, -1, 1]'$; $X_2 = [0, 1, 1]'$ and $X_3 = [2, 0, -2]'$

Here X_1, X_2, X_3 are not pairwise orthogonal.

$\because X_1.X_2 = 0; X_2.X_3 \neq 0$ and $X_3 . X_1 = 0$

To get X_3 orthogonal to X_2 assume a vector $[u, v, w]'$ orthogonal to X_2 also satisfying

$2x_1 - x_2 + x_3 = 0$ i.e. $2u - v + w = 0$ and $0.u + 1.v + 1.w = 0$

Solving $[u, v, w]' = [1, 1, -1]' = X_3$ so that $X_1.X_2 = X_3 = X_3.X_1 = 0$.

$$\therefore \text{ The normalized modal matrix is } B = \begin{bmatrix} \frac{1}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \end{bmatrix}.$$

Now B is orthogonal matrix and $|B| = 1$.

i.e. $B' = B^{-1}$ and $B^{-1}AB = D$, where $D = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

$$\therefore X'AX = Y'\left(B^{-1}AB\right)Y = Y'AB = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}\begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = 8y_1^2 + 2y_2^2 + 2y_3^2,$$

which is the required canonical form.

Here Rank of the quadratic form is 3, Index = 3, signature = 3. It is positive definite.

Q.No.3.: Find the orthogonal transformation which transforms the quadratic form $x_1^2 + 3x_2^2 + 3x_3^2 - 2x_2x_3$ to canonical form (or "sum of squares form" or "principal axes form"). Determine the index, signature and nature of the quadratic form.

Sol.: Let $X = [x_1 x_2 x_3]^T$, $Y = [y_1 y_2 y_3]^T$.

Let P be the non-singular orthogonal matrix, containing the three eigen vectors of the coefficient matrix A of the given quadratic form. Then $X = \hat{P}Y$ is the required non-singular linear transformation that transforms (reduces) the given quadratics form to canonical form. Here $\hat{P}$ is the normalized modal matrix P.

The coefficient matrix A of the given quadratic form is $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$.

Its characteristic equation is $|A - \lambda I| = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 3-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{vmatrix} = 0$

$\Rightarrow \lambda^3 - 7\lambda^2 + 14\lambda - 8 = (\lambda - 1)(\lambda - 2)(\lambda - 4) = 0$.

So there are three distinct real eigen values $\lambda = 1,\ 2,\ 4$ of A.

For $\lambda = 1$:

$$\begin{matrix} 0 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{matrix} \sim \begin{matrix} 2x_2 = x_3 \\ x_2 = 2x_3 \end{matrix}$$

$\therefore x_2 = x_3 = 0$, x_1 = arbitrary,

The eigen vector X_1 associated with $\lambda = 1$ is $X_1 = [1 \quad 0 \quad 0]^T$.

For $\lambda = 2$:

$-x_1 + 0 + 0 = 0$, $x_2 - x_3 = 0$, $-x_2 + x_3 = 0$

$\therefore \quad x_1 = 0,\ x_2 = x_3$

The eigen vector X_1 associated with $\lambda = 2$ is $X_2 = [0 \quad 1 \quad 1]^T$.

For $\lambda = 3$

$$\begin{matrix} -3 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{matrix} \sim \begin{matrix} x_1 = 0 \\ x_2 = -x_3 \end{matrix}$$

The eigen vector X_1 associated with $\lambda = 3$ is $X_3 = [0 \quad 1 \quad -1]^T$.

Thus, the nodal matrix P is $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$.

The norm of eigen vector X_1 is

$\|X_1\| = \sqrt{1^2 + 0 + 0} = 1$,

$\|X_2\| = \sqrt{0 + 1^2 + 1^2} = \sqrt{2}$,

$\|X_3\| = \sqrt{1^2 + 1^2} = \sqrt{2}$.

Then, the normalized modal matrix $\hat{P}$ is $\hat{P} = \begin{bmatrix} \frac{1}{1} & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$.

To find inverse of P:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \sim \begin{matrix} R_{32(-1)} \\ R_{3\left(-\frac{1}{2}\right)} \\ R_{23(-1)} \end{matrix} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} \end{array}\right]$$

Thus $P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$ and the normalized P^{-1} is $P^{-1} = \frac{1}{\sqrt{2}}\begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$.

