Differential Calculus

Partial Differentiation

(Partial Differential Coefficient)

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Introduction

- **Partial differentiation is the process of finding partial derivatives.**

  Let u be a function of x and y i.e. u = f(x, y).

- **A partial derivative of several variables is the ordinary derivative with respect to one of the variables when all the remaining variables are held constant.**
- **All the rules of differentiation applicable to function of a single independent variable are also applicable in partial differentiation with the only difference that while differentiating (partially) with respect to one variable, all the other variables are treated (temporarily) as constants.**

Differential Coefficient:

If y is a function of only one independent variable, say x, then we can write

$$y = f(x).$$

Then, the rate of change of y w.r.t. x i.e. the derivative of y w.r.t. x is defined as

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{(y+\delta y)-y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x+\delta x)-f(x)}{\delta x}$$

where δy is the change or increment of y corresponding to the increment δx of the independent variable x.

Partial Differential Coefficient:

Let u be a function of x and y i.e. u = f(x, y).

Then the partial differential coefficient of u (i.e. f(x, y) w.r.t. x (keeping y as constant) is defined and written as

$$\frac{\partial u}{\partial x} = \lim_{\delta x \to 0} \frac{f(x+\delta x, y) - f(x, y)}{\delta x} = u_x = f_x = \frac{\partial f}{\partial x}.$$

Similarly, the partial differential coefficient of u (i.e. f(x, y) w.r.t. y (keeping x as constant) is defined and written as

$$\frac{\partial u}{\partial y} = \lim_{\delta y \to 0} \frac{f(x, y+\delta y) - f(x, y)}{\delta y} = u_y = f_y = \frac{\partial f}{\partial y}.$$

Similarly, we can find

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right), \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right), \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right), \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right).$$

Also, it can be verified that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$.

Notation:

The partial derivative $\frac{\partial u}{\partial x}$ is also denoted by $\frac{\partial f}{\partial x}$ or $f_x(x, y, z)$ or f_x or $D_x f$ or $f_1(x, y, z)$, where the subscripts x and 1 denote the variable w.r.t. x which the partial differentiation is carried out.

Thus, we can have $\frac{\partial u}{\partial y} = \frac{\partial f}{\partial y} = f_y(x, y, z) = f_y = D_y f = f_2(x, y, z)$ etc.

The value of a partial derivative at a point (a, b, c) is denoted by

$$\left.\frac{\partial u}{\partial x}\right|_{x=a, y=b, z=c} = \left.\frac{\partial u}{\partial x}\right|_{(a,b,c)} = f_x(a, b, c).$$

Geometrical Interpretation of partial derivatives:

(Geometrical interpretation of a partial derivative of a function of two variables)

$z = f(x, y)$ represents the **equation of surface** in xyz-coordinate system. Let APB be the curve, which is drawn on a plane through any point P on the surface parallel to the xz-plane.

As point P moves along the curve APB, its coordinates z and x vary while y remains constant. The slope of the tangent line at P to APB represents the 'rate at which z changes w.r.t. x'.

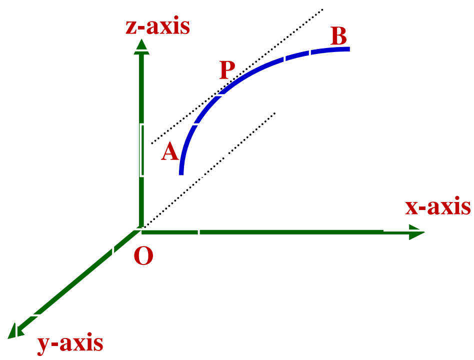

Figure 1

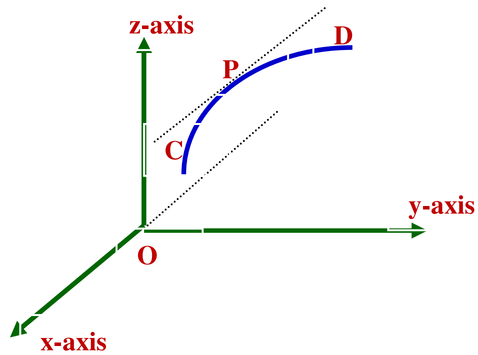

Figure 2

Thus $\frac{\partial z}{\partial x} = \tan\alpha$ = slope of the curve APB at the point P (see fig.1).

Similarly, $\frac{\partial z}{\partial y} = \tan\beta$ = slope of the curve CPD at the point P (see fig.2).

Higher Order Parallel Derivatives:

Partial derivatives of higher order, of a function f(x, y, z) are calculated by successive differentiate. Thus, if u = f(x, y, z) then

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = f_{xx} = f_{11}, \quad \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_{yx} = f_{21},$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{xy} = f_{12}, \quad \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = f_{yy} = f_{22},$$

$$\frac{\partial^3 u}{\partial z^2 \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial^2 f}{\partial z \partial y}\right) = \frac{\partial}{\partial z}\left[\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial y}\right)\right] = f_{yzz} = f_{233},$$

$$\frac{\partial^4 u}{\partial x \partial y \partial z^2} = \frac{\partial}{\partial x}\left(\frac{\partial^3 f}{\partial y \partial z^2}\right) = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial z^2}\right)\right] = f_{zzyx} = f_{3321}.$$

The partial derivative $\frac{\partial f}{\partial x}$ obtained by differentiating once in known as first order partial derivative, while $\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y \partial x}$ which are obtained by differentiating twice are known as second order derivatives. 3rd order, 4th order derivatives involve 3, 4, times differentiation respectively.

Note 1: The crossed or mixed partial derivatives $\frac{\partial^2 f}{\partial y \partial x}$ and $\frac{\partial^2 f}{\partial x \partial y}$ are, in general, equal

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}.$$

i.e. the order of differentiation is immaterial if the derivatives involved are continuous.

Note 2: In the subscript notation, the subscript are written in the same order in which differentiation is carried out, while in '∂' notation the order is opposite, for example

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = f_{xy}.$$

Note 3: A function of 2 variables has two first order derivatives, four second order derivatives and 2^n of n^{th} order derivatives. A function of m independent variables will have m^n derivatives of order n.

Now let us solve some problems related to the above-mentioned topics:

Q.No.1.: If $u = \tan^{-1}\left(\frac{y}{x}\right)$, then prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

Sol.: Here $u = \tan^{-1}\left(\frac{y}{x}\right)$.

Since $\frac{\partial u}{\partial x}$ = the p. d. coefficient of u w. r. t. x (keeping y as constant)

$$= \frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right) = \frac{-y}{x^2+y^2}.$$

$$\therefore \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{-y}{x^2+y^2}\right) = \frac{\left(x^2+y^2\right).0-(2x)(-y)}{\left(x^2+y^2\right)^2} = \frac{2xy}{\left(x^2+y^2\right)^2} \qquad \text{....(i)}$$

Similarly, $\frac{\partial u}{\partial y} =$ the p. d. coefficient of u w. r. t. y (keeping x as constant)

$$= \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2}.$$

$$\therefore \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right) = \frac{\left(x^2+y^2\right).0-(2y)(x)}{\left(x^2+y^2\right)^2} = \frac{-2xy}{\left(x^2+y^2\right)^2} \qquad \text{....(ii)}$$

Adding (i) and (ii), we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{2xy}{\left(x^2+y^2\right)^2} - \frac{2xy}{\left(x^2+y^2\right)^2} = 0.$$

This completes the proof.

Q.No.2.: If $u = f(x+ay)+\phi(x-ay)$, then prove that $\frac{\partial^2 u}{\partial y^2} = a^2.\frac{\partial^2 u}{\partial x^2}$.

Sol.: Here $u = f(x+ay)+\phi(x-ay)$.

$$\therefore \frac{\partial u}{\partial x} = f'(x+ay)+\phi'(x-ay) \text{ and } \frac{\partial^2 u}{\partial x^2} = f''(x+ay)+\phi''(x-ay)$$

Also $\frac{\partial u}{\partial y} = f'(x+ay)(a)+\phi'(x-ay)(-a)$

and $\frac{\partial^2 u}{\partial y^2} = f''(x+ay)\left(a^2\right)+\varphi''(x-ay)(-a)^2$.

$$\frac{\partial^2 u}{\partial y^2} = \left(a^2\right)\left[f''(x+ay)+\phi''(x-ay)\right] = a^2.\frac{\partial^2 u}{\partial x^2}.$$

$$\Rightarrow \frac{\partial^2 u}{\partial y^2} = a^2.\frac{\partial^2 u}{\partial x^2}.$$

This completes the proof.

Q.No.3: Show that $\lim\limits_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^6}$ does not exist.

Sol.: Now $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^3}{x^2+y^6} = \lim\limits_{\substack{x\to0\\y\to0}} \dfrac{xy^3}{x^2+y^6} = \lim\limits_{y\to0} \dfrac{0.y^3}{0+y^6} = \lim\limits_{y\to0} 0 = 0$. ...(i)

Again $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^3}{x^2+y^6} = \lim\limits_{\substack{x\to0\\y\to0}} \dfrac{xy^3}{x^2+y^6} = \lim\limits_{x\to0} \dfrac{x.0}{x^2+0} = \lim\limits_{x\to0} 0 = 0$ ....(ii)

Let $(x,y)\to(0,0)$ along the curve $x = my^3$, where m is a constant.

$\therefore \lim\limits_{(x,y)\to(0,0)} \dfrac{xy^3}{x^2+y^6} = \lim\limits_{y\to0} \dfrac{my^3.y^3}{m^2y^6+y^6} = m\lim\limits_{y\to0} \dfrac{y^6}{y^6(m^2+1)} = \dfrac{m}{m^2+1}\lim\limits_{y\to0} 1 = \dfrac{m}{m^2+1}$. (iii)

From (i) and (ii) given limit is zero as $(x, y)\to(0, 0)$ separately.

But from (iii) limit is not zero, but is different for different values of m.

Hence the given limit does not exist**.**

Q.No.4: Show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y}{x^4+y^2}$ does not exist**.**

Sol.: Now $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y}{x^4+y^2} = \lim\limits_{\substack{x\to0\\y\to0}} \dfrac{x^2y}{x^4+y^2} = \lim\limits_{y\to0} \dfrac{0.y}{0+y^2} = \lim\limits_{y\to0} 0 = 0$. ...(i)

Again $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y}{x^4+y^2} = \lim\limits_{\substack{x\to0\\y\to0}} \dfrac{x^2y}{x^4+y^2} = \lim\limits_{x\to0} \dfrac{x^2.0}{x^4+0} = \lim\limits_{x\to0} 0 = 0$. ....(ii)

Let $(x,y)\to(0,0)$ along the curve $x = \sqrt{my}$,where m is a constant.

$\therefore \lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y}{x^4+y^2} = \lim\limits_{y\to0} \dfrac{my.y}{m^2y^2+y^2} = m\lim\limits_{y\to0} \dfrac{y^2}{y^2(m^2+1)} = \dfrac{m}{m^2+1}\lim\limits_{y\to0} 1 = \dfrac{m}{m^2+1}$ (iii)

From (i) and (ii) given limit is zero as $(x, y)\to(0, 0)$ separately.

But from (iii) limit is not zero, but is different for different values of m.

Hence the given limit does not exist**.**

Q.No.5: If $f(x,y) = \dfrac{y^2+x^2}{y^2-x^2}$, find the limit of f(x, y) when approaches origin (0, 0) along the line y = mx, where m is constant**.**

Sol.: Let $(x,y)\to(0,0)$ along the curve $y = mx$ where m is a constant.

$$\therefore \lim_{(x,y)\to(0,0)} \frac{y^2+x^2}{y^2-x^2} = \lim_{x\to 0} \frac{m^2x^2+.x^2}{m^2x^2-x^2} = \frac{m^2+1}{m^2-1} \lim_{x\to 0} \frac{x^2}{x^2} = \frac{m^2+1}{m^2-1} \lim_{x\to 0} 1 = \frac{m^2+1}{m^2-1}.\ \text{Ans.}$$

Q.No.6.: If $u = \frac{1}{r}$, where $r^2 = x^2 + y^2 + z^2$. Show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$.

Sol.: Since $r^2 = x^2 + y^2 + z^2$.

Differential partially w. r. t. x , we get $2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$.

Now here $u = \frac{1}{r}$,

Differential partially w. r. t. x , we get $\frac{\partial u}{\partial x} = -\frac{1}{r^2}\frac{\partial r}{\partial x} = -\frac{1}{r^2}.\frac{x}{r} = -\frac{x}{r^3}$.

$$\therefore \frac{\partial^2 u}{\partial x^2} = -\frac{r^3.1 - x.3r^2.\frac{\partial r}{\partial x}}{r^6} = -\frac{r^3 - 3r^2.\frac{x^2}{r}}{r^6} = -\frac{r^3 - 3rx^2}{r^6} = \frac{3x^2}{r^5} - \frac{1}{r^3} \qquad ...(i)$$

Similarly, $\frac{\partial^2 u}{\partial y^2} = \frac{3y^2}{r^5} - \frac{1}{r^3}$ ...(ii),

$$\frac{\partial^2 u}{\partial z^2} = \frac{3z^2}{r^5} - \frac{1}{r^3} \qquad ...(iii)$$

Adding (i), (ii) and (iii), we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{3}{r^5}\left[x^2 + y^2 + z^2\right] - \frac{3}{r^3} = \frac{3}{r^5}.r^2 - \frac{3}{r^3} = \frac{3}{r^3} - \frac{3}{r^3} = 0.$$

This completes the proof.

Q.No.7: If u = xyz, find d^2u.

Sol.: We know that if u = f(x, y, z), then

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz = \left(dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y} + dz\frac{\partial}{\partial z}\right)u$$

$$\therefore d^2 = d(du)$$

$$= \left(dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y} + dz\frac{\partial}{\partial z}\right)\left(dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y} + dz\frac{\partial}{\partial z}\right)u = \left(dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y} + dz\frac{\partial}{\partial z}\right)^2 u$$

$$=\left[(dx)^2\frac{\partial^2}{\partial x^2}+(dy)^2\frac{\partial^2}{\partial y^2}+(dz)^2\frac{\partial^2}{\partial z^2}+2dxdy\frac{\partial^2}{\partial x\partial y}+2dydz\frac{\partial^2}{\partial y\partial z}+2dzdx\frac{\partial^2}{\partial z\partial x}\right]u$$

$$=\frac{\partial^2 u}{\partial x^2}(dx)^2+\frac{\partial^2 u}{\partial y^2}(dy)^2+\frac{\partial^2 u}{\partial z^2}(dz)^2+2\frac{\partial^2 u}{\partial x\partial y}dxdy+2\frac{\partial^2 u}{\partial y\partial z}dydz+2\frac{\partial^2 u}{\partial z\partial x}dzdx \qquad \text{(i)}$$

Here $u = xyz$

$$\frac{\partial u}{\partial x}=yz,\quad \frac{\partial u}{\partial y}=zx,\quad \frac{\partial u}{\partial z}=xy.$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 u}{\partial z^2}=0.$$

$$\frac{\partial^2 u}{\partial x\partial y}=z,\quad \frac{\partial^2 u}{\partial y\partial z}=x,\quad \frac{\partial^2 u}{\partial z\partial x}=y.$$

$\therefore$ From (i), we have $d^2u = 2zdxdy + 2xdydz + 2ydzdx$.

Q.No.8: Evaluate $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$, when **(a)** $u = x^y$ and **(b)** $xy + yu + ux = 1$.

Sol.: **(a)** Given $u = x^y$. ...(i)

Differentiate (i) partially w. r. t. x and y separately, we get

$$\frac{\partial u}{\partial x}=\frac{\partial}{\partial x}\left(x^y\right)=yx^{y-1} \text{ and } \frac{\partial u}{\partial y}=\frac{\partial}{\partial y}\left(x^y\right)=x^y\log x.$$ Ans.