Diagonalisation:

$$\hat{P}^{-1} A \hat{P} = \frac{1}{\sqrt{2}}\begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix} \times \frac{1}{\sqrt{2}}\begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 4 & -4 \end{bmatrix}\begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

Then $\hat{P}^{-1} A \hat{P} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} = D =$ diagonal matrix

with the eigen values of A as the diagonal elements.

Transformation (Reduction) to canonical form:

Quadratic form (QF)

$$Q = x_1^2 + 3x_2^2 + 3x_3^2 - 2x_2x_3 = [x_1 \quad x_2 \quad x_3]\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = X^T A X$$

Put $X = \hat{P} Y$ and $X^T = \left(\hat{P} Y\right)^T = Y^T \hat{P}^T$.

So $Q = X^T A X = Y^T \hat{P}^T A \hat{P} Y = Y^T \left(\hat{P}^T A \hat{P}\right) Y$.

But we know that $\hat{P}$ is an orthogonal matrix, because

$$\hat{P}\hat{P}^T = \frac{1}{\sqrt{2}}\begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \frac{1}{\sqrt{2}}\begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Thus $\hat{P}^T = \hat{P}^{-1}$

So $Q.F. = X^T AX = Y^T\left(\hat{P}^{-1} A\hat{P}\right)Y$.

But through Diagonalisation $\hat{P}^{-1} A\hat{P} = D$.

Therefore $Q = X^T A X = Y^T DY = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$$= \begin{bmatrix} y_1 & 2.y_2 & 4y_3 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = y_1^2 + 2y_2^2 + 4y_3^2.$$

This is the required canonical form (or sum of squares form).

Orthogonal transformation:

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x3 \end{bmatrix} = \hat{P}Y = \frac{1}{\sqrt{2}}\begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

So $x_1 = y_1$, $x_2 = \frac{1}{\sqrt{2}}(y_2 + y_3)$, $x_3 = \frac{1}{\sqrt{2}}(y_2 - y_3)$ is the orthogonal transformation which reduces the QF to the canonical form.

Index is 3 for the QF since the number of positive terms in canonical form is 3 i.e. S = 3, Rank $r = 3$. The number of variables is n = 3.

Signature of the QF is $2s - r = 6 - 3 = 3$ (difference between number of positive terms and negative terms in CF).

The given QF is positive definite because r = 3 = n and s = 3 = n.

Q.No.4.: Reduce the quadratic form $3x^2 + 5y^2 + 3z^2 - 2yz + 2zx - 2xy$ to the **canonical form**. Also specify the matrix of transformation.

Sol.: The matrix of the given quadratic form is $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$.

Its characteristic equation is $|A-\lambda I|=0$ i.e. $\begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix}$,

which gives $\lambda = 2, 3, 6$ as its eigen values.

Hence, the given quadratic form reduces to the canonical form

$\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2$ i.e. $2x^2 + 3y^2 + 6z^2$.

To find the matrix of transformation from $[A-\lambda I]X = 0$, we obtain the equations

$(3-\lambda)x - y + z = 0$; $-x + (5-\lambda)y - z = 0$; $x - y + (3-\lambda)z = 0$

Now corresponding to $\lambda = 2$, we get $x - y + z = 0$, $-x + 3y - z = 0$ and $x - y + z = 0$,

whence $\dfrac{x}{1} = \dfrac{y}{0} = \dfrac{z}{-1}$

$\therefore$ The eigen vector is $(1, 0, -1)$ and its normalized form is $\left(\dfrac{1}{\sqrt{2}},\ 0,\ -\dfrac{1}{\sqrt{2}}\right)$

Similarly corresponding to $\lambda = 3$, the eigen vectors is (1, 1, 1) and its normalized form is $\left(\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}\right)$.

Finally, corresponding to $\lambda = 6$, the eigen vectors is $(1,\ -2,\ 1)$ and its normalized form is $\left(\dfrac{1}{\sqrt{6}}, -\dfrac{2}{\sqrt{6}}, \dfrac{1}{\sqrt{6}}\right)$..

Hence, the matrix of transformation is $P = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}$

Index of the quadratic form = 3. Its signature is also 3.

Q.No.5.: If $X_1 = \frac{1}{3}[2 \quad -1 \quad 2]^T$ and $X_2 = k[3 \quad -4 \quad -5]^T$,

where $k = \dfrac{1}{\sqrt{50}}$, construct an orthogonal matrix $A = [X_1 \quad X_2 \quad X_3]$.

Sol.: Let $X_3 = [a_1 \quad a_2 \quad a_3]^T$ be the undetermined vector. Since A is orthogonal, the columns vectors of A form an orthogonal system $X_i^T X_j = \delta_{ij}$

$$X_1^T X_2 = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 3k \\ -4k \\ -5k \end{bmatrix} = 2k + \frac{4}{3} - \frac{10}{3}k = 0, \text{ true}$$

$\therefore$ X_1 and X_2 are orthogonal.

$$X_1^T X_3 = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \frac{1}{3}[2a_1 - a_2 + 2a_3] = 0. \qquad \text{(i)}$$

$$X_2^T X_3 = [3k \quad -4k \quad -5k] \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = [3a_1 - 4a_2 - 5a_3]k = 0. \qquad \text{(ii)}$$

Since X_3 should b e normalized

$$X_3^T X_3 = [a_1 \quad a_2 \quad a_3] \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = a_1^2 + a_2^2 + a_3^2$$

$$I = \|X_3\| = \sqrt{X_3^T X_3} = \sqrt{a_1^2 + a_2^2 + a_3^2}. \qquad \text{(iii)}$$

Solving (i), (ii), (iii), we get a_1, a_2, a_3

$$2a_1 - a_2 + 2a_3 = 0$$

$$3a_1 - 4a_2 + 5a_3 = 0$$

$$a_1^2 + a_2^2 + a_3^2 = 1$$

So $a_1 = -\frac{13}{5}a_3$, $\quad a_2 = -\frac{16}{5}a_3$, $\quad a_3^2 = \frac{25}{550}$ $\quad a_3 = \frac{1}{\sqrt{22}}$

$\therefore a_1 = -\frac{13}{5}k_1$, $\quad a_2 = -\frac{16}{5}k_1$, $a_3 = k_1$, $\quad$ where $k = \frac{1}{\sqrt{50}}$

Thus, the required orthogonal matrix A is $A = \begin{bmatrix} \frac{2}{3} & 3k & -\frac{13}{5}k_1 \\ -\frac{1}{3} & -4k & -\frac{16}{5}k_1 \\ \frac{2}{3} & -5k & k_1 \end{bmatrix}$.

Reduction of quadratic form to Canonical form by Lagrange's reduction trasformation:

Q.No.13.: By Lagrange's reduction transform the quadratic form X^TAX to "sum of squares" form for $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & -2 \\ 4 & -2 & 18 \end{bmatrix}$.

Sol.: $QF = X^TAX = [x_1 \quad x_2 \quad x_3]\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & -2 \\ 4 & -2 & 18 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$QF = [x_1 + 2x_2 + 4x_3 \quad 2x_1 + 6x_2 - 2x_3 \quad 4x_1 - 2x_2 + 18x_3] \times \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= x_1^2 + 6x_2^2 + 18x_3^2 + 4x_1x_2 + 8x_1x_3 - 4x_2x_3$$

$$= \left[x_1^2 + 4x_1(x_2 + 2x_3)\right] + 6x_2^2 + 18x_3^2 - 4x_2x_3$$

$$= \left[x_1^2 + 4x_1(x_2 + 2x_3) + 2^2(x_2 + 2x_3)^2\right] - 2^2(x_2 + 2x_3)^2 + 6x_2^2 + 18x_3^2 - 4x_2x_3$$

$$= [x_1 + 2(x_2 + 2x_3)]^2 + 2x_2^2 + 2x_3^2 - 20x_2x_3$$

$$= [x_1 + 2(x_2 + 2x_3)]^2 + 2\left[x_2^2 - 10x_2x_3\right] + 2x_3^2$$

$$= [x_1 + 2(x_2 + 2x_3)]^2 + 2\left[x_2^2 - 10x_2x_3 + 5^2x_3^2\right] - 2.5^2x_3^2 + 2x_3^2$$

$$= [x_1 + 2(x_2 + 2x_3)]^2 + 2[x_2 - 5x_3]^2 - 48x_3^2$$

$QF = y_1^2 + 2y_2^2 - 48y_3^2$,

where $y_1 = x_1 + 2(x_2 + 2x_3)$, $y_2 = x_2 - 5x_3$, $y_3 = x_3$.