(b) Given $xy + yu + ux = 1 \Rightarrow u(x+y) = 1 - xy \Rightarrow u = \frac{1-xy}{x+y}$ ...(ii)

Differentiate (ii) partially w. r. t. x and y separately, we get

$$\frac{\partial u}{\partial x}=\frac{\partial}{\partial x}\left(\frac{1-xy}{x+y}\right)=\frac{(x+y)(-y)-(1-xy).1}{(x+y)^2}=-\frac{\left(1+y^2\right)}{(x+y)^2}$$

and $$\frac{\partial u}{\partial y}=\frac{\partial}{\partial y}\left(\frac{1-xy}{x+y}\right)=\frac{(x+y)(-x)-(1-xy).1}{(x+y)^2}=-\frac{\left(1+x^2\right)}{(x+y)^2}.$$ Ans.

Q.No.9: Verify that $\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial^2 u}{\partial y\partial x}$, where u is equal to

(i) $\log(y\sin x + x\sin y)$, **(ii)** $\log\left(\frac{x^2+y^2}{xy}\right)$,

(iii) $\log\tan\left(\frac{x}{y}\right)$ and **(iv)** $x^2\tan^{-1}\left(\frac{y}{x}\right)-y^2\tan^{-1}\left(\frac{x}{y}\right)$.

Sol.:(i) Here $u=\log(y\sin x+x\sin y)$. ...(i)

Differentiate (i) partially w. r. t. x, we get

$$\frac{\partial u}{\partial x}=\frac{(y\cos x+\sin y)}{(y\sin x+x\sin y)}. \quad ....(ii)$$

Differentiate (ii) partially w. r. t. y, we get

$$\frac{\partial^2 u}{\partial y\partial x}=\frac{\partial}{\partial y}\left[\frac{\partial u}{\partial x}\right]=\frac{(y\sin x+x\sin y)(\cos x+\cos y)-(y\cos x+\sin y)(\sin x+x\cos y)}{(y\sin x+x\sin y)^2}. \quad (iii)$$

Differentiate (i) partially w. r. t. y, we get

$$\frac{\partial u}{\partial y}=\frac{(\sin x+x\cos y)}{(y\sin x+x\sin y)}. \quad (iv)$$

Differentiate (iv) partially w. r. t. x, we get

$$\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial}{\partial x}\left[\frac{\partial u}{\partial y}\right]=\frac{(y\sin x+x\sin y)(\cos x+\cos y)-(\sin x+x\cos y)(y\cos x+\sin y)}{(y\sin x+x\sin y)^2}. \quad (v)$$

Hence from (iii) and (v), we get $\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial^2 u}{\partial y\partial x}$.

This completes the proof.

(ii) Here $u=\log\left(\frac{x^2+y^2}{xy}\right)$. ...(i)

Differentiate (i) partially w. r. t. x, we get

$$\frac{\partial u}{\partial x}=\frac{1}{\frac{x^2+y^2}{xy}}.\frac{xy(2x)-(x^2+y^2)y}{(xy)^2}=\frac{1}{x^2+y^2}.\frac{x^2y-y^3}{xy}=\frac{x^2-y^2}{x(x^2+y^2)}. \quad (ii)$$

Differentiate (ii) partially w. r. t. y, we get

$$\frac{\partial^2 u}{\partial y\partial x}=\frac{\partial}{\partial y}\left[\frac{\partial u}{\partial x}\right]=\frac{(x^3+y^2x)(-2y)-(x^2-y^2)(2xy)}{(x^3+y^2x)^2}=-\frac{4x^3y}{(x^3+y^2x)^2}=-\frac{4xy}{(x^2+y^2)^2}. \quad (iii)$$

Differentiate (i) partially w. r. t. y, we get

$$\frac{\partial u}{\partial y}=\frac{1}{\frac{x^2+y^2}{xy}}\cdot\frac{xy(2y)-(x^2+y^2)x}{(xy)^2}=\frac{1}{x^2+y^2}\cdot\frac{xy^2-x^3}{xy}=\frac{y^2-x^2}{y(x^2+y^2)}. \qquad \text{(iv)}$$

Differentiate (iv) partially w. r. t. x, we get

$$\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial}{\partial x}\left[\frac{\partial u}{\partial y}\right]=\frac{(yx^2+y^3)(-2x)-(y^2-x^2)(2xy)}{(yx^2+y^3)^2}=-\frac{4xy^3}{(yx^2+y^3)^2}=-\frac{4xy}{(x^2+y^2)^2}. \qquad \text{..(v)}$$

Hence from (iii) and (v), we get $\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial^2 u}{\partial y\partial x}$.

This completes the proof.

(iii) Here $u=\log\tan\left(\frac{x}{y}\right)$. ....(i)

Differentiate (i) partially w. r. t. x, we get

$$\frac{\partial u}{\partial x}=\frac{1}{\tan\frac{x}{y}}.\sec^2\frac{x}{y}.\frac{1}{y}=\frac{\sec^2\frac{x}{y}}{y\tan\frac{x}{y}}. \qquad \text{....(ii)}$$

Differentiate (ii) partially w. r. t. y, we get

$$\frac{\partial^2 u}{\partial y\partial x}=\frac{\partial}{\partial y}\left[\frac{\partial u}{\partial x}\right]=\frac{y\tan\frac{x}{y}.\frac{\partial}{\partial y}\left(\sec^2\frac{x}{y}\right)-\sec^2\frac{x}{y}.\frac{\partial}{\partial y}\left(y\tan\frac{x}{y}\right)}{y^2\tan^2\frac{x}{y}}$$

$$=\frac{x\sec^2\frac{x}{y}\tan\frac{x}{y}-3x\sec^2\frac{x}{y}\tan^2\frac{x}{y}}{y^3\tan^2\frac{x}{y}}. \qquad \text{(iii)}$$

Differentiate (i) partially w. r. t. y, we get

$$\frac{\partial u}{\partial y}=\frac{1}{\tan\frac{x}{y}}.\sec^2\frac{x}{y}.\left(-\frac{x}{y^2}\right)=-\frac{x}{y^2}.\frac{\sec^2\frac{x}{y}}{\tan\frac{x}{y}}. \qquad \text{(iv)}$$

Differentiate (iv) partially w. r. t. y, we get

$$\frac{\partial^2 u}{\partial x \partial y}=\frac{\partial}{\partial x}\left[\frac{\partial u}{\partial y}\right]=-\frac{y^2\tan\frac{x}{y}.\frac{\partial}{\partial x}\left(x\sec^2\frac{x}{y}\right)-x\sec^2\frac{x}{y}.\frac{\partial}{\partial y}\left(y^2\tan\frac{x}{y}\right)}{y^4\tan^2\frac{x}{y}}$$

$$=\frac{x\sec^2\frac{x}{y}\tan\frac{x}{y}-3x\sec^2\frac{x}{y}\tan^2\frac{x}{y}}{y^3\tan^2\frac{x}{y}}. \qquad \text{(v)}$$

Hence from (iii) and (v), we get $\frac{\partial^2 u}{\partial x \partial y}=\frac{\partial^2 u}{\partial y \partial x}$.

This completes the proof.

(iv) Here $u=x^2\tan^{-1}\left(\frac{y}{x}\right)-y^2\tan^{-1}\left(\frac{x}{y}\right)$. (i)

Differentiate (i) partially w. r. t. x, we get

$$\frac{\partial u}{\partial x}=x^2\frac{1}{1+\frac{y^2}{x^2}}.\left(-\frac{y}{x^2}\right)+\left[2x\tan^{-1}\frac{y}{x}-y^2.\frac{1}{1+\frac{x^2}{y^2}}\left(\frac{1}{y}\right)\right]$$

$$=-\frac{x^2y}{x^2+y^2}+2x\tan^{-1}\frac{y}{x}-\frac{y^3}{x^2+y^2}=2x\tan^{-1}\frac{y}{x}-\frac{x^2y+y^3}{x^2+y^2}=2x\tan^{-1}\frac{y}{x}-y. \qquad \text{(ii)}$$

Differentiate (ii) partially w. r. t. y, we get

$$\frac{\partial^2 u}{\partial y \partial x}=\frac{\partial}{\partial y}\left[\frac{\partial u}{\partial x}\right]=2x.\frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right)-1=\frac{2x^2}{x^2+y^2}-1=\frac{2x^2-x^2-y^2}{x^2+y^2}=\frac{x^2-y^2}{x^2+y^2} \qquad \text{(iii)}$$

Differentiate (i) partially w. r. t. y, we get

$$\frac{\partial u}{\partial y}=x^2\frac{1}{1+\frac{y^2}{x^2}}.\frac{1}{x}-\left[2y\tan^{-1}\frac{x}{y}+y^2.\frac{1}{1+\frac{x^2}{y^2}}\left(-\frac{x}{y^2}\right)\right].$$

$$=\frac{x^3}{x^2+y^2}-2y\tan^{-1}\frac{x}{y}+\frac{xy^2}{x^2+y^2}=\frac{x^3+xy^2}{x^2+y^2}-2y\tan^{-1}\frac{x}{y}=\frac{x\left(x^2+y^2\right)}{x^2+y^2}-2y\tan^{-1}\frac{x}{y}$$

$$\therefore \frac{\partial u}{\partial y} = x - 2y\tan^{-1}\frac{x}{y}. \qquad \text{(iv)}$$

Differentiate (iv) partially w. r. t. x, we get

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial u}{\partial y}\right] = \frac{\partial}{\partial x}\left[x - 2y\tan^{-1}\frac{x}{y}\right] = 1 - 2y\frac{1}{1+\frac{x^2}{y^2}}.\frac{1}{y} = 1 - \frac{2y^2}{x^2+y^2} = \frac{x^2-y^2}{x^2+y^2}. \qquad \text{(v)}$$

Hence from (iii) and (v), we get $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$**.**

This completes the proof**.**

Q.No.10: If $u = \log\left(x^3 + y^3 + z^3 - 3xyz\right)$, show that $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$**.**

Sol.: Since $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right)$.

Here $u = \log\left(x^3 + y^3 + z^3\right)$. .(i)

Differentiate (i) partially w. r. t. x ,y and z separately, we get

$$\frac{\partial u}{\partial x} = \frac{3\left(x^2 - yz\right)}{x^3+y^3+z^3-3xyz}, \quad \frac{\partial u}{\partial y} = \frac{3\left(y^2 - xz\right)}{x^3+y^3+z^3-3xyz} \text{ and } \frac{\partial u}{\partial z} = \frac{3\left(z^2 - xy\right)}{x^3+y^3+z^3-3xyz},$$

$$\therefore \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right) = \frac{3\left(x^2+y^2+z^2-xy-yz-zx\right)}{\left(x^3+y^3+z^3-3xyz\right)} = \frac{3}{(x+y+z)}.$$

$$\text{Hence } \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right) = \frac{\partial}{\partial x}\left(\frac{3}{x+y+z}\right) + \frac{\partial}{\partial y}\left(\frac{3}{x+y+z}\right) + \frac{\partial}{\partial z}\left(\frac{3}{x+y+z}\right)$$

$$= \frac{-3}{(x+y+z)^2} + \frac{-3}{(x+y+z)^2} + \frac{-3}{(x+y+z)^2}$$

$$= \frac{-9}{(x+y+z)^2}.$$

Hence $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$**.**

This completes the proof**.**

Q.No.11: If $u = e^{xyz}$, show that $\dfrac{\partial^3 u}{\partial x \partial y \partial z} = \left(1 + 3xyz + x^2y^2z^2\right)e^{xyz}$**.**

Sol.: Here $u = e^{xyz}$. Now $\dfrac{\partial u}{\partial z} = \dfrac{\partial}{\partial z}\left(e^{xyz}\right) = e^{xyz}xy$.

$$\therefore \frac{\partial^2 u}{\partial y \partial z} = \frac{\partial}{\partial y}\left[\frac{\partial u}{\partial z}\right] = \frac{\partial}{\partial y}\left[e^{xyz}xy\right] = xy\left(e^{xyz}xz\right) + e^{xyz}x = x^2yze^{xyz} + e^{xyz}x = \left(x^2yz + x\right)e^{xyz}$$

And hence $\dfrac{\partial^3 u}{\partial x \partial y \partial z} = \dfrac{\partial}{\partial x}\left[\dfrac{\partial^2 u}{\partial y \partial z}\right] = \dfrac{\partial}{\partial x}\left[\left(x^2yz + x\right)e^{xyz}\right] = [2xyz + 1]e^{xyz} + \left[x^2yz + x\right]e^{xyz}yz$

$$= \left[2xyz + 1 + x^2y^2z^2 + xyz\right]e^{xyz} = \left[x^2y^2z^2 + 3xyz + 1\right]e^{xyz}.$$

This completes the proof**.**

Q.No.12: If $u = z = \left(1 - 2xy + y^2\right)^{-1/2}$, prove that

(i) $x\dfrac{\partial z}{\partial x} - y\dfrac{\partial z}{\partial y} = y^2z^3$, **(ii)** $\dfrac{\partial}{\partial x}\left[\left(1 - x^2\right)\dfrac{\partial u}{\partial x}\right] + \dfrac{\partial}{\partial y}\left(y^2\dfrac{\partial u}{\partial y}\right) = 0$**.**

Sol.: (i) Here $z = \left(1 - 2xy + y^2\right)^{-1/2}$. ....(i)

Differentiate (i) partially w. r. t. x and y separately, we get

$$\frac{\partial z}{\partial x} = -\frac{1}{2}\left(1 - 2xy + y^2\right)^{-3/2}(-2y) = y\left(1 - 2xy + y^2\right)^{-3/2}.$$

and $\dfrac{\partial z}{\partial y} = -\dfrac{1}{2}\left(1 - 2xy + y^2\right)^{-3/2}(-2x + 2y) = (x - y)\left(1 - 2xy + y^2\right)^{-3/2}$.

Hence $x\dfrac{\partial z}{\partial x} - y\dfrac{\partial z}{\partial y} = x\left[y\left(1 - 2xy + y^2\right)^{-3/2}\right] - y\left[(x - y)\left(1 - 2xy + y^2\right)^{-3/2}\right]$

$$= \left(1 - 2xy + y^2\right)^{-3/2}\left[xy - xy + y^2\right] = y^2z^3.$$

This completes the proof**.**

(ii) To show: $\dfrac{\partial}{\partial x}\left[\left(1 - x^2\right)\dfrac{\partial u}{\partial x}\right] + \dfrac{\partial}{\partial y}\left(y^2\dfrac{\partial u}{\partial y}\right) = 0$.