Index: $S = 2$, $(n = 3,\ r = 3)$,

Signature: $2s - r = 2.2 - 3 = 1$ (or $2 - 1 = 1$).

Home Assignments

Reduction to diagonal form

Q.No.1.: Diagonalise the matrices. Find the modal matrix P, which diagonalises (transforms) $A = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$.

Ans.: $P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $D = \begin{bmatrix} 8 & 0 \\ 0 & 2 \end{bmatrix}$.

Q.No.2.: Diagonalise the matrices. Find the modal matrix P, which diagonalises (transforms) $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

Ans.: Not diagonalizable since only one eigen vector $\begin{bmatrix} k \\ 0 \end{bmatrix}$ exists.

Q.No.3.: Diagonalise the matrices. Find the modal matrix P, which diagonalises (transforms) $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$.

Ans.: $P = \begin{bmatrix} 4 & 1 \\ 1 & -1 \end{bmatrix}$, $D = \begin{bmatrix} 6 & 0 \\ 0 & 1 \end{bmatrix}$.

Q.No.4.: Diagonalise the matrices. Find the modal matrix P, which diagonalises (transforms) $A = \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}$.

Ans.: $P = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 6 & 0 \\ 0 & 1 \end{bmatrix}$.

Q.No.5.: Diagonalise the matrices. Find the modal matrix P, which diagonalises (transforms) $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$.

Ans.: $P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $D = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$.

Q.No.6.: Diagonalise the matrices. Find the modal matrix P, which diagonalises

(transforms) $A = \begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix}$.

Ans.: $P = \begin{bmatrix} 1 & 2 \\ -1 & 5 \end{bmatrix}$, $D = \begin{bmatrix} -1 & 0 \\ 0 & 6 \end{bmatrix}$.

Q.No.7.: Diagonalise the matrices. Find the modal matrix P which diagonalises

(transforms) $A = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$, hence find A^5

Ans.: $P = \begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$, $A^5 = \begin{bmatrix} 2344 & 781 \\ 2343 & 782 \end{bmatrix}$.

Q.No.8.: Diagonalise the matrices. Find the modal matrix P which diagonalises

(transforms) $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$.

Ans.: No real eigen values, $\lambda = 1 + i$, so not diagonalizable over real.

Modal matrix over complex $\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}$, $D = \begin{bmatrix} 1+i & 0 \\ 0 & 1+i \end{bmatrix}$.

Q.No.9.: Diagonalise the matrices. Find the modal matrix P, which diagonalises

(transforms) $A = \begin{bmatrix} 7.3 & 0.2 & -3.7 \\ -11.5 & 1.0 & 5.5 \\ 17.7 & 1.8 & -9.3 \end{bmatrix}$.

Ans.: Characteristic equation $\lambda^3 + \lambda^2 - 12\lambda = 0$, eigen values $3, -4, 0$.

Modal matrix $= \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$, $D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Q.No.10.: Diagonalise the matrices. Find the modal matrix P which diagonalises

(transforms) $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$, hence find A^4.

Ans.: Characteristic equation $(\lambda+2)(\lambda-3)(\lambda-6) = 0$, $\lambda = -2, 3, 6$, Modal matrix $P=\begin{bmatrix}-1 & 1 & 1\\ 0 & -1 & 2\\ 1 & 1 & 1\end{bmatrix}$, $D=\begin{bmatrix}-2 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 6\end{bmatrix}$, $A^4=\begin{bmatrix}251 & 485 & 235\\ 485 & 1051 & 485\\ 235 & 485 & 251\end{bmatrix}$.

Q.No.11.: Diagonalise the matrices. Find the modal matrix P, which diagonalises

(transforms) $A=\begin{bmatrix}8 & -6 & 2\\ -6 & 7 & -4\\ 2 & -4 & 3\end{bmatrix}$.

Ans.: $\lambda^3-18\lambda^2+45\lambda=0$**,** $\lambda=0,3,15$**,** $P=\begin{bmatrix}1 & 2 & 2\\ 2 & 1 & -2\\ 2 & -2 & 1\end{bmatrix}$, $D=\begin{bmatrix}0 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 15\end{bmatrix}$.

Q.No.12.: Diagonalise the matrices. Find the modal matrix P, which diagonalises

(transforms) $A=\begin{bmatrix}7 & -2 & 1\\ -2 & 10 & -2\\ 1 & -2 & 7\end{bmatrix}$.

Ans.: $\lambda^3-24\lambda^2+180\lambda-432=0$**,** $\lambda=6,6,12$**,** $P=\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}}\\ 0 & \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}}\\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}}\end{bmatrix}$.

Q.No.13.: Diagonalise the matrices. Find the modal matrix P, which diagonalises

(transforms) $\begin{bmatrix}+1 & -3 & 2\\ -3 & 7 & -5\\ 2 & -5 & 8\end{bmatrix}$.

Ans.: $\lambda=1,-2,18$**,** $P=\begin{bmatrix}1 & 3 & -1\\ 0 & 1 & 1\\ 0 & 0 & 2\end{bmatrix}$**.**

Q.No.14.: Find A^8 for $A=\begin{bmatrix}1 & 1 & 1\\ 0 & 2 & 1\\ -4 & 4 & 3\end{bmatrix}$.

Ans.: $(1-\lambda)(\lambda-2)(\lambda-3)=0$, $\lambda=1,2,3$, $P=\begin{bmatrix}1 & 1 & 1\\ 2 & 1 & 1\\ -2 & 0 & 1\end{bmatrix}$, $A^8=\begin{bmatrix}-12099 & 12355 & 6305\\ -12100 & 12356 & 6305\\ -13120 & 13120 & 6561\end{bmatrix}$.

Q.No.15.: Find A^5 for $A=\begin{bmatrix}11 & -4 & -7\\ 7 & -2 & -5\\ 10 & -4 & -6\end{bmatrix}$.

Ans.: $\lambda=0,1,2$, $P=\begin{bmatrix}1 & 1 & -2\\ 1 & -1 & -1\\ 1 & 2 & -2\end{bmatrix}$, $A^5=\begin{bmatrix}191 & -64 & -127\\ 97 & -32 & -65\\ 190 & -64 & -126\end{bmatrix}$.

Q.No.16.: Find A^4 for $A=\begin{bmatrix}3 & -1 & 1\\ -1 & 5 & -1\\ 1 & -1 & 3\end{bmatrix}$.

Ans.: $\lambda=2,3,6$, $P=\begin{bmatrix}1 & 1 & 1\\ 0 & 1 & -2\\ -1 & 1 & 1\end{bmatrix}$, $A^4=\begin{bmatrix}251 & -405 & 235\\ -405 & 891 & -405\\ 235 & -405 & 251\end{bmatrix}$.

Problems related to quadratic form and canonical forms:

Q.No.1.: Write down the **quadratic forms** corresponding to following matrices:

(i) $\begin{bmatrix}2 & 4 & 5\\ 4 & 3 & 1\\ 6 & 1 & 1\end{bmatrix}$, **(ii)** $\begin{bmatrix}1 & 1 & -2 & 0\\ 1 & -4 & 0 & 0\\ -2 & 0 & 6 & -3\\ 0 & 0 & -3 & 2\end{bmatrix}$.