Here $u = \left(1 - 2xy + y^2\right)^{-1/2}$. ....(i)

Differentiate (i) partially w. r. t. x and y separately, we get

$$\frac{\partial u}{\partial x} = y\left(1 - 2xy + y^2\right)^{-3/2} \text{ and } \frac{\partial u}{\partial y} = (x - y)\left(1 - 2xy + y^2\right)^{-3/2}.$$

Now $\frac{\partial}{\partial x}\left[\left(1-x^2\right)\frac{\partial u}{\partial x}\right]=\frac{\partial}{\partial x}\left[\left(1-x^2\right)y\left(1-2xy+y^2\right)^{-3/2}\right]$

$$=y\left[\left(1-x^2\right)\frac{\partial}{\partial x}\left(1-2xy+y^2\right)^{-3/2}+\left(1-2xy+y^2\right)^{-3/2}\frac{\partial}{\partial x}\left(1-x^2\right)\right]$$

$$=y\left[\left(1-x^2\right)\left(-\frac{3}{2}\right)\left(1-2xy+y^2\right)^{-5/2}(-2y)+\left(1-2xy+y^2\right)^{-3/2}(-2x)\right]$$

$$=y\left[\frac{3y\left(1-x^2\right)}{\left(1-2xy+y^2\right)^{5/2}}-\frac{2x}{\left(1-2xy+y^2\right)^{3/2}}\right]=y\left[\frac{3y-3x^2y-2x+4x^2y-2xy^2}{\left(1-2xy+y^2\right)^{5/2}}\right]$$

$$\therefore \frac{\partial}{\partial x}\left[\left(1-x^2\right)\frac{\partial u}{\partial x}\right]=\frac{y\left(3y-2x+x^2y-2xy^2\right)}{\left(1-2xy+y^2\right)^{5/2}}.$$

Again $\frac{\partial}{\partial y}\left[y^2\frac{\partial u}{\partial y}\right]=\frac{\partial}{\partial y}\left[y^2(x-y)\left(1-2xy+y^2\right)^{-3/2}\right]$

$$=\frac{\partial}{\partial y}\left(xy^2-y^3\right)\left(1-2xy+y^2\right)^{-3/2}$$

$$=\left(1-2xy+y^2\right)^{-3/2}\left(2xy-3y^2\right)+\left(xy^2-y^3\right)\left(-\frac{3}{2}\right)\left(1-2xy+y^2\right)^{-5/2}(-2x+2y)$$

$$=\frac{2xy-3y^2}{\left(1-2xy+y^2\right)^{3/2}}+\frac{3\left(xy^2-y^3\right)(x-y)}{\left(1-2xy+y^2\right)^{5/2}}=\frac{\left(2xy-3y^2\right)\left(1-2xy+y^2\right)+3\left(xy^2-y^3\right)(x-y)}{\left(1-2xy+y^2\right)^{5/2}}$$

$$=\frac{2xy-4x^2y^2+2xy^3-3y^2+6xy^3-3y^4+3y^2x^2-6xy^3+3y^4}{\left(1-2xy+y^2\right)^{5/2}}$$

$$=\frac{2xy-3y^2-x^2y^2+2xy^3}{\left(1-2xy+y^2\right)^{5/2}}=\frac{-y\left(3y-2x+x^2y-2xy^2\right)}{\left(1-2xy+y^2\right)^{5/2}}$$

or $\frac{\partial}{\partial y}\left(y^2\frac{\partial u}{\partial y}\right)=-\frac{\partial}{\partial x}\left[\left(1-x^2\right)\frac{\partial u}{\partial x}\right].$

Hence $\frac{\partial}{\partial x}\left[\left(1-x^2\right)\frac{\partial u}{\partial x}\right]+\frac{\partial}{\partial y}\left(y^2\frac{\partial u}{\partial y}\right)=0$**.**

This completes the proof **.**

Q.No.13: If $u=\tan^{-1}\left[\frac{xy}{\sqrt{1+x^2+y^2}}\right]$, prove that $\frac{\partial^2 u}{\partial x\partial y}=\left(1+x^2+y^2\right)^{-3/2}$**.**

Sol.: Here $u = \tan^{-1}\left[\dfrac{xy}{\sqrt{1+x^2+y^2}}\right]$. ...(i)

Differentiate (i) partially w. r. t. y, we get

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\tan^{-1}\left[\frac{xy}{\sqrt{(1+x^2+y^2)}}\right]$$

$$= \frac{1}{1+\dfrac{x^2y^2}{1+x^2+y^2}} \cdot \frac{\sqrt{1+x^2+y^2}.x - xy.\dfrac{1}{2\sqrt{1+x^2+y^2}}.2y}{(1+x^2+y^2)}$$

$$= \frac{(1+x^2+y^2)}{1+x^2+y^2+x^2y^2} \cdot \frac{x(1+x^2+y^2)-xy^2}{(1+x^2+y^2)\sqrt{1+x^2+y^2}} = \frac{x+x^3+xy^2-xy^2}{(1+x^2+y^2+x^2y^2)\sqrt{1+x^2+y^2}}$$

$$= \frac{x+x^3}{(1+x^2+y^2+x^2y^2)\sqrt{1+x^2+y^2}} = \frac{x(1+x^2)}{\{(1+x^2)+y^2(1+x^2)\}\sqrt{1+x^2+y^2}}$$

$$= \frac{x}{(1+y^2)\sqrt{1+x^2+y^2}}. \qquad ...(ii)$$

Differentiate (ii) partially w. r. t. x ,we get

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial z}{\partial y}\right] = \frac{\partial}{\partial x}\left[\frac{x}{(1+y^2)\sqrt{1+x^2+y^2}}\right]$$

$$= \frac{\sqrt{1+x^2+y^2}(1+y^2).1 - x\left\{(1+y^2).\dfrac{2x}{2\sqrt{1+x^2+y^2}}\right\}}{(1+y^2)^2(1+x^2+y^2)} = \frac{(1+x^2+y^2)(1+y^2)-x^2(1+y^2)}{(1+y^2)^2(1+x^2+y^2)\sqrt{1+x^2+y^2}}$$

$$= \frac{(1+y^2)(1+x^2+y^2-x^2)}{(1+y^2)^2(1+x^2+y^2)^{3/2}} = \frac{(1+y^2)^2}{(1+y^2)^2(1+x^2+y^2)^{3/2}} = \frac{1}{(1+x^2+y^2)^{3/2}}.$$

Hence $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{1}{(1+x^2+y^2)^{3/2}}$.

This completes the proof**.**

Q.No.14: If $z^2+t^2-4x+y^2=0$ and $z^3+t^3-2x^3+3y=0$;

Evaluate $\frac{\partial z}{\partial x}$ and $\frac{\partial t}{\partial x}$.

Sol.: Here $z^2+t^2-4x+y^2=0$ and $z^3+t^3-2x^3+3y=0$.

Differentiate partially the given equations w. r. t. x, considering z and t as function of x, we get

$$2z\frac{\partial z}{\partial x}+2t\frac{\partial t}{\partial x}-4=0$$

and $3z^2\frac{\partial z}{\partial x}+3t^2\frac{\partial t}{\partial x}--6x^2=0$.

Solve these equations simultaneously for $\frac{\partial z}{\partial x}$ and $\frac{\partial t}{\partial x}$.

$$\frac{\frac{\partial z}{\partial x}}{2t.\left(-6x^2\right)+4.3.t^2}=\frac{\frac{\partial t}{\partial x}}{-12z^2+12zx^2}=\frac{1}{6zt^2-6tz^2}.$$

$$\Rightarrow \frac{\frac{\partial z}{\partial x}}{12t\left(t-x^2\right)}=\frac{\frac{\partial t}{\partial x}}{12z\left(x^2-z\right)}=\frac{1}{6tz(t-z)}.$$

Considering $\frac{\frac{\partial z}{\partial x}}{12t\left(t-x^2\right)}=\frac{1}{6tz(t-z)}$ and $\frac{\frac{\partial t}{\partial x}}{12z\left(x^2-z\right)}=\frac{1}{6tz(t-z)}$.

We get $\frac{\partial z}{\partial x}=\frac{12t\left(t-x^2\right)}{6tz(t-z)}=\frac{2\left(x^2-t\right)}{z(z-t)}$ and $\frac{\partial t}{\partial x}=\frac{12z\left(x^2-z\right)}{6tz(t-z)}=\frac{2\left(x^2-z\right)}{t(t-z)}$. Ans.

Q.No.15: If $u=\frac{ke^{\frac{-x^2}{4a^2y}}}{\sqrt{y}}$, then prove that $\frac{\partial u}{\partial y}=a^2\frac{\partial^2 u}{\partial x^2}$ **.**

Sol.: Here $u=\frac{ke^{\frac{-x^2}{4a^2y}}}{\sqrt{y}}$, then $\frac{\partial u}{\partial y}=\frac{k}{\sqrt{y}}.e^{\frac{-x^2}{4a^2y}}\left(\frac{x^2}{4a^2y^2}\right)+k\left(-\frac{1}{2y^{3/2}}\right).e^{\frac{-x^2}{4a^2y}}$

$$=ke^{\frac{-x^2}{4a^2y}}\left[\frac{x^2}{4a^2y^{5/2}}-\frac{1}{2y^{3/2}}\right].$$

Also $\dfrac{\partial u}{\partial x} = \dfrac{k}{\sqrt{y}} e^{\frac{-x^2}{4a^2y}} \left(\dfrac{-2x}{4a^2y} \right) = -\dfrac{kx}{2a^2y^{3/2}} e^{\frac{-x^2}{4a^2y}}$

and $\therefore \dfrac{\partial^2 u}{\partial x^2} = -\dfrac{k}{2a^2y^{3/2}}.e^{\frac{-x^2}{4a^2y}} - \dfrac{kx}{2a^2y^{3/2}}.e^{\frac{-x^2}{4a^2y}} \left(\dfrac{-2x}{4a^2y} \right) = ke^{\frac{-x^2}{4a^2y}} \left[\dfrac{x^2}{4a^4y^{5/2}} - \dfrac{1}{2a^2y^{3/2}} \right]$

$\Rightarrow \dfrac{\partial^2 u}{\partial x^2} = \dfrac{1}{a^2} \dfrac{\partial u}{\partial y}$, hence $\dfrac{\partial u}{\partial y} = a^2 \dfrac{\partial^2 u}{\partial x^2}$.

This completes the proof**.**

Q.No.16: If $\theta = t^n e^{-\frac{r^2}{4t}}$, find what value of n will make $\dfrac{1}{r^2} \dfrac{\partial}{\partial r} \left(r^2 \dfrac{\partial \theta}{\partial r} \right) = \dfrac{\partial \theta}{\partial t}$**.**

Sol.: Here $\theta = t^n e^{-\frac{r^2}{4t}}$. ....(i)

Differentiate (i) partially w. r. t. r, we get

$$\frac{\partial \theta}{\partial r} = \frac{\partial}{\partial r}\left[t^n e^{-\frac{r^2}{4t}} \right] = t^n.\frac{\partial}{\partial r}\left[e^{-\frac{r^2}{4t}} \right] = t^n.e^{-\frac{r^2}{4t}}\left(-\frac{2r}{4t} \right) = -\frac{1}{2} t^{n-1} r.e^{-\frac{r^2}{4t}}.$$

$$r^2 \frac{\partial \theta}{\partial r} = -\frac{1}{2} t^{n-1} r^3.e^{-\frac{r^2}{4t}}. \qquad \text{...(ii)}$$

Differentiate (ii) partially w. r. t. r, we get

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r} \right) = \frac{\partial}{\partial r}\left[-\frac{1}{2} t^{n-1} r^3 e^{-\frac{r^2}{4t}} \right] = -\frac{t^{n-1}}{2} \frac{\partial}{\partial r}\left[r^3 e^{-\frac{r^2}{4t}} \right]$$

$$= -\frac{t^{n-1}}{2}\left\{ 3r^2.e^{-\frac{r^2}{4t}} + r^3.e^{-\frac{r^2}{4t}}\left(-\frac{2r}{4t} \right) \right\} = -\frac{t^{n-1}}{2}\left\{ \left(3r^2 - \frac{r^4}{2t} \right) e^{-\frac{r^2}{4t}} \right\}$$

$$\therefore \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r} \right) = -\frac{t^{n-1}}{2r^2}\left\{ \left(3r^2 - \frac{r^4}{2t} \right) e^{-\frac{r^2}{4t}} \right\} \qquad \text{...(iii)}$$

$$\text{Now } \frac{\partial \theta}{\partial t} = \frac{\partial}{\partial t}\left[t^n e^{-\frac{r^2}{4t}} \right] = t^n.e^{-\frac{r^2}{4t}}\left(\frac{r^2}{4t^2} \right) + nt^{n-1}.e^{-\frac{r^2}{4t}} = e^{-\frac{r^2}{4t}}\left[\frac{r^2}{4} t^{n-2} + nt^{n-1} \right]$$

$$= e^{-\frac{r^2}{4t}}\left[t^{n-1}\left(\frac{r^2}{4t}+n\right)\right]. \qquad ...(iv)$$

But $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right)=\frac{\partial\theta}{\partial t} \Rightarrow -\frac{t^{n-1}}{2r^2}\left\{\left(3r^2-\frac{r^4}{2t}\right)e^{-\frac{r^2}{4t}}\right\}=e^{-\frac{r^2}{4t}}\left[t^{n-1}\left(\frac{r^2}{4t}+n\right)\right]$

$\Rightarrow -\frac{1}{2r^2}\left\{\left(3r^2-\frac{r^4}{2t}\right)\right\}=\left[\left(\frac{r^2}{4t}+n\right)\right]. \Rightarrow -\frac{1}{2}\left\{\left(3-\frac{r^2}{2t}\right)\right\}=\left[\left(\frac{r^2}{4t}+n\right)\right]$

$\Rightarrow -\frac{3}{2}=n$. Hence $n=-\frac{3}{2}$. Ans.

Q.No.17: If $u = Ae^{-gx}\sin(nt-gx)$, where A , g , n are positive constants, satisfies the heat conduction equation $\frac{\partial u}{\partial t}=\mu\frac{\partial^2 u}{\partial x^2}$, then prove that $g=\sqrt{\frac{n}{2\mu}}$.

or

The equation $\frac{\partial u}{\partial t}=\mu\frac{\partial^2 u}{\partial x^2}$ refers to the conduction of heat along a bar without radiation, show that if $u = Ae^{-gx}\sin(nt-gx)$, where A , g , n are positive constants then $g=\sqrt{\frac{n}{2\mu}}$.

Sol.: Here $u = Ae^{-gx}\sin(nt-gx)$,we have $\frac{\partial u}{\partial t}=Ae^{-gx}\cos(nt-gx)n$.

Also $\frac{\partial u}{\partial x}=A\left[e^{-gx}(-g)\sin(nt-gx)+e^{-gx}\cos(nt-gx)(-g)\right]$

$$=A(-g)\left[e^{-gx}\sin(nt-gx)+e^{-gx}\cos(nt-gx)\right]$$

$$=-Age^{-gx}[\sin(nt-gx)+\cos(nt-gx)]$$

and $\frac{\partial^2 u}{\partial x^2}=-Ag\left[e^{-gx}\{\cos(nt-gx)(-g)-\sin(nt-gx)(-g)\}\right.$

$$\left.+\{\sin(nt-gx)+\cos(nt-gx)\}e^{-gx}(-g)\right]$$

$$=-Age^{-gx}(-g)[\cos(nt-gx)-\sin(nt-gx)+\sin(nt-gx)+\cos(nt-gx)]$$

$$=-Age^{-gx}(-g)[2\cos(nt-gx)]=2Ag^2e^{-gx}\cos(nt-gx).$$

Also given $\frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial x^2} \Rightarrow Ae^{-gx}\cos(nt-gx)n = \mu 2Ag^2e^{-gx}\cos(nt-gx)$

$$\Rightarrow g^2 = \frac{n}{2\mu}\text{ . Hence } \therefore g = \sqrt{\frac{n}{2\mu}}\,.$$

This completes the proof**.**

Q.No.18: (a) Show that at the point for surface $x^x y^y z^z = \text{const.}$, where $x = y = z$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{-1}{x\log(ex)}.$$

(b) If $u = e^{xyz}$; find the value of $\frac{\partial^3 u}{\partial x \partial y \partial z}$.

Sol.: (a) Given $x^x y^y z^z = \text{const.}$, where $x = y = z$.

Taking log both sides, we get

$x\log x + y\log y + z\log z = \log c$

Differentiating z partially w. r. t. x [keeping y as constant] , we get

$$(1+\log x)+(1+\log z)\frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial z}{\partial x} = -\frac{1+\log x}{1+\log z}.\ \text{ Similarly, } \frac{\partial z}{\partial y} = -\frac{1+\log y}{1+\log z}.$$

$$\text{Now } \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial y}\left[\frac{\partial z}{\partial x}\right] = \frac{\partial}{\partial z}\left[\frac{\partial z}{\partial x}\right]\frac{\partial z}{\partial y} = \frac{\partial}{\partial z}\left[-\frac{1+\log x}{1+\log z}\right]\times\left[-\frac{1+\log y}{1+\log z}\right]$$

$$= \frac{(1+\log z).0-(1+\log x)\frac{1}{z}}{(1+\log z)^2}\times\left[\frac{1+\log y}{1+\log z}\right] = -\frac{1}{z}\frac{(1+\log x)(1+\log y)}{(1+\log z)^3}$$

Since $x = y = z$,

$$\therefore \frac{\partial^2 z}{\partial x \partial y} = -\frac{1}{x}\frac{(1+\log x)^2}{(1+\log x)^3} = -\frac{1}{x(1+\log x)} = \frac{-1}{x(\log e+\log x)} = \frac{-1}{x\log(ex)}.$$

Hence $\frac{\partial^2 z}{\partial x \partial y} = \frac{-1}{x\log(ex)}$**.** This completes the proof**.**

(b) Here $u = e^{xyz}$.