Ans.: **(i)** $2x^2+3y^2+z^2+8xy+2yz+10zx$

(ii) $x_1^2-4x_2^2+6x_3^2+2x_4^2+2x_1x_2-4x_1x_3-6x_1x_4$

Q.No.2.: Write down the matrices of the following **quadratic form**:

(i) $2x^2+3y^2+6xy$

(ii) $2x^2+5y^2-6z^2-2xy-yz+8zx$

(iii) $x_1^2+2x_2^2+3x_3^2+4x_4^2+2x_1x_2+4x_1x_3-6x_1x_4-4x_2x_3-8x_2x_4+12x_3x_4$

Ans.: **(i)** $\begin{bmatrix} 2 & 3 \\ 3 & 3 \end{bmatrix}$, **(ii)** $\begin{bmatrix} 2 & -1 & 4 \\ -1 & 5 & -\frac{1}{2} \\ 4 & -\frac{1}{2} & -6 \end{bmatrix}$, **(iii)** $\begin{bmatrix} 1 & 1 & 2 & -3 \\ 1 & 2 & -2 & -4 \\ 2 & -2 & 3 & 6 \\ -3 & -4 & 6 & 4 \end{bmatrix}$.

Q.No.3.: Find real symmetric matrix C such that $Q = X^TCX$, where

$$Q = 6x_1^2 - 4x_1x_2 + 2x_2^2.$$

Ans.: $\begin{pmatrix} 6 & -2 \\ -2 & 2 \end{pmatrix}$.

Q.No.4.: Find real symmetric matrix C such that $Q = X^TCX$, where $Q = 2(x_1 - x_2)^2$.

Ans.: $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$.

Q.No.5.: Find real symmetric matrix C such that $Q = X^TCX$, where

$$Q = (x_1 + x_2 + x_3)^2.$$

Ans.: $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

Q.No.6.: Find real symmetric matrix C such that $Q = X^TCX$, where

$$Q = 4x_1x_3 + 2x_2x_3 + x_3^2.$$

Ans.: $\begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix}$.

Determine the nature, index and signature of the quadratic form

Q.No.1.: Determine the nature, index and signature of the quadratic form

$$x_1^2 + 2x_2^2 + 3x_3^2 + 2x_2x_3 - 2x_3x_1 + 2x_1x_2.$$

Ans.: Indefinite, Eigen value: $1,\ 1,\ -2$, Index : 2, Signature : 1.

Q.No.2.: Determine the nature, index and signature of the quadratic form

$$5x_1^2 + 26x_2^2 + 10x_3^2 + 4x_2x_3 + 14x_3x_1 + 6x_1x_2.$$

Ans.: Positive semi-definite, Eigen value: $5,\ 0,\ 5$, Index : 3, Signature : 3.

Q.No.3.: Determine the nature, index and signature of the quadratic form

$$x_1^2 + 5x_2^2 + x_3^2 + 2x_1x_2 + 2x_2x_3 + 6x_3x_1.$$

Ans.: Indefinite, Eigen value: $-2,\ 3,\ 6$, Index: 2, Signature : 1.

Q.No.4.: Determine the nature, index and signature of the quadratic form

$$3x_1^2 + 5x_2^2 + 3x_3^2 - 2x_2x_3 + 2x_3x_1 - 2x_1x_2.$$

Ans.: Positive definite, Eigen value: $2,\ 3,\ 6$, Index : 3, Signature : 3.

Q.No.5.: Determine the nature, index and signature of the quadratic form

$$8x_1^2 + 7x_2^2 + 3x_3^2 - 12x_1x_2 - 8x_2x_3 + 4x_3x_2.$$

Ans.: Positive semi-definite, Eigen value: $3,\ 0,\ 15$, Index : 3, Signature : 3.

Q.No.6.: Determine the nature, index and signature of the quadratic form

$$6x_1^2 + 3x_2^2 + 3x_3^2 - 4x_1x_2 - 2x_2x_3 + 4x_1x_3$$

Ans.: Positive definite, Eigen value: $8,\ 2,\ 2$, Index : 3, Signature : 3.

Q.No.7.: Determine the nature, index and signature of the quadratic form

$$-4x_1^2 - 2x_2^2 - 13x_3^2 - 4x_1x_2 - 8x_2x_3 - 4x_1x_3.$$

Ans.: Negative definite, Index: 0, Signature : -3.

Q.No.8.: Determine the nature, index and signature of the quadratic form

$$-3x_1^2 - 3x_2^2 - 7x_3^2 - 6x_1x_2 - 6x_2x_3 - 6x_1x_3.$$

Ans.: Negative definite, Index : 0, Signature : -3.

Reduction of quadratic form to canonical form (or "sum of squares form" or "principal axes form") by orthogonal transformation:

Q.No.1.: Transform (reduce) the quadratic form to canonical form (or "sum of squares form" or "principal axes form") by orthogonal transformation. State matrix for transformation (i.e., modal matrix) $17x_1^2 - 30x_1x_2 + 17x_2^2$.

Ans.: $A = \begin{bmatrix} 17 & -15 \\ -15 & +17 \end{bmatrix}$, $\lambda = 2,\ 32$,

$P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \frac{1}{\sqrt{2}}$, CF : $2y_1^2 + 32y_2^2$.

Q.No.2.: Transform (reduce) the quadratic form to canonical form (or "sum of squares form" or "principal axes form") by orthogonal transformation. State matrix for transformation (i.e., modal matrix)

$5x_1^2 + 26x_2^2 + 10x_3^2 + 4x_2x_3 + 6x_1x_2 + 14x_1x_3$

Ans.: $A = \begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix}$, $\lambda = 5, \ \frac{121}{3}, \ 0$, $P = \begin{bmatrix} 1 & -\frac{3}{5} & -\frac{16}{11} \\ 0 & 1 & \frac{1}{11} \\ 0 & 0 & 1 \end{bmatrix}$, CF $5y_1^2 + \frac{121}{3}y_2^2$.

Q.No.3.: Transform (reduce) the quadratic form to canonical form (or "sum of squares form" or "principal axes form") by orthogonal transformation. State matrix for transformation (i.e., modal matrix) $2(x_1x_2 + x_2x_3 + x_3x_1)$; nature of QF.

Ans.: $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & +1 \\ 1 & +1 & 0 \end{bmatrix}$, $\lambda = 2, \ -1, \ -1$, $P = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \end{bmatrix}$,

CF : $2y_1^2 - y_2^2 - y_3^2$

Nature: Indefinite.

Q.No.4.: Transform (reduce) the quadratic form to canonical form (or "sum of squares form" or "principal axes form") by orthogonal transformation. State matrix for transformation (i.e., modal matrix) $2(x_1^2 + x_1x_2 + x_2^2)$.

Ans.: $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, $\lambda = 1, \ 3$,

$P = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, CF: $y_1^2 + 3y_2^2$.

Q.No.5.: Transform (reduce) the quadratic form to canonical form (or "sum of squares form" or "principal axes form") by orthogonal transformation. State matrix for transformation (i.e., modal matrix) $2x_1^2 + x_2^2 - 3x_3^2 - 8x_2x_3 - 4x_3x_1 + 12x_1x_2$, find index.

Ans.: $A = \begin{bmatrix} 2 & 6 & -2 \\ 6 & 1 & -4 \\ -2 & -4 & -3 \end{bmatrix}$, $\lambda = 1, -1, -1$, $P = \begin{bmatrix} a & -3b & \frac{11c}{17} \\ 0 & b & \frac{2b}{17} \\ 0 & 0 & c \end{bmatrix}$,

where $a = \frac{1}{\sqrt{2}}$, $b = \frac{1}{\sqrt{17}}$, $c = \sqrt{\left(\frac{17}{81}\right)}$,

CF: $y_1^2 - y_2^2 - y_3^2$, Index = 1.

Q.No.6.: Transform (reduce) the quadratic form to canonical form (or "sum of squares form" or "principal axes form") by orthogonal transformation. State matrix for transformation (i.e., modal matrix) $3x_1^2 - 2x_2^2 - x_3^2 - 4x_1x_2 + 12x_2x_3 + 8x_1x_3$.

Ans.: $A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & -2 & 6 \\ 4 & 6 & -1 \end{bmatrix}$, $\lambda = 3, 6, -9$,

$P = \frac{1}{3}\begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ -1 & 2 & 2 \end{bmatrix}$, CF: $3y_1^2 + 6y_2^2 - 9y_3^2$.

Q.No.7.: Reduce the following **quadratic forms to canonical forms** or to sum of squares by **orthogonal transformation**. Write also the rank, index and signature.