Now $\frac{\partial u}{\partial z} = \frac{\partial}{\partial z}\left(e^{xyz}\right) = e^{xyz}xy$.

$\therefore \dfrac{\partial^2 u}{\partial y \partial z} = \dfrac{\partial}{\partial y}\left[\dfrac{\partial u}{\partial z}\right] = \dfrac{\partial}{\partial y}\left[e^{xyz}xy\right] = xy\left(e^{xyz}xz\right) + e^{xyz}x = x^2yze^{xyz} + e^{xyz}x = \left(x^2yz + x\right)e^{xyz}$

And hence $\dfrac{\partial^3 u}{\partial x \partial y \partial z} = \dfrac{\partial}{\partial x}\left[\dfrac{\partial^2 u}{\partial y \partial z}\right] = \dfrac{\partial}{\partial x}\left[\left(x^2yz + x\right)e^{xyz}\right] = [2xyz + 1]e^{xyz} + \left[x^2yz + x\right]e^{xyz}yz$

$= \left[2xyz + 1 + x^2y^2z^2 + xyz\right]e^{xyz} = \left[x^2y^2z^2 + 3xyz + 1\right]e^{xyz}$. Ans.

Q.No.19: If $z = xf(x+y) + yg(x+y)$, show that $\dfrac{\partial^2 z}{\partial x^2} - 2\dfrac{\partial^2 z}{\partial x \partial y} + \dfrac{\partial^2 z}{\partial y^2} = 0$.

Sol.: Since $z = xf(x+y) + yg(x+y)$. ...(i)

$\therefore \dfrac{\partial z}{\partial x} = xf'(x+y) + f(x+y) + yg'(x+y)$.

and $\therefore \dfrac{\partial^2 z}{\partial x^2} = f'(x+y) + xf''(x+y) + f'(x+y) + yg''(x+y)$. ...(ii)

Also $\dfrac{\partial z}{\partial y} = xf'(x+y) + yg'(x+y) + g(x+y)$.

and $\therefore \dfrac{\partial^2 z}{\partial y^2} = xf''(x+y) + yg''(x+y) + g'(x+y) + g'(x+y)$. ...(iii)

Now since $\dfrac{\partial z}{\partial x} = xf'(x+y) + f(x+y) + yg'(x+y)$.

$\therefore \dfrac{\partial^2 z}{\partial x \partial y} = xf''(x+y) + f'(x+y) + g'(x+y) + yg''(x+y)$. ...(iv)

Putting these values in $\dfrac{\partial^2 z}{\partial x^2} - 2\dfrac{\partial^2 z}{\partial x \partial y} + \dfrac{\partial^2 z}{\partial y^2}$, we get

$\dfrac{\partial^2 z}{\partial x^2} - 2\dfrac{\partial^2 z}{\partial x \partial y} + \dfrac{\partial^2 z}{\partial y^2} = 0$. This completes the proof.

Q.No.20: If $u = \dfrac{y}{z} + \dfrac{z}{x} + \dfrac{x}{y}$, then show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = 0$.

Sol.: Since $u = \dfrac{y}{z} + \dfrac{z}{x} + \dfrac{x}{y}$. ...(i)

Differentiate (i) partially w. r. t. x, y and z separately, we get

$$\frac{\partial u}{\partial x} = \left(\frac{1}{y} - \frac{z}{x^2}\right), \quad \frac{\partial u}{\partial y} = \left(\frac{1}{z} - \frac{x}{y^2}\right) \text{ and } \frac{\partial u}{\partial z} = \left(\frac{1}{x} - \frac{y}{z^2}\right).$$

Hence $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = x\left(\frac{1}{y} - \frac{z}{x^2}\right) + y\left(\frac{1}{z} - \frac{x}{y^2}\right) + z\left(\frac{1}{x} - \frac{y}{z^2}\right) = 0$.

This completes the proof.

Q.No.21: If $u = e^{ax+by}\phi(ax - by)$, then prove that $b\frac{\partial u}{\partial x} + a\frac{\partial u}{\partial y} = 2abu$.

Sol.: Since $u = e^{ax+by}\phi(ax - by)$. ....(i)

Differentiate (i) partially w. r. t. x and y separately, we get

$$\frac{\partial u}{\partial x} = e^{ax+by}a.\phi(ax - by) + e^{ax+by}.\phi'(ax - by)a,$$

and $\frac{\partial u}{\partial y} = e^{ax+by}b.\phi(ax - by) + e^{ax+by}.\phi'(ax - by)(-b)$

Now $b\frac{\partial u}{\partial x} + a\frac{\partial u}{\partial y} = 2abe^{ax+by}\phi(ax - by) = 2abu$.

This completes the proof.

Q.No.22: If $x = r\cos\theta$, $y = r\sin\theta$, then show that

(i) $\frac{\partial x}{\partial r} = \frac{\partial r}{\partial x}$, **(ii)** $r\frac{\partial \theta}{\partial x} = \frac{1}{r}\frac{\partial x}{\partial \theta}$.

Sol.: (i) Given $x = r\cos\theta$, $y = r\sin\theta \Rightarrow x^2 + y^2 = r^2$ ......(i)

Differentiating (i) w. r. t. x partially (keeping y as constant), we get

$$2x + 0 = 2r\frac{\partial r}{\partial x} \Rightarrow r\frac{\partial r}{\partial x} = x = r\cos\theta \Rightarrow \frac{\partial r}{\partial x} = \cos\theta \quad ......(ii)$$

Also since $x = r\cos\theta \Rightarrow \frac{\partial x}{\partial r} = \cos\theta$. ....(iii)

Comparing (ii) and (iii) ,we get $\frac{\partial x}{\partial r} = \frac{\partial r}{\partial x}$. Ans.

This completes the proof.

(ii) To show : $r\frac{\partial \theta}{\partial x} = \frac{1}{r}\frac{\partial x}{\partial \theta}$.

Now since $x = r\cos\theta$, $y = r\sin\theta \Rightarrow \tan\theta = \frac{y}{x} \Rightarrow \theta = \tan^{-1}\frac{y}{x}$

$$\therefore \frac{\partial\theta}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}}\cdot\left(\frac{-y}{x^2}\right) = \frac{-y}{x^2+y^2}.$$

Now $r\frac{\partial\theta}{\partial x} = r.\left(\frac{-y}{x^2+y^2}\right) = r.\left(\frac{-y}{r^2}\right) = \frac{-y}{r}$. ...(i)

since $x = r\cos\theta \;\therefore \frac{\partial x}{\partial\theta} = -r\sin\theta \Rightarrow \frac{1}{r}\frac{\partial x}{\partial\theta} = -\sin\theta = -\frac{y}{r}$. .....(ii)

Comparing (i) and (ii) ,we get $r\frac{\partial\theta}{\partial x} = \frac{1}{r}\frac{\partial x}{\partial\theta}$. This completes the proof**.**

Q.No.23: If $x = r\cos\theta$, $y = r\sin\theta$, prove that

(i) $\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right]$

(ii) $\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = 0 \quad (x \neq 0, y \neq 0)$

Sol.: (i) Given $x = r\cos\theta$, $y = r\sin\theta$.

[By looking at the answer we find that we need partial derivative of r w. r. t. x and y. Therefore, let us express r as an explicit function of x and y]

Squaring and adding $x = r\cos\theta$, $y = r\sin\theta$; we find that

$r^2 = x^2 + y^2$ i.e. $r = \sqrt{x^2+y^2}$. ...(i)

Differentiating (i) w. r. t. x partially (keeping y as constant), we get

$$\frac{\partial r}{\partial x} = \frac{1}{2}\left(x^2+y^2\right)^{-1/2}.2x = \left(x^2+y^2\right)^{-1/2}.x = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{r}. \quad ...(ii)$$

Similarly, differentiating (i) w. r. t. y partially (keeping x as constant), we get

$$\frac{\partial r}{\partial y} = \frac{1}{2}\left(x^2+y^2\right)^{-1/2}.2y = \left(x^2+y^2\right)^{-1/2}.y = \frac{y}{\sqrt{x^2+y^2}} = \frac{y}{r}. \quad .(iii)$$

Again differentiating(ii) w. r. t. x partially (keeping y as constant), we get

$$\frac{\partial^2 r}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{x}{r}\right)=\frac{r\frac{\partial}{\partial x}(x)-x\frac{\partial}{\partial x}(r)}{r^2}=\frac{r-x\frac{\partial r}{\partial x}}{r^2}=\frac{r-x.\frac{x}{r}}{r^2}=\frac{r^2-x^2}{r^3}=\frac{y^2}{r^3}.$$

Again differentiating(iii) w. r. t. y partially (keeping x as constant), we get

$$\frac{\partial^2 r}{\partial y^2}=\frac{\partial}{\partial x}\left(\frac{y}{r}\right)=\frac{r\frac{\partial}{\partial y}(y)-y\frac{\partial}{\partial y}(r)}{r^2}=\frac{r-y\frac{\partial r}{\partial y}}{r^2}=\frac{r-y.\frac{y}{r}}{r^2}=\frac{r^2-y^2}{r^3}=\frac{x^2}{r^3}.$$

$$\text{L.H.S.}=\frac{\partial^2 r}{\partial x^2}+\frac{\partial^2 r}{\partial y^2}=\frac{y^2}{r^3}+\frac{x^2}{r^3}=\frac{x^2+y^2}{r^3}=\frac{r^2}{r^3}=\frac{1}{r}.$$

$$\text{R.H.S.}=\frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2+\left(\frac{\partial r}{\partial y}\right)^2\right]=\frac{1}{r}\left[\frac{x^2}{r^2}+\frac{y^2}{r^2}\right]=\frac{1}{r}\left[\frac{x^2+y^2}{r^2}\right]=\frac{1}{r}\left[\frac{r^2}{r^2}\right]=\frac{1}{r}.$$

$\therefore$ L.H.S.= R.H.S. This completes the proof**.**

(ii) It is given that $x=r\cos\theta$, $y=r\sin\theta$. Dividing ,we get $\tan\theta=\frac{y}{x}$

$$\therefore \theta=\tan^{-1}\frac{y}{x}\ . \qquad \text{...(i)}$$

Differentiating (i) w. r. t. x partially (keeping y as constant), we get

$$\frac{\partial\theta}{\partial x}=\frac{\partial}{\partial x}\tan^{-1}\frac{y}{x}=\frac{1}{1+\frac{y^2}{x^2}}\frac{\partial}{\partial x}\left(\frac{y}{x}\right)=\frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right)=-\frac{y}{x^2+y^2}. \qquad \text{...(ii)}$$

Again differentiating (ii) w. r. t. x partially (keeping y as constant), we get

$$\frac{\partial^2\theta}{\partial x^2}=\frac{\partial}{\partial x}\left(-\frac{y}{x^2+y^2}\right)=\frac{(x^2+y^2)(0)-(-y).2x}{(x^2+y^2)^2}=\frac{2xy}{(x^2+y^2)^2}. \qquad \text{...(iii)}$$

Differentiating (i) w. r. t. y partially (keeping x as constant), we get

$$\frac{\partial\theta}{\partial y}=\frac{\partial}{\partial y}\tan^{-1}\frac{y}{x}=\frac{1}{1+\frac{y^2}{x^2}}\frac{\partial}{\partial y}\left(\frac{y}{x}\right)=\frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right)=\frac{x}{x^2+y^2}. \qquad \text{...(iv)}$$

Again differentiating (iv) w. r. t. y partially (keeping x as constant), we get

$$\frac{\partial^2\theta}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right)=\frac{(x^2+y^2)(0)-(x).2y}{(x^2+y^2)^2}=-\frac{2xy}{(x^2+y^2)^2}. \qquad \text{...(v)}$$

Adding (iv) and (v), we get

$$\text{L.H.S.} = \frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = \frac{2xy}{\left(x^2+y^2\right)^2} - \frac{2xy}{\left(x^2+y^2\right)^2} = 0 = \text{R.H.S.}$$ This completes the proof.

Q.No.24: If $u = f\left(ax^2 + 2hxy + by^2\right)$ and $v = \varphi\left(ax^2 + 2hxy + by^2\right)$, prove that

$$\frac{\partial}{\partial y}\left(u\frac{\partial v}{\partial x}\right) = \frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial y}\right).$$

Sol.: Given $u = f\left(ax^2 + 2hxy + by^2\right)$ ...(i)

and $v = \varphi\left(ax^2 + 2hxy + by^2\right)$ ....(ii)

Differentiating (ii) partially w. r. t. x and y separately, we get

$$\frac{\partial v}{\partial x} = \varphi'\left(ax^2 + 2hxy + by^2\right)(2ax + 2hy) = \varphi'.(2ax + 2hy)$$

$$\frac{\partial v}{\partial y} = \varphi'\left(ax^2 + 2hxy + by^2\right)(2by + 2hx) = \varphi'.(2by + 2hx)$$

Now L.H.S.= $\frac{\partial}{\partial y}\left(u\frac{\partial v}{\partial x}\right) = \frac{\partial}{\partial y}\left[f.\varphi'.(2ax + 2hy)\right]$

$$= f'.(2by + 2hx).\varphi'.(2ax + 2hy) + f.\varphi''.(2by + 2hx).(2ax + 2hy) + f.\varphi'.2h$$

$$= (2ax + 2hy).(2by + 2hx).\left[f'\varphi' + f\varphi''\right] + 2h.f.\varphi'. \quad ....(iii)$$

R.H.S.= $\frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial y}\right) = \frac{\partial}{\partial x}\left[f.\varphi'.(2by + 2hx)\right]$

$$= f'.(2ax + 2hy).\varphi'.(2by + 2hx) + f.\varphi''.(2ax + 2hy).(2by + 2hx) + f.\varphi'.2h$$

$$= (2ax + 2hy).(2by + 2hx).\left[f'\varphi' + f\varphi''\right] + 2h.f.\varphi'. \quad (iv)$$

From (iii) and (iv), we have $\frac{\partial}{\partial y}\left(u\frac{\partial v}{\partial x}\right) = \frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial y}\right)$. This completes the proof.

Q.No.25: If $u = \left(x^2 - y^2\right)f(t)$, where t = x y, prove that

$$\frac{\partial^2 u}{\partial x \partial y} = \left(x^2 - y^2\right)\left[tf''(t) + 3f'(t)\right]$$

Sol.: Given $u = \left(x^2 - y^2\right)f(t) = \left(x^2 - y^2\right)f(xy) = x^2 f(xy) - y^2 f(xy)$. (i)

Differentiating (i) partially w. r. t. x and y separately, we get

$$\frac{\partial u}{\partial x}=\left[2x.f(xy)+x^2.f'(xy)y\right]-\left[y^2.f'(xy)y\right]=2xf(xy)+x^2yf'(xy)-y^3f'(xy)$$

$$\therefore \frac{\partial^2 u}{\partial x\partial y}=\frac{\partial^2 u}{\partial y\partial x}=\frac{\partial}{\partial y}\left[\frac{\partial u}{\partial x}\right]=\frac{\partial}{\partial y}\left[2xf(xy)+x^2yf'(xy)-y^3f'(xy)\right]$$

$$=\left[2xf'(xy)x\right]+\left[x^2y.f''(xy)x+x^2.f'(xy)\right]-\left[y^3.f''(xy)x+3y^2.f'(xy)\right]$$

$$=\left[2x^2f'(t)\right]+\left[x^3yf''(t)+x^2f'(t)\right]-\left[y^3xf''(t)+3y^2f'(t)\right]$$

$$=3x^2f'(t)-3y^2f'(t)+\left(x^3y-y^3x\right)f''(t)$$

$$=3\left(x^2-y^2\right)f'(t)+xy\left(x^2-y^2\right)f''(t)$$

$$=\left(x^2-y^2\right)tf''(t)+\left(x^2-y^2\right)3f'(t)$$

Hence $\dfrac{\partial^2 u}{\partial x\partial y}=\left(x^2-y^2\right)\left[tf''(t)+3f'(t)\right]$**.** This completes the proof**.**

Q.No.26: If u and v are functions of x and y defined by $x=u+e^{-v}\sin u$,

$y=v+e^{-v}\cos u$, then prove that $\dfrac{\partial u}{\partial y}=\dfrac{\partial v}{\partial x}$.

Sol.: Given $x=u+e^{-v}\sin u$ and $y=v+e^{-v}\cos u$.

Differentiating both the equations partially w. r .t. x and y separately, we get

$$1=\frac{\partial u}{\partial x}+e^{-v}\cos u\frac{\partial u}{\partial x}+e^{-v}\left(-\frac{\partial v}{\partial x}\right)\sin u \Rightarrow 1=\frac{\partial u}{\partial x}\left[1+e^{-v}\cos u\right]-e^{-v}\frac{\partial v}{\partial x}\sin u \qquad \text{(i)}$$

$$0=\frac{\partial u}{\partial y}+e^{-v}\cos u\frac{\partial u}{\partial y}+e^{-v}\left(-\frac{\partial v}{\partial y}\right)\sin u \Rightarrow 0=\frac{\partial u}{\partial y}\left[1+e^{-v}\cos u\right]-e^{-v}\frac{\partial v}{\partial y}\sin u \qquad \text{(ii)}$$

$$0=\frac{\partial v}{\partial x}+e^{-v}(-\sin u)\frac{\partial u}{\partial x}+e^{-v}\left(-\frac{\partial v}{\partial x}\right)\cos u \Rightarrow 0=\frac{\partial v}{\partial x}\left[1-e^{-v}\cos u\right]-e^{-v}\frac{\partial u}{\partial x}\sin u \qquad \text{(iii)}$$

$$1=\frac{\partial v}{\partial y}+e^{-v}(-\sin u)\frac{\partial u}{\partial y}+e^{-v}\left(-\frac{\partial v}{\partial y}\right)\cos u \Rightarrow 1=\frac{\partial v}{\partial y}\left[1-e^{-v}\cos u\right]-e^{-v}\frac{\partial u}{\partial y}\sin u \qquad \text{(iv)}$$

Multiplying (i) by $e^{-v}\sin u$ and (iii) by $\left[1+e^{-v}\cos u\right]$ and then adding, we get

$$\frac{\partial v}{\partial x}=\frac{e^{-v}\sin u}{1-e^{-2v}} \qquad \text{(v)}$$

Multiplying (ii) by $\left[1-e^{-v}\cos u\right]$and (iv) by $e^{-v}\sin u$ and then adding, we get

$$\frac{\partial u}{\partial y} = \frac{e^{-v}\sin u}{1-e^{-2v}} \qquad \text{(vi)}$$

From (v) and (vi), we get

$\dfrac{\partial u}{\partial y} = \dfrac{\partial v}{\partial x}$. This completes the proof.

Q.No.27: If $z(x+y) = x^2 + y^2$, show that $\left(\dfrac{\partial z}{\partial x} - \dfrac{\partial z}{\partial y}\right)^2 = 4\left(1 - \dfrac{\partial z}{\partial x} - \dfrac{\partial z}{\partial y}\right)$.

Sol.: Since $z(x+y) = x^2 + y^2 \Rightarrow z = \dfrac{x^2+y^2}{x+y}$. ...(i)

Differentiating (i) partially w. r. t. x and y separately, we get

$$\frac{\partial z}{\partial x} = \frac{(x+y).2x - \left(x^2+y^2\right).1}{(x+y)^2} = \frac{x^2 - y^2 + 2xy}{(x+y)^2}$$

$$\frac{\partial z}{\partial y} = \frac{(x+y).2y - \left(x^2+y^2\right).1}{(x+y)^2} = \frac{y^2 - x^2 + 2xy}{(x+y)^2}$$

Now L.H.S.$= \left(\dfrac{\partial z}{\partial x} - \dfrac{\partial z}{\partial y}\right)^2 = \left[\dfrac{x^2-y^2+2xy}{(x+y)^2} - \dfrac{y^2-x^2+2xy}{(x+y)^2}\right]^2$

$$= \left[\frac{(x^2-y^2+2xy)-(y^2-x^2+2xy)}{(x+y)^2}\right]^2 = \left[\frac{2x^2-2y^2}{(x+y)^2}\right]^2 = \left[\frac{2(x-y)(x+y)}{(x+y)^2}\right]^2$$

$$= \left[\frac{2(x-y)}{(x+y)}\right]^2 = \frac{4(x-y)^2}{(x+y)^2}. \qquad \text{(ii)}$$

R.H.S.$= 4\left(1 - \dfrac{\partial z}{\partial x} - \dfrac{\partial z}{\partial y}\right) = 4\left[1 - \dfrac{\left(x^2-y^2+2xy\right)}{(x+y)^2} - \dfrac{\left(y^2-x^2+2xy\right)}{(x+y)^2}\right]$

$$= 4\left[\frac{(x^2+y^2+2xy)-(x^2-y^2+2xy)-(y^2-x^2+2xy)}{(x+y)^2}\right] = 4\left[\frac{x^2+y^2-2xy}{(x+y)^2}\right]$$

$$= \frac{4(x-y)^2}{(x+y)^2}. \qquad \text{(iii)}$$

From (ii) and (iii), we have L.H.S.=R.H.S**.** This completes the proof**.**

Q.No.28: If $u = x^y$, show that $\dfrac{\partial^3 u}{\partial x^2 \partial y} = \dfrac{\partial^3 u}{\partial x \partial y \partial x}$.

Sol.: Since $u = x^y$. (i)

For $\dfrac{\partial^3 u}{\partial x^2 \partial y}$, first differentiate (i) partially w. r. t. y and then twice w. r. t. x

$\therefore \dfrac{\partial u}{\partial y} = x^y \log x$. Now differentiate twice w. r. t. x, we get

$$\frac{\partial^2 u}{\partial x \partial y} = x^y.\frac{1}{x} + \log x . y x^{y-1} = x^{y-1} + y \log x^{y-1} = x^{y-1}(1 + y \log x) \text{ and}$$

$$\frac{\partial^3 u}{\partial x^2 \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial^2 u}{\partial x \partial y}\right] = (1 + y \log x).(y-1)x^{y-2} + x^{y-1}.\frac{y}{x} = x^{y-2}[(1 + y \log x)(y-1) + y]. \quad \text{(ii)}$$

For $\dfrac{\partial^3 u}{\partial x \partial y \partial x}$, first differentiate (i) partially w. r. t. x , then y and then x

$\therefore \dfrac{\partial u}{\partial x} = y x^{y-1}$. Now differentiate partially w. r t. y, we get

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left[\frac{\partial u}{\partial x}\right] = y.x^{y-1} \log x + x^{y-1} = (1 + y \log x) x^{y-1}.$$

Now again differentiate partially w. r. t. x, we get

$$\frac{\partial}{\partial x}\left[\frac{\partial^2 u}{\partial y \partial x}\right] = x^{y-2}[(1 + y \log x)(y-1) + y]. \quad \text{(iii)}$$

Hence from (ii) and (iii), $\dfrac{\partial^3 u}{\partial x^2 \partial y} = \dfrac{\partial^3 u}{\partial x \partial y \partial x}$. This completes the proof.

Q.No.29: If $\dfrac{x^2}{a^2 + u} + \dfrac{y^2}{b^2 + u} + \dfrac{z^2}{c^2 + u} = 1$, where u is a function of x , y , z ; prove that

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = 2\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}\right).$$

Sol.: Since $\dfrac{x^2}{a^2 + u} + \dfrac{y^2}{b^2 + u} + \dfrac{z^2}{c^2 + u} = 1$.

Now differentiate partially w. r. t. x, we get

$$\frac{(a^2+u)2x - x^2\left(\frac{\partial u}{\partial x}\right)}{(a^2+u)^2} + \frac{-y^2\left(\frac{\partial u}{\partial x}\right)}{(b^2+u)^2} + \frac{-z^2\left(\frac{\partial u}{\partial x}\right)}{(c^2+u)^2} = 0$$

$$\Rightarrow \frac{(a^2+u)2x - x^2\left(\frac{\partial u}{\partial x}\right)}{(a^2+u)^2} - \frac{y^2\left(\frac{\partial u}{\partial x}\right)}{(b^2+u)^2} - \frac{z^2\left(\frac{\partial u}{\partial x}\right)}{(c^2+u)^2} = 0$$

$$\Rightarrow \frac{(a^2+u)2x - x^2\left(\frac{\partial u}{\partial x}\right)}{(a^2+u)^2} = \frac{y^2\left(\frac{\partial u}{\partial x}\right)}{(b^2+u)^2} + \frac{z^2\left(\frac{\partial u}{\partial x}\right)}{(c^2+u)^2}$$

$$\Rightarrow \frac{2x}{(a^2+u)} = \frac{x^2\left(\frac{\partial u}{\partial x}\right)}{(a^2+u)^2} + \frac{y^2\left(\frac{\partial u}{\partial x}\right)}{(b^2+u)^2} + \frac{z^2\left(\frac{\partial u}{\partial x}\right)}{(c^2+u)^2}$$

$$\Rightarrow \frac{2x}{(a^2+u)} = \left[\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right]\left(\frac{\partial u}{\partial x}\right)$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{2x}{(a^2+u)} \div \left[\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right]$$

Similarly $\frac{\partial u}{\partial y} = \frac{2y}{(b^2+u)} \div \left[\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right]$,

$$\frac{\partial u}{\partial z} = \frac{2z}{(c^2+u)} \div \left[\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right]$$

Now L.H.S.= $\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = \frac{\left\{\frac{2x}{(a^2+u)}\right\}^2 + \left\{\frac{2y}{(b^2+u)}\right\}^2 + \left\{\frac{2z}{(c^2+u)}\right\}^2}{\left[\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right]^2}$

$$= \frac{4}{\left[\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right]}$$

$$\text{R.H.S.}= 2\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}\right)=\frac{2\left[x.\frac{2x}{(a^2+u)^2}+y.\frac{2y}{(b^2+u)^2}+z.\frac{2z}{(c^2+u)^2}\right]}{\left[\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right]}$$

$$=\frac{4}{\left[\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right]}$$

$$=\text{L.H.S.}$$

Hence $\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2=2\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}\right)$**.** This completes the proof**.**

Q.No.30: If $v=\left(x^2+y^2+z^2\right)^{-1/2}$. Show that $\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}=0$ **.**

Sol.: Since $v=\left(x^2+y^2+z^2\right)^{-1/2}$, we have

$$\frac{\partial v}{\partial x}=-\frac{1}{2}\left(x^2+y^2+z^2\right)^{-3/2}.2x=-x\left(x^2+y^2+z^2\right)^{-3/2}.$$

and

$$\frac{\partial^2 v}{\partial x^2}=\frac{\partial}{\partial x}\left[-x\left(x^2+y^2+z^2\right)^{-3/2}\right]=-\left[1.\left(x^2+y^2+z^2\right)^{-3/2}+x\left(-\frac{3}{2}\right)\left(x^2+y^2+z^2\right)^{-5/2}.2x\right]$$

$$=-\left(x^2+y^2+z^2\right)^{-5/2}\left[x^2+y^2+z^2-3x^2\right]=\left(x^2+y^2+z^2\right)^{-5/2}\left(2x^2-y^2-z^2\right) \quad ..(i)$$

Similarly, $\frac{\partial^2 v}{\partial y^2}=\left(x^2+y^2+z^2\right)^{-5/2}\left(-x^2+2y^2-z^2\right).$ ...(ii)

and $\frac{\partial^2 v}{\partial z^2}=\left(x^2+y^2+z^2\right)^{-5/2}\left(-x^2-y^2+2z^2\right).$ ...(iii)

Adding (i), (ii) and (iii), we have

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}=\left(x^2+y^2+z^2\right)^{-5/2}(0)=0 \text{ .}$$

This completes the proof**.**

Q.No.31: If $V = r^m$, $r^2 = x^2 + y^2 + z^2$, then show that

$$V_{xx} + V_{yy} + V_{zz} = m(m+1)r^{m-2}.$$

Sol.: Since $r^2 = x^2 + y^2 + z^2$ $\therefore 2r\dfrac{\partial r}{\partial x} = 2x \Rightarrow \dfrac{\partial r}{\partial x} = \dfrac{x}{r}$

Now $V = r^m$ $\therefore \dfrac{\partial V}{\partial x} = mr^{m-1}.\dfrac{x}{r} = mxr^{m-2}$ and

$$\therefore \frac{\partial^2 V}{\partial x^2} = m\left[r^{m-2} + x(m-2)r^{m-3}\frac{\partial r}{\partial x}\right] = m\left[r^{m-2} + x(m-2)r^{m-3}\frac{x}{r}\right]$$

$$\Rightarrow \frac{\partial^2 V}{\partial x^2} = m\left[r^{m-2} + (m-2)x^2 r^{m-4}\right]. \qquad \text{......(i)}$$

Similarly, $\dfrac{\partial^2 V}{\partial y^2} = m\left[r^{m-2} + (m-2)y^2 r^{m-4}\right]$ ......(ii)

and $\dfrac{\partial^2 V}{\partial z^2} = m\left[r^{m-2} + (m-2)z^2 r^{m-4}\right].$ .....(iii)

Adding (i), (ii) and (iii), we get

$$V_{xx} + V_{yy} + V_{zz} = m\left[3r^{m-2} + (m-2)r^2 r^{m-4}\right] = m\left[r^{m-2}(3+m-2)\right] = m(m+1)r^{m-2}.$$

This completes the proof.

Q.No.32: If $u = \log(\tan x + \tan y + \tan z)$, then prove that

$$\sin 2x\frac{\partial u}{\partial x} + \sin 2y\frac{\partial u}{\partial y} + \sin 2z\frac{\partial u}{\partial z} = 2.$$

Sol.: Here $u = \log(\tan x + \tan y + \tan z)$. ....(i)

Differentiate (i) partially w. r. t. x, y and z separately, we get

$$\frac{\partial u}{\partial x} = \frac{\sec^2 x}{\tan x + \tan y + \tan z}, \quad \frac{\partial u}{\partial y} = \frac{\sec^2 y}{\tan x + \tan y + \tan z} \quad \text{and} \quad \frac{\partial u}{\partial z} = \frac{\sec^2 z}{\tan x + \tan y + \tan z}.$$

Now L.H.S.= $\sin 2x\dfrac{\partial u}{\partial x} + \sin 2y\dfrac{\partial u}{\partial y} + \sin 2z\dfrac{\partial u}{\partial z}$

$$= \frac{2\sin x\cos x.\dfrac{1}{\cos^2 x} + 2\sin y\cos y.\dfrac{1}{\cos^2 y} + 2\sin z\cos z.\dfrac{1}{\cos^2 z}}{\tan x + \tan y + \tan z}$$

$$= \frac{2(\tan x + \tan y + \tan z)}{(\tan x + \tan y + \tan z)} = 2 = \text{R.H.S.}$$

This completes the proof.

Q.No.33: If $u = \dfrac{xy(x^2 - y^2)}{x^2 + y^2}$; $u(0,0) = 0$, show that $\dfrac{\partial^2 u}{\partial x \partial y} \neq \dfrac{\partial^2 u}{\partial y \partial x}$ at $\begin{matrix} x = 0 \\ y = 0 \end{matrix}$.

Sol.: For $(x, y) \neq (0,0)$, $u(x, y) = \dfrac{xy(x^2 - y^2)}{x^2 + y^2}$ (given) ....(i)

Differentiating (i) partially w. r. t. x, we get

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left[\frac{xy(x^2 - y^2)}{x^2 + y^2}\right] = y\frac{\partial}{\partial x}\left[\frac{x^3 - xy^2}{x^2 + y^2}\right] = y\left[\frac{(x^2 + y^2)(3x^2 - y^2) - (x^3 - xy^2).2x}{(x^2 + y^2)^2}\right]$$

$$= y\left[\frac{3x^4 + 2x^2y^2 - y^4 - 2x^4 + 2x^2y^2}{(x^2 + y^2)^2}\right] = y\left[\frac{x^4 + 4x^2y^2 - y^4}{(x^2 + y^2)^2}\right]$$

$\therefore$ For $(x, y) \neq (0,0)$, $\dfrac{\partial u}{\partial x} = u_x(x, y) = \dfrac{y(x^4 + 4x^2y^2 - y^4)}{(x^2 + y^2)^2}$. ...(ii)

For $\dfrac{\partial u}{\partial x}(0,0)$, let us consider $\dfrac{\partial u}{\partial x}(0,0) = \lim\limits_{\delta x \to 0} \dfrac{u(\delta x, 0) - u(0,0)}{\delta x} = \lim\limits_{\delta x \to 0} \dfrac{0 - 0}{\delta x} = 0$.

which exists. $\therefore \dfrac{\partial u}{\partial x}(0,0) = 0$.

For the existence of $u_{yx}(0,0)$, i.e. $\dfrac{\partial}{\partial y}\left[\dfrac{\partial u}{\partial x}\right]_{(0,0)}$

Consider $\dfrac{\partial}{\partial y}\left[\dfrac{\partial u}{\partial x}\right]_{(0,0)} = \lim\limits_{\delta y \to 0} \dfrac{u_x(0, \delta y) - u_x(0,0)}{\delta y} = \lim\limits_{\delta y \to 0} \dfrac{-\delta y - 0}{\delta y} = -1$, which exists.

$\therefore \dfrac{\partial}{\partial y}\left[\dfrac{\partial u}{\partial x}\right]_{(0,0)} = -1$. ...(iii)

Again because for $(x, y) \neq (0,0)$, $u(x, y) = \dfrac{xy(x^2 - y^2)}{x^2 + y^2}$ (given) ....(i)

Differentiating (i) partially w. r. t. x, we get

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left[\frac{xy(x^2 - y^2)}{x^2 + y^2}\right] = x\frac{\partial}{\partial y}\left[\frac{yx^2 - y^3}{x^2 + y^2}\right] = x\left[\frac{(x^2 + y^2)(x^2 - 3y^2) - (x^2y - y^3).2y}{(x^2 + y^2)^2}\right]$$

$$= x\left[\frac{x^4-2x^2y^2-3y^4-2x^2x^2+2y^4}{\left(x^2+y^2\right)^2}\right] = x\left[\frac{x^4-4x^2y^2-y^4}{\left(x^2+y^2\right)^2}\right]$$

$\therefore$ For $(x,y)\neq(0,0)$, $\frac{\partial u}{\partial y}=u_y(x,y)=\frac{x\left(x^4-4x^2y^2-y^4\right)}{\left(x^2+y^2\right)^2}$. ...(iv)

For $\frac{\partial u}{\partial y}(0,0)$, let us consider $\frac{\partial u}{\partial y}(0,0)=\lim\limits_{\delta y\to 0}\frac{f(0,\delta y)-f(0,0)}{\delta y}=\lim\limits_{\delta y\to 0}\frac{0-0}{\delta y}=0$.

which exists. $\therefore \frac{\partial u}{\partial y}(0,0)=0$. For the existence of $u_{xy}(0,0)$, i.e. $\frac{\partial}{\partial x}\left[\frac{\partial u}{\partial y}\right]_{(0,0)}$

Consider $\frac{\partial}{\partial x}\left[\frac{\partial u}{\partial y}\right]_{(0,0)} = \lim\limits_{\delta x\to 0}\frac{u_y(\delta x,0)-u_y(0,0)}{\delta x}=\lim\limits_{\delta y\to 0}\frac{\delta x-0}{\delta x}=1$, which exists.

$\therefore \frac{\partial}{\partial x}\left[\frac{\partial u}{\partial y}\right]_{(0,0)}=1$. ...(v)

$\therefore$ From (iii) and (v), we get $\frac{\partial^2 u}{\partial x\partial y}\neq\frac{\partial^2 u}{\partial y\partial x}$ at $\begin{matrix}x=0\\y=0\end{matrix}$.

i.e. $u_{yx}(0,0)\neq u_{xy}(0,0)$.

This completes the proof.

Q.No.34: If $\theta=t^n e^{-\frac{r^2}{4t}}$, find the value of n which will make $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right)=\frac{\partial\theta}{\partial t}$.

Sol.: Given $\theta=t^n e^{-\frac{r^2}{4t}}$.

$$\frac{\partial\theta}{\partial r}=t^n.e^{-\frac{r^2}{4t}}.\left(-\frac{2r}{4t}\right)=-\frac{1}{2}rt^{n-1}e^{-\frac{r^2}{4t}}$$

$$\therefore r^2\frac{\partial\theta}{\partial r}=-\frac{1}{2}r^3.t^{n-1}e^{-\frac{r^2}{4t}}$$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right)=-\frac{1}{2}t^{n-1}\left[3r^2e^{-\frac{r^2}{4t}}+r^3e^{-\frac{r^2}{4t}}\left(-\frac{2r}{4t}\right)\right]=-\frac{1}{2}t^{n-1}r^2e^{-\frac{r^2}{4t}}\left[3-\frac{r^2}{2t}\right]$$

$$\therefore \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right)=\frac{1}{2}t^{n-1}e^{-\frac{r^2}{4t}}\left(\frac{r^2}{2t}-3\right)$$

Also $\frac{\partial\theta}{\partial t}=nt^{n-1}e^{-\frac{r^2}{4t}}+t^n e^{-\frac{r^2}{4t}}.\left(\frac{r^2}{4t^2}\right)=t^{n-1}e^{-\frac{r^2}{4t}}.\left(n+\frac{r^2}{4t^2}\right).$

Since $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right)=\frac{\partial\theta}{\partial t}$ is given

$$\therefore \frac{1}{2}t^{n-1}e^{-\frac{r^2}{4t}}.\left(\frac{r^2}{2t}-3\right)=t^{n-1}e^{-\frac{r^2}{4t}}.\left(n+\frac{r^2}{4t}\right)$$

$\Rightarrow \frac{r^2}{4t}-\frac{3}{2}=n+\frac{r^2}{4t}$ $\qquad\qquad \therefore n=-\frac{3}{2}$. Ans.

Q.No.35: If $u=f(r)$, where $r^2=x^2+y^2$, prove that $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=f''(r)+\frac{1}{r}f'(r)$.

Sol.: Given $r^2=x^2+y^2$. (i)

Differentiating partially w. r. t., we get $2r\frac{\partial r}{\partial x}=2x\Rightarrow\frac{\partial r}{\partial x}=\frac{x}{r}$

Similarly, $\frac{\partial r}{\partial y}=\frac{x}{r}$.

Now $u=f(r)$ $\therefore \frac{\partial u}{\partial x}=f'(r).\frac{\partial r}{\partial x}=\frac{x}{r}f'(r)$

Differentiating again w. r. t. x, we get

$$\frac{\partial^2 u}{\partial x^2}=\frac{1}{r}f'(r)+x.\left(-\frac{1}{r^2}\frac{\partial r}{\partial x}\right)f'(r)+\frac{x}{r}f''(r).\frac{\partial r}{\partial x}$$

$$\left[\because -\frac{\partial}{\partial x}(uvw)=vw\frac{\partial}{\partial x}(u)+uw\frac{\partial}{\partial x}(v)+uv\frac{\partial}{\partial x}(w)\right]$$

$$=\frac{1}{r}f'(r)-\frac{x}{r^2}.\frac{x}{r}f'(r)+\frac{x}{r}.f''(r).\frac{x}{r}=\frac{1}{r}f'(r)-\frac{x^2}{r^3}f'(r)+\frac{x^2}{r^2}f''(r)$$

$$=\frac{r^2-x^2}{r^3}f'(r)+\frac{x^2}{r^2}f''(r)=\frac{y^2}{r^3}f'(r)+\frac{x^2}{r^2}f''(r) \qquad \text{[using (i)]}$$

Similarly, $\dfrac{\partial^2 u}{\partial y^2} = \dfrac{x^2}{r^3} f'(r) + \dfrac{y^2}{r^2} f''(r)$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{x^2 + y^2}{r^3} f'(r) + \frac{x^2 + y^2}{r^2} f''(r) = \frac{r^2}{r^3} f'(r) + \frac{r^2}{r^2} f''(r)$$

$\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = f''(r) + \dfrac{1}{r} f'(r)$. Hence prove.

Q.No.36: If $x = e^{r\cos\theta}\cos(r\sin\theta)$ and $y = e^{r\cos\theta}\sin(r\sin\theta)$,

prove that $\dfrac{\partial x}{\partial r} = \dfrac{1}{r}.\dfrac{\partial y}{\partial \theta}$, $\dfrac{\partial y}{\partial r} = -\dfrac{1}{r}.\dfrac{\partial x}{\partial \theta}$.

Hence deduce that $\dfrac{\partial^2 x}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial x}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 x}{\partial \theta^2} = 0$.

Sol.: Given $x = e^{r\cos\theta}\cos(r\sin\theta)$**.**

$$\therefore \frac{\partial x}{\partial r} = e^{r\cos\theta}.\cos\theta.\cos(r\sin\theta) - e^{r\cos\theta}.\sin(r\sin\theta).\sin\theta$$

$$= e^{r\cos\theta}[\cos\theta\cos(r\sin\theta) - \sin\theta\sin(r\sin\theta)]$$

$$= e^{r\cos\theta}\cos(\theta + r\sin\theta) \qquad \text{(i)}$$

$$\frac{\partial x}{\partial \theta} = e^{r\cos\theta}.(-r\sin\theta).\cos(r\sin\theta) - e^{r\cos\theta}.\sin(r\sin\theta).r\cos\theta$$

$$= -re^{r\cos\theta}[\sin\theta\cos(r\sin\theta) + \cos\theta\sin(r\sin\theta)]$$

$$= -re^{r\cos\theta}\sin(\theta + r\sin\theta) \qquad \text{(ii)}$$

Also $y = e^{r\cos\theta}\sin(r\sin\theta)$

$$\frac{\partial y}{\partial r} = e^{r\cos\theta}.\cos\theta.\sin(r\sin\theta) + e^{r\cos\theta}.\cos(r\sin\theta).\sin\theta$$

$$= e^{r\cos\theta}[\sin\theta\cos(r\sin\theta) + \cos\theta\sin(r\sin\theta)]$$

$$= e^{r\cos\theta}\sin(\theta + r\sin\theta) \qquad \text{(iii)}$$

$$\frac{\partial y}{\partial \theta} = e^{r\cos\theta}.(-r\sin\theta).\sin(r\sin\theta) + e^{r\cos\theta}.\cos(r\sin\theta).r\cos\theta$$

$$= re^{r\cos\theta}[\cos\theta\cos(r\sin\theta) - \sin\theta\sin(r\sin\theta)]$$

$$= re^{r\cos\theta}\cos(\theta + r\sin\theta) \qquad \text{(iv)}$$

From (i) and (iv), we get $\frac{\partial x}{\partial r} = \frac{1}{r}.\frac{\partial y}{\partial \theta}$ (v)

From (ii) and (iii), we get $\frac{\partial y}{\partial r} = -\frac{1}{r}.\frac{\partial x}{\partial \theta}$ (vi)

From (v), we get $\frac{\partial^2 x}{\partial r^2} = -\frac{1}{r^2}.\frac{\partial y}{\partial \theta} + \frac{1}{r}.\frac{\partial^2 y}{\partial r \partial \theta}$

From(vi), we get $\frac{\partial x}{\partial \theta} = -r\frac{\partial y}{\partial r}$

$$\therefore \frac{\partial^2 x}{\partial \theta^2} = -r\frac{\partial^2 y}{\partial r \partial \theta} = -r\frac{\partial^2 y}{\partial r \partial \theta}$$

$$\frac{\partial^2 x}{\partial r^2} + \frac{1}{r}\frac{\partial x}{\partial r} + \frac{1}{r^2}\frac{\partial^2 x}{\partial \theta^2} = -\frac{1}{r^2}.\frac{\partial y}{\partial \theta} + \frac{1}{r}.\frac{\partial^2 y}{\partial r \partial \theta} + \frac{1}{r^2}.\frac{\partial y}{\partial \theta} - \frac{1}{r}.\frac{\partial^2 y}{\partial r \partial \theta} = 0.$$

Q.No.37: Prove that if $f(x, y) = \frac{1}{\sqrt{y}}.e^{-\frac{(x-a)^2}{4y}}$, then $f_{xy} = f_{yx}$.

Sol.: Given $f(x, y) = \frac{1}{\sqrt{y}}.e^{-\frac{(x-a)^2}{4y}} = y^{-\frac{1}{2}}.e^{-\frac{(x-a)^2}{4y}}$.

$$f_x = \frac{\partial f}{\partial x} = y^{-\frac{1}{2}}.e^{-\frac{(x-a)^2}{4y}}.\frac{\partial}{\partial x}\left[-\frac{(x-a)^2}{4y}\right]$$

$$= y^{-\frac{1}{2}}.e^{-\frac{(x-a)^2}{4y}}.\left[-\frac{2(x-a)}{4y}\right] = -\frac{1}{2}y^{-\frac{3}{2}}(x-a).e^{-\frac{(x-a)^2}{4y}}$$

$$f_y = \frac{\partial f}{\partial y} = -\frac{1}{2}y^{-\frac{3}{2}}.e^{-\frac{(x-a)^2}{4y}} + y^{-\frac{1}{2}}.e^{-\frac{(x-a)^2}{4y}}.\frac{\partial}{\partial y}\left[-\frac{(x-a)^2}{4y}\right]$$

$$= e^{-\frac{(x-a)^2}{4y}}.\left[-\frac{1}{2}y^{-\frac{3}{2}} + y^{-\frac{1}{2}}.\frac{(x-a)^2}{4y^2}\right] = \frac{1}{4}y^{-\frac{3}{2}}.e^{-\frac{(x-a)^2}{4y}}\left[-2 + y^{-1}(x-a)^2\right]$$

$$f_{xy} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$=\frac{1}{4}y^{-\frac{3}{2}}\left\{e^{-\frac{(x-a)^2}{4y}}.\frac{\partial}{\partial x}\left[-\frac{(x-a)^2}{4y}\right].\left[-2+y^{-1}(x-a)^2\right]+e^{-\frac{(x-a)^2}{4y}}.2y^{-1}(x-a)\right\}$$

$$=\frac{1}{4}y^{-\frac{3}{2}}.e^{-\frac{(x-a)^2}{4y}}\left\{-\frac{2(x-a)}{4y}\left[-2+y^{-1}(x-a)^2\right]+2y^{-1}(x-a)\right\}$$

$$=\frac{1}{4}y^{-\frac{3}{2}}.e^{-\frac{(x-a)^2}{4y}}.\frac{x-a}{y}\left\{-\frac{1}{2}\left[-2+y^{-1}(x-a)^2\right]+2\right\}$$

$$=\frac{1}{4}y^{-\frac{5}{2}}(x-a).e^{-\frac{(x-a)^2}{4y}}\left[3-\frac{(x-a)^2}{2y}\right].$$

$$f_{yx}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)=-\frac{1}{2}(x-a)\left[-\frac{3}{2}y^{-\frac{5}{2}}.e^{-\frac{(x-a)^2}{4y}}+y^{-\frac{3}{2}}.e^{-\frac{(x-a)^2}{4y}}.\frac{(x-a)^2}{4y^2}\right]$$

$$=-\frac{1}{4}(x-a).y^{-\frac{5}{2}}.e^{-\frac{(x-a)^2}{4y}}\left[-3+\frac{(x-a)^2}{2y}\right]$$

$$=\frac{1}{4}y^{-\frac{5}{2}}(x-a).e^{-\frac{(x-a)^2}{4y}}\left[3-\frac{(x-a)^2}{2y}\right].$$

Hence $f_{xy}=f_{yx}$.

Q.No38.: Find the value of $\frac{1}{a^2}\frac{\partial^2 z}{\partial x^2}+\frac{1}{b^2}\frac{\partial^2 z}{\partial y^2}$ when $a^2x^2+b^2y^2-c^2z^2=0$.

Sol.: Here $a^2x^2+b^2y^2-c^2z^2=0\Rightarrow c^2z^2=a^2x^2+b^2y^2$

$$\therefore z^2=\frac{1}{c^2}\left(a^2x^2+b^2y^2\right) \qquad \text{(i)}$$

Differentiating (i) partially w.r.t. x, we get

$$2z\frac{\partial z}{\partial x}=\frac{1}{c^2}.2a^2x\Rightarrow\frac{\partial z}{\partial z}=\frac{a^2}{c^2}\left(\frac{x}{z}\right) \qquad \text{(ii)}$$

Differentiating (ii) partially w.r.t. x, we get

$$\frac{\partial^2 z}{\partial x^2}=\frac{a^2}{c^2}\left[\frac{z.1-x\dfrac{\partial z}{\partial x}}{z^2}\right]=\frac{a^2}{c^2z^2}\left[z-x\frac{a^2}{c^2}\left(\frac{x}{z}\right)\right]=\frac{a^2}{c^2z^2}\left[z-\frac{a^2x^2}{c^2z}\right]=\frac{a^2}{c^2z^2.c^2z}\left[c^2z^2-a^2x^2\right]$$

$$=\frac{a^2}{c^4z^3}\left(b^2y^2\right)\quad\left[\because a^2x^2+b^2y^2-c^2z^2=0\right]$$

$$\Rightarrow\frac{\partial^2 z}{\partial x^2}=\frac{a^2b^2}{c^4}\frac{y^2}{z^3}\qquad\text{(iii)}$$

Similarly, $\therefore\dfrac{\partial^2 z}{\partial y^2}=\dfrac{a^2b^2}{c^4}\dfrac{x^2}{z^3}$ (iv)

Consider $\dfrac{1}{a^2}\dfrac{\partial^2 z}{\partial x^2}+\dfrac{1}{b^2}\dfrac{\partial^2 z}{\partial y^2}=\dfrac{1}{a^2}.\dfrac{a^2b^2}{c^4}\dfrac{y^2}{z^3}+\dfrac{1}{b^2}.\dfrac{a^2b^2}{c^4}\dfrac{x^2}{z^3}=\dfrac{1}{c^4z^3}\left[b^2y^2+a^2x^2\right]$

$$=\frac{1}{c^4z^3}\left(c^2z^2\right)\quad\left[\because a^2x^2+b^2y^2-c^2z^2=0\right]$$

$\dfrac{1}{a^2}\dfrac{\partial^2 z}{\partial x^2}+\dfrac{1}{b^2}\dfrac{\partial^2 z}{\partial y^2}=\dfrac{1}{c^2z}$. Ans.

Thank you

NEXT TOPIC

Homogeneous Functions and Euler’s Theorem

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Differential Calculus

Indeterminate Forms

$$\frac{0}{0},\ 0\times\infty,\ \frac{\infty}{\infty},\ \infty-\infty,\ 0^0,\ \infty^0,\ 1^\infty$$

Prepared by

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Indeterminate forms:

If the value of a function f(x) when x = a takes one of the following forms, i.e.

$$\frac{0}{0},\ 0\times\infty,\ \frac{\infty}{\infty},\ \infty-\infty,\ 0^0,\ \infty^0,\ 1^\infty,$$

then the function is said to be in an indeterminate form. The value of the function is obtained by finding the limit of f(x) as x approaches or tends to a. All the indeterminate forms, by a little arrangements or (simplification), can be brought to the form $\frac{0}{0}$.

(i) Problems solved by using different algebraic laws

First of all, we will discuss some problems which can be solved easily by simplifying the given function with the help of different algebraic laws:

Q.No.1.: Prove that $\underset{x\to\infty}{\text{Lt}}\left[\sqrt{x^2+3x+1}-\sqrt{x^2-2x+8}\right]=\frac{5}{2}$ **.**

Sol.: L.H.S. $=\underset{x\to\infty}{\text{Lt}}\left[\sqrt{x^2+3x+1}-\sqrt{x^2-2x+8}\right]$ $\qquad [\infty-\infty \text{ form}]$

Multiplying and dividing by $\left[\sqrt{x^2+3x+1}+\sqrt{x^2-2x+8}\right]$, we get

$$\text{L.H.S.} = \underset{x\to\infty}{\text{Lt}} \frac{\left[\sqrt{x^2+3x+1}-\sqrt{x^2-2x+8}\right]\left[\sqrt{x^2+3x+1}+\sqrt{x^2-2x+8}\right]}{\left[\sqrt{x^2+3x+1}+\sqrt{x^2-2x+8}\right]}$$

$$= \underset{x\to\infty}{\text{Lt}} \frac{\left(x^2+3x+1\right)-\left(x^2-2x+8\right)}{\sqrt{\left(x^2+3x+1\right)}+\sqrt{\left(x^2-2x+8\right)}}$$

$$= \underset{x\to\infty}{\text{Lt}} \frac{5x-7}{\sqrt{\left(x^2+3x+1\right)}+\sqrt{\left(x^2-2x+8\right)}}.$$

Divide the numerator and denominator by x, we get

$$\text{L.H.S.} = \underset{x\to\infty}{\text{Lt}} \frac{5-\frac{7}{x}}{\sqrt{1+\frac{3}{x}+\frac{1}{x^2}}+\sqrt{1-\frac{2}{x}+\frac{8}{x^2}}} = \frac{5-0}{\sqrt{1+0+0}+\sqrt{1-0+0}} = \frac{5}{1+1} = \frac{5}{2} = \text{R.H.S.}$$

This completes the proof**.**

Q.No.2.: Prove that $\lim\limits_{x\to 2} \frac{\sqrt{1+4x}-\sqrt{5+2x}}{x-2} = \frac{1}{3}$**.**

Sol.: L.H.S.= $\lim\limits_{x\to 2} \frac{\sqrt{1+4x}-\sqrt{5+2x}}{x-2}$**.** $\qquad [\frac{0}{0}$ form]

Multiplying and dividing by $\sqrt{1+4x}+\sqrt{5+2x}$, we get

$$\text{L.H.S.} = \lim_{x\to 2} \frac{\sqrt{1+4x}-\sqrt{5+2x}}{x-2} \times \frac{\sqrt{1+4x}+\sqrt{5+2x}}{\sqrt{1+4x}+\sqrt{5+2x}}$$

$$= \lim_{x\to 2} \frac{(1+4x)-(5+2x)}{(x-2)\left[\sqrt{1+4x}+\sqrt{5+2x}\right]} = \lim_{x\to 2} \frac{2(x-2)}{(x-2)\left[\sqrt{1+4x}+\sqrt{5+2x}\right]}$$

$$= \lim_{x\to 2} \frac{2}{\left[\sqrt{1+4x}+\sqrt{5+2x}\right]} = \frac{2}{6} = \frac{1}{3} = \text{R.H.S.}$$

This completes the proof**.**

Q.No.3.: Prove that $\lim\limits_{x\to 5} \frac{5-x}{\sqrt{6x-5}-\sqrt{4x+5}} = -5$**.**

Sol.: L.H.S.= $\lim\limits_{x\to 5} \frac{5-x}{\sqrt{6x-5}-\sqrt{4x+5}}$**.**

Multiplying and dividing by $\sqrt{6x-5}+\sqrt{4x+5}$, we get

$$\text{L.H.S.}= \lim_{x\to 5}\frac{5-x}{\sqrt{6x-5}-\sqrt{4x+5}}\times\frac{\sqrt{6x-5}+\sqrt{4x+5}}{\sqrt{6x-5}+\sqrt{4x+5}}$$

$$=\lim_{x\to 5}\frac{(5-x)\left[\sqrt{6x-5}+\sqrt{4x+5}\right]}{(6x-5)-(4x+5)}=\lim_{x\to 5}\frac{(5-x)\left[\sqrt{6x-5}+\sqrt{4x-5}\right]}{-2(5-x)}$$

$$=\lim_{x\to 5}\frac{\left[\sqrt{6x-5}+\sqrt{4x-5}\right]}{-2}=\frac{5+5}{-2}=-5=\text{R.H.S.}$$

This completes the proof.

Q.No.4.: Evaluate $\lim\limits_{x\to\infty}\dfrac{1^2+2^2+3^2+......+x^2}{x^3}$.

Sol.: $\lim\limits_{x\to\infty}\dfrac{1^2+2^2+3^2+......+x^2}{x^3}=\lim\limits_{x\to\infty}\dfrac{x(x+1)(2x+1)}{6x^3}$ $\left[\dfrac{\infty}{\infty}\text{form}\right]$

$$=\lim_{x\to\infty}\frac{x^3\left(1+\frac{1}{x}\right)\left(2+\frac{1}{x}\right)}{6x^3}=\frac{1}{3}.\ \text{Ans.}$$

Q.No.5.: Evaluate $\lim\limits_{n\to\infty}\dfrac{\sum n^3}{n^4}$.

Sol.: $\lim\limits_{n\to\infty}\dfrac{\sum n^3}{n^4}=\lim\limits_{n\to\infty}\dfrac{\left[\frac{n(n+1)}{2}\right]^2}{n^4}=\lim\limits_{n\to\infty}\dfrac{n^2(n+1)^2}{4n^4}=\lim\limits_{n\to\infty}\dfrac{n^4\left(1+\frac{1}{n}\right)^2}{4n^4}=\dfrac{1}{4}$. Ans.

Q.No.6.: Evaluate $\lim\limits_{x\to a}\dfrac{x^3-a^3}{x^2-a^2}$.

Sol.: Since $\lim\limits_{x\to a}\dfrac{x^3-a^3}{x^2-a^2}=\lim\limits_{x\to a}\dfrac{(x-a)\left(x^2+ax+a^2\right)}{(x-a)(x+a)}$

$$=\lim_{x\to a}\frac{\left(x^2+ax+a^2\right)}{(x+a)}$$

$$=\frac{3a^2}{2a}=\frac{3}{2}a.\quad\text{Ans.}$$

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Differential Calculus

Partial Differentiation

(Homogeneous Functions and Euler's Theorem)

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Homogeneous Expression:

An expression of the form $a_0x^n + a_1x^{n-1}y^1 + a_2x^{n-2}y^2 + ... + a_ny^n$, where each term of degree 'n' , is called **Homogeneous expression** in x and y and of degree or order 'n'.

Homogeneous Function:

If this expression equal to some quantity 'u' , then 'u' is called **Homogeneous Function** in x and y of degree 'n' .

Now $u = a_0x^n + a_1x^{n-1}y^1 + a_2x^{n-2}y^2 + ... + a_ny^n$

$$= x^n\left[a_0 + a_1\left(\frac{y}{x}\right) + a_2\left(\frac{y}{x}\right)^2 + ... + a_n\left(\frac{y}{x}\right)^n\right]$$

$$\Rightarrow u = x^n f\left(\frac{y}{x}\right).$$

Also, we can write a Homogeneous Function in x and y of degree 'n' as $u = y^n f\left(\frac{x}{y}\right)$.

Similarly, a Homogeneous Function in x ,y and z of degree 'n' can be written as

$$u = x^n F\left(\frac{y}{x}, \frac{z}{x}\right) \text{ or } u = y^n F\left(\frac{x}{y}, \frac{z}{y}\right) \text{ or } u = z^n F\left(\frac{x}{z}, \frac{y}{z}\right).$$

Here 'u' is dependent variable and x, y, z are independent variables.

Euler's Theorem:

Statement: If 'u' is a homogeneous function of x and y of degree 'n', then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu .$$

Proof: Given 'u' is a homogeneous function in x and y of degree 'n'.

Then we may write $u = x^n f\left(\frac{y}{x}\right)$. (i)

Differentiating (i) partially w.r.t. x [keeping y as constant], we get

$$\Rightarrow \frac{\partial u}{\partial x} = x^n f'\left(\frac{y}{x}\right) \times \left(-\frac{y}{x^2}\right) + nx^{n-1} f\left(\frac{y}{x}\right)$$

$$= -x^{n-2} y f'\left(\frac{y}{x}\right) + nx^{n-1} f\left(\frac{y}{x}\right).$$

Similarly, differentiating (i) partially w.r.t. y [keeping x as constant], we get

$$\frac{\partial u}{\partial y} = x^n f'\left(\frac{y}{x}\right)\frac{1}{x} = x^{n-1} f'\left(\frac{y}{x}\right).$$

Now $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x\left[-x^{n-2} y f'\left(\frac{y}{x}\right) + nx^{n-1} f\left(\frac{y}{x}\right)\right] + y\left[x^{n-1} f'\left(\frac{y}{x}\right)\right]$

$$= n\left[x^n f\left(\frac{y}{x}\right)\right] = nu .$$

This completes the proof.

Extension of Euler's Theorem:

Statement: If 'u' is a homogeneous function of x and y of degree 'n', then show that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u .$$

Proof: Since 'u' is a homogeneous function in x and y of degree 'n' then

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu .$ [by Euler's Theorem] ..... (i)

Differentiating (i) partially w. r. t. x [keeping y as constant], we get

$$x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}+y\frac{\partial^2 u}{\partial x\partial y}=n\frac{\partial u}{\partial x}.$$

Multiplying by x, we get

$$x^2\frac{\partial^2 u}{\partial x^2}+x\frac{\partial u}{\partial x}+xy\frac{\partial^2 u}{\partial x\partial y}=nx\frac{\partial u}{\partial x}\ \Rightarrow x^2\frac{\partial^2 u}{\partial x^2}+xy\frac{\partial^2 u}{\partial x\partial y}=(n-1)x\frac{\partial u}{\partial x}. \qquad \text{......(ii)}$$

Again, Differentiating (i) partially w. r. t. y [keeping x as constant], we get

$$x\frac{\partial^2 u}{\partial y\partial x}+y\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial y}=n\frac{\partial u}{\partial y}.$$

Multiplying by y, we get

$$xy\frac{\partial^2 u}{\partial y\partial x}+y^2\frac{\partial^2 u}{\partial y^2}+y\frac{\partial u}{\partial y}=ny\frac{\partial u}{\partial y}\ \Rightarrow xy\frac{\partial^2 u}{\partial y\partial x}+y^2\frac{\partial^2 u}{\partial y^2}=(n-1)y\frac{\partial u}{\partial y}. \qquad \text{.....(iii)}$$

Adding (ii) and (iii), we get

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=(n-1)\left[x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right]=(n-1)nu=n(n-1)u\ .$$

This completes the proof.

Now let us solve some problems related to the above-mentioned topics:

Q.No.1.: Verify Euler's theorem, when $z=x^3-3x^2y-y^3$.

Sol.: Since $z=x^3-3x^2y-y^3$

$$\therefore \frac{\partial z}{\partial x}=3x^2-6xy \text{ and } \frac{\partial z}{\partial y}=-3x^2-3y^2.$$

$$\therefore x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=x\left(3x^2-6xy\right)+y\left(-3x^2-3y^2\right)=3\left(x^3-3x^2y-y^3\right)=3z.$$

$$\text{Also } z=x^3\left[1-3\left(\frac{y}{x}\right)-\left(\frac{y}{x}\right)^3\right]=x^3 f\left(\frac{y}{x}\right).$$

$\Rightarrow$ z is a homogeneous function of x and y of degree 3.

$\therefore$ By Euler's theorem, we get $x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y}=nz=3z$.

Hence, Euler's theorem is verified.

Q.No.2.: If $u=\left[\dfrac{x^{1/3}+y^{1/3}}{x^{1/2}+y^{1/2}}\right]^{1/2}$, then prove that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=-\dfrac{1}{12}u$.

Sol.: Here $u=\left[\dfrac{x^{1/3}+y^{1/3}}{x^{1/2}+y^{1/2}}\right]^{1/2}=\left[\dfrac{x^{1/3}\left(1+\dfrac{y^{1/3}}{x^{1/3}}\right)}{x^{1/2}\left(1+\dfrac{y^{1/2}}{x^{1/2}}\right)}\right]^{1/2}=\left[x^{-1/6}\left\{\dfrac{1+\left(\dfrac{y}{x}\right)^{1/3}}{1+\left(\dfrac{y}{x}\right)^{1/2}}\right\}\right]^{1/2}$

$$=x^{-1/12}\left[\frac{1+\left(\frac{y}{x}\right)^{1/3}}{1+\left(\frac{y}{x}\right)^{1/2}}\right]^{1/2}=x^{-1/12}f\left(\frac{y}{x}\right).$$

$\Rightarrow$ u is a homogeneous function of x and y of degree $-\dfrac{1}{12}$.

$\therefore$ By Euler's theorem, we have $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=nu=-\dfrac{1}{12}u$.

Hence the result.

Q.No.3.: If $u=f\left(\dfrac{y}{x}\right)$, then show that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=0$.

Sol.: Here $u=f\left(\dfrac{y}{x}\right)=x^0 f\left(\dfrac{y}{x}\right)$.

$\Rightarrow$ u is a homogeneous function of x and y of degree 0.

Hence by Euler's theorem, we have

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu=0.u=0.$$

Hence the result.

Q.No.4.: If $u=xyf\left(\dfrac{y}{x}\right)$, then show that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=2\left[xyf\left(\dfrac{y}{x}\right)\right]$.

Sol.: Here $u=xyf\left(\dfrac{y}{x}\right)=x^2\left[\dfrac{y}{x}f\left(\dfrac{y}{x}\right)\right]=x^2F\left(\dfrac{y}{x}\right)$.

$\Rightarrow$ u is a homogeneous function of x and y of degree 2.

Hence by Euler's theorem, we have

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu=2.u=2\left[xyf\left(\frac{y}{x}\right)\right].$$

Hence the result.

Q.No.5.: If $u=\sin^{-1}\left(\frac{x}{y}\right)+\tan^{-1}\left(\frac{y}{x}\right)$, find the value of $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}$.

Sol.: Here $u=\sin^{-1}\left(\frac{x}{y}\right)+\tan^{-1}\left(\frac{y}{x}\right)=x^0\left[\sin^{-1}\left\{\frac{1}{\frac{y}{x}}\right\}+\tan^{-1}\left(\frac{y}{x}\right)\right]=x^0f\left(\frac{y}{x}\right)$.

$\Rightarrow$ u is a homogeneous function of x and y of degree 0.

Hence by Euler’s theorem, we have

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu=0u=0.$$

Hence the result.

Q.No.6.: Verify Euler’s Theorem on homogeneous functions in the following cases:-

(i) $f(x,y)=\frac{\left(x^{1/4}+y^{1/4}\right)}{\left(x^{1/5}+y^{1/5}\right)}$, **(ii)** $u=f(x,y,z)=\sqrt{x}+\sqrt{y}+\sqrt{z}$

(iii) $z=\tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{y}\right)$, **(iv)** $u=\sin^{-1}\left(\frac{x}{y}\right)+\tan^{-1}\left(\frac{y}{x}\right)$

(v) $z=x^4\log\left(\frac{y}{x}\right)$.

Sol.: (i) Here $f(x,y)=\frac{\left(x^{1/4}+y^{1/4}\right)}{\left(x^{1/5}+y^{1/5}\right)}=x^{1/20}\left[\frac{1+\left(\frac{y}{x}\right)^{1/4}}{1+\left(\frac{y}{x}\right)^{1/5}}\right]=x^{1/20}f\left(\frac{y}{x}\right)$

$\Rightarrow f(x,y)$ is a homogeneous function of x and y of degree $\frac{1}{20}$.

Hence by Euler’s theorem, we have

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=nf=\frac{1}{20}\frac{\left(x^{1/4}+y^{1/4}\right)}{\left(x^{1/5}+y^{1/5}\right)}. \qquad \text{(i)}$$

Again since $f(x,y)=\dfrac{\left(x^{1/4}+y^{1/4}\right)}{\left(x^{1/5}+y^{1/5}\right)}$

Differentiating partially w. r. t x and y respectively, we get

$$\frac{\partial f}{\partial x}=\frac{\left(x^{1/5}+y^{1/5}\right)\left(\frac{1}{4}x^{-3/4}\right)-\left(x^{1/4}+y^{1/4}\right)\left(\frac{1}{5}x^{-4/5}\right)}{\left(x^{1/5}+y^{1/5}\right)^2}$$

Multiplying by x, we get

$$x\frac{\partial f}{\partial x}=\frac{\left(x^{1/5}+y^{1/5}\right)\left(\frac{1}{4}x^{1/4}\right)-\left(x^{1/4}+y^{1/4}\right)\left(\frac{1}{5}x^{1/5}\right)}{\left(x^{1/5}+y^{1/5}\right)^2} \qquad \text{(ii)}$$

$$\frac{\partial f}{\partial y}=\frac{\left(x^{1/5}+y^{1/5}\right)\left(\frac{1}{4}y^{-3/4}\right)-\left(x^{1/4}+y^{1/4}\right)\left(\frac{1}{5}y^{-4/5}\right)}{\left(x^{1/5}+y^{1/5}\right)^2}$$

Multiplying by y, we get

$$y\frac{\partial f}{\partial y}=\frac{\left(x^{1/5}+y^{1/5}\right)\left(\frac{1}{4}y^{1/4}\right)-\left(x^{1/4}+y^{1/4}\right)\left(\frac{1}{5}y^{1/5}\right)}{\left(x^{1/5}+y^{1/5}\right)^2} \qquad \text{(iii)}$$

Adding (ii) and (iii), we get

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=\frac{\left(x^{1/5}+y^{1/5}\right)\frac{1}{4}\left(x^{1/4}+y^{1/4}\right)-\left(x^{1/4}+y^{1/4}\right)\frac{1}{5}\left(x^{1/5}+y^{1/5}\right)}{\left(x^{1/5}+y^{1/5}\right)^2}$$

$$=\frac{1}{20}\frac{\left(x^{1/4}+y^{1/4}\right)}{\left(x^{1/5}+y^{1/5}\right)} \qquad \text{(iv)}$$

Hence, the Euler's Theorem is verified**.**

(ii) Here $u=f(x,y,z)=\sqrt{x}+\sqrt{y}+\sqrt{z}=x^{1/2}\left[1+\left(\frac{y}{x}\right)^{1/2}+\left(\frac{z}{x}\right)^{1/2}\right]=x^{1/2}f\left(\frac{y}{x},\frac{z}{x}\right)$

$\Rightarrow u=f(x,y,z)$ is a homogeneous function of x ,y and z of degree $\dfrac{1}{2}$.

Hence by Euler's theorem, we have

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=nu=\frac{1}{2}u=\frac{1}{2}\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right).$$

Again since $u = f(x,y,z) = \sqrt{x} + \sqrt{y} + \sqrt{z}$

Differentiating partially w. r. t x , y and z respectively, we get

$$\frac{\partial u}{\partial x} = \frac{1}{2\sqrt{x}}\ ,\ \frac{\partial u}{\partial y} = \frac{1}{2\sqrt{y}}\ \text{and}\ \frac{\partial u}{\partial z} = \frac{1}{2\sqrt{z}}$$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = x\frac{1}{2\sqrt{x}} + y\frac{1}{2\sqrt{y}} + z\frac{1}{2\sqrt{z}} = \frac{1}{2}\left(\sqrt{x} + \sqrt{y} + \sqrt{z}\right).$$

Hence, the Euler's Theorem is verified**.**

(iii) Here $z = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{y}\right) = y^0 \tan^{-1}\left[\sqrt{\left(\frac{x}{y}\right)^2 + 1}\right] = y^0 f\left(\frac{x}{y}\right).$

$\Rightarrow z$ is a homogeneous function of x ,y and z of degree 0.

Hence by Euler's theorem, we have

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz = 0.z = 0.$$

Again since $z = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{y}\right).$

Differentiating partially w. r. t x and y respectively, we get

$$\frac{\partial z}{\partial x} = \frac{1}{1+\frac{x^2+y^2}{y^2}}\cdot\frac{1}{y}\left(\frac{1}{2\sqrt{x^2+y^2}}.2x\right) = \frac{xy}{\left(x^2+2y^2\right)}\cdot\frac{1}{\sqrt{x^2+y^2}}$$

$$\frac{\partial z}{\partial y} = \frac{1}{1+\frac{x^2+y^2}{y^2}}\cdot\left(\frac{\frac{y}{2}\frac{1}{\sqrt{x^2+y^2}}.2y - \sqrt{x^2+y^2}}{y^2}\right) = \frac{1}{\left(x^2+2y^2\right)}\cdot\frac{y^2 - \left(x^2+y^2\right)}{\sqrt{x^2+y^2}}$$

$$= \frac{1}{\left(x^2+2y^2\right)}\cdot\frac{\left(-x^2\right)}{\sqrt{x^2+y^2}}.$$

$$\therefore x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = x\left[\frac{xy}{\left(x^2+2y^2\right)}\cdot\frac{1}{\sqrt{x^2+y^2}}\right] + y\left[\frac{1}{\left(x^2+2y^2\right)}\cdot\frac{\left(-x^2\right)}{\sqrt{x^2+y^2}}\right] = 0.$$

Hence, the Euler's Theorem is verified**.**

(iv) Here $u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right) = x^0\left[\sin^{-1}\left\{\frac{1}{\frac{y}{x}}\right\} + \tan^{-1}\left(\frac{y}{x}\right)\right] = x^0 f\left(\frac{y}{x}\right)$.

$\Rightarrow$ u is a homogeneous function of x and y of degree 0.

Hence by Euler's theorem, we have

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu = 0u = 0.$$

Again since $u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$.

Differentiating partially w. r. t x and y respectively, we get

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}.\frac{1}{y} + \frac{1}{1+\frac{y^2}{x^2}}.\left(\frac{-y}{x^2}\right) = \frac{1}{\sqrt{y^2-x^2}} - \frac{y}{x^2+y^2}.$$

$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}.\left(\frac{-x}{y^2}\right) + \frac{1}{1+\frac{y^2}{x^2}}.\frac{1}{x} = \frac{-x}{y\sqrt{y^2-x^2}} + \frac{x}{x^2+y^2}.$$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x\left[\frac{1}{\sqrt{y^2-x^2}} - \frac{y}{x^2+y^2}\right] + y\left[\frac{-x}{y\sqrt{y^2-x^2}} + \frac{x}{x^2+y^2}\right] = 0.$$

Hence, the Euler's Theorem is verified**.**

(v) Here $z = x^4\log\left(\frac{y}{x}\right) = x^4 f\left(\frac{y}{x}\right)$.

$\Rightarrow$ z is a homogeneous function of x and y of degree 4.

Hence by Euler's theorem, we have

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz = 4z = 4x^4\log\left(\frac{y}{x}\right).$$

Again since $z = x^4\log\left(\frac{y}{x}\right)$.

Differentiating partially w. r. t x and y respectively, we get

$$\frac{\partial z}{\partial x} = 4x^3.\log\left(\frac{y}{x}\right) + x^4.\frac{1}{\frac{y}{x}}\left(\frac{-y}{x^2}\right) = 4x^3.\log\left(\frac{y}{x}\right) - x^3 \text{ and } \frac{\partial z}{\partial y} = x^4.\frac{1}{\frac{y}{x}}\left(\frac{1}{x}\right) = \frac{x^4}{y}$$

Hence, the Euler's Theorem is verified**.**

Q.No.7.: If $u = \log\dfrac{x^4 + y^4}{x + y}$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 3$.

Sol.: Here $u = \log\dfrac{x^4 + y^4}{x + y} \Rightarrow e^u = \dfrac{x^4 + y^4}{x + y}$.

e^u is a homogeneous function of x and y of degree 3.

Hence by Euler's theorem, we have $x\dfrac{\partial\left(e^u\right)}{\partial x} + y\dfrac{\partial\left(e^u\right)}{\partial y} = ne^u = 3e^u$.

$$\Rightarrow xe^u\frac{\partial u}{\partial x} + ye^u\frac{\partial u}{\partial y} = 3e^u.$$

Hence $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 3$.

Hence the result**.**

Q.No.8.: If $u = \sin^{-1}\left(\dfrac{\sqrt{x} - \sqrt{y}}{\sqrt{x} + \sqrt{y}}\right)$, show that $\dfrac{\partial u}{\partial x} = -\dfrac{y}{x}\dfrac{\partial u}{\partial y}$.

Sol.: Here $u = \sin^{-1}\left(\dfrac{\sqrt{x} - \sqrt{y}}{\sqrt{x} + \sqrt{y}}\right) \Rightarrow \sin u = \left(\dfrac{\sqrt{x} - \sqrt{y}}{\sqrt{x} + \sqrt{y}}\right) = x^0\left[\dfrac{1 - \left(\dfrac{y}{x}\right)^{1/2}}{1 + \left(\dfrac{y}{x}\right)^{1/2}}\right] = x^0 f\left(\dfrac{y}{x}\right)$.

$\Rightarrow \sin u$ is a homogeneous function of x and y of degree 0.

Hence by Euler's theorem, we have

$$x\frac{\partial(\sin u)}{\partial x} + y\frac{\partial(\sin u)}{\partial y} = n\sin u = 0.\sin u = 0.$$

$$\Rightarrow x\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} = 0 \Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0.$$

Hence $\dfrac{\partial u}{\partial x} = -\dfrac{y}{x}\dfrac{\partial u}{\partial y}$.

Hence the result**.**

Q.No.9.: If $u = x\varphi\left(\frac{y}{x}\right) + \psi\left(\frac{y}{x}\right)$, show that

(i) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x\varphi\left(\frac{y}{x}\right)$.

(ii) $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0$ **.**

Sol.:(i) Here $u = x\varphi\left(\frac{y}{x}\right) + \psi\left(\frac{y}{x}\right) = u_1 + u_2$ (say)

Then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x\frac{\partial u_1}{\partial x} + y\frac{\partial u_1}{\partial y} + x\frac{\partial u_2}{\partial x} + y\frac{\partial u_2}{\partial y}$.

Now since u_1 and u_2 are homogeneous function of x and y of degree 1 and 0 respectively. Then by Euler's theorem, we have

$x\frac{\partial u_1}{\partial x} + y\frac{\partial u_1}{\partial y} = nu_1 = x\varphi\left(\frac{y}{x}\right)$ and $x\frac{\partial u_2}{\partial x} + y\frac{\partial u_2}{\partial y} = nu_2 = 0$.

Hence $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x\frac{\partial u_1}{\partial x} + y\frac{\partial u_1}{\partial y} + x\frac{\partial u_2}{\partial x} + y\frac{\partial u_2}{\partial y} = x\varphi\left(\frac{y}{x}\right) + 0$.

$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x\varphi\left(\frac{y}{x}\right)$.

Hence the result**.**

(ii) Again, since u_1 and u_2 are homogeneous function of x and y of degree 1 and 0 respectively. Then, by extension of Euler's theorem, we have

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u = 1(1-1).x\varphi\left(\frac{y}{x}\right) = 0.$$

Hence $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0$.

Hence the result**.**

Q.No.10.: If $u = \left(x^2 + y^2\right)^{1/3}$, prove that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -\frac{2}{9}u$ **.**

Sol.: Here $u = \left(x^2 + y^2\right)^{1/3} = x^{2/3}\left[1 + \left(\frac{y}{x}\right)^2\right]^{1/3} = x^{2/3}f\left(\frac{y}{x}\right)$.

$\Rightarrow$ u is a homogeneous function of x and y of degree $\frac{2}{3}$.

Hence by extension of Euler's theorem, we have

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=n(n-1)u=\frac{2}{3}\left(\frac{2}{3}-1\right)=-\frac{2}{9}u\ .$$

Hence the result.

Q.No.11.: If $u=\sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$, then show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\tan u$.

Sol.: Here $u=\sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)\Rightarrow \sin u(=z,\text{say})=\left(\frac{x^2+y^2}{x+y}\right)=x\left[\frac{1+\frac{y^2}{x^2}}{1+\frac{y}{x}}\right]=x^1 f\left(\frac{y}{x}\right)$.

$\Rightarrow$ z is a homogeneous function of x and y of degree 1.

Then, by Euler's theorem, we have $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=nz=z$.

$$\Rightarrow x\cos u\frac{\partial u}{\partial x}+y\cos u\frac{\partial u}{\partial y}=\sin u\ .$$

$$\Rightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\tan u\ .$$

Hence the result.

Q.No.12.: If $u=\sin^{-1}\left[\frac