(i) $2x^2 + 5y^2 + 3z^2 - 2xy - 2yz + zx$

(ii) $2x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 + 2x_1x_3 - 2x_1x_3$

(iii) $3x^2 - 2y^2 - z^2 - 4xy + 8xz + 12yz$

(iv) $x^2 + 3y^2 + 3z^2 - 2yz$.

Ans.: Ans.: (i). $2y_1^2 + 3y_2^2 + 6y_3^2$**;** Rank = 3, Index = 3, signature = 3

(ii). $4y_1^2 + y_2^2 + y_3^2$**;** Rank = 3, Index = 3, signature = 3

(iii). $3y_1^2 + 6y_2^2 - 9y_3^2$**;** Rank = 3, Index = 2, signature = 1

(iv). $y_1^2 + 2y_2^2 - +4y_3^2$**;** Rank = 3, Index = 3, signature = 3

Reduction of quadratic form to canonical form (or "sum of squares form" or "principal axes form") by linear transformation:

Q.No.1.: Reduce the following **quadratic forms** to **canonical forms** or to sum of squares by linear transformation. Write also the rank, index and signature.

(i) $2x^2 + 2y^2 + 3z^2 + 2xy - 4yz - 4zx$

(ii) $12x_1^2 + 4x_2^2 + 5x_3^2 - 4x_2x_3 + 6x_1x_3 - 6x_1x_2$

(iii) $2x^2 + 9y^2 + 6z^2 + 8xy + 8yz + 6zx$

(iv) $x^2 + 4y^2 + z^2 + 4xy + 6yz + 2zx$**.**

Ans.: (i). $2y_1^2 + \frac{3}{2}y_2^2 + \frac{1}{3}y_3^2$**;** Rank = 3, Index = 3, signature = 3

(ii). $12y_1^2 + \frac{13}{4}y_2^2 + \frac{49}{13}y_3^2$**;** Rank = 3, Index = 3, signature = 3

(iii). $2y_1^2 + y_2^2 - \frac{5}{2}y_3^2$**;** Rank = 3, Index = 2, signature = 1

(iv). $y_1^2 + 2y_2^2 - \frac{1}{2}y_3^2$**;** Rank = 3, Index = 2, signature = 1.

Reduction of quadratic form to canonical form (or "sum of squares form" or "principal axes form") by Lagrange's Reduction method:

Q.No.1.: Transform (reduce) the quadratic form to canonical form (or "sum of squares form" or "principal axes form") by Lagrange's Reduction method

$x_1^2 + 2x_2^2 - 7x_3^2 - 4x_1x_2 + 8x_1x_3$.

Ans.: $(x_1 - 2x_2 + 4x_3)^2 - 2(x_2 - 4x_3)^2 + 9x_3^2$**.**

Q.No.2.: Transform (reduce) the quadratic form to canonical form (or "sum of squares form" or "principal axes form") by Lagrange's Reduction method

$$2x_1^2 + 5x_2^2 + 19x_3^2 - 24x_4^2 + 8x_1x_2 + 12x_1x_3 + 8x_1x_4 + 18x_2x_3 - 8x_2x_4 - 16x_3x_4$$

.

Ans.: $2(x_1+2x_2+3x_3+2x_4)^2-3(x_2+x_3+4x_4)^2+4(x_3-2x_4)^2$.

Q.No.3.: Transform (reduce) the quadratic form to canonical form (or "sum of squares form" or "principal axes form") by Lagrange's Reduction method

$$2x_1^2+7x_2^2+5x_3^2-8x_1x_2-10x_2x_3+4x_1x_3.$$

Ans.: $2(x_1-2x_2-x_3)^2-(x_2+x_3)^2+4x_3^2$.

Q.No.4.: By Lagrange's reduction transform the quadratic form X^TAX to sum of the squares form for $A=\begin{bmatrix}1 & -1 & 0 & 2\\ -1 & 4 & 6 & 4\\ 0 & 6 & 11 & 8\\ 2 & 4 & 8 & 8\end{bmatrix}$.

Ans.: $(x_1-x_2+2x_3)^2+3(x_2+2x_3+2x_4)^2-(x_3+4x_4)^2+8x_4^2$.

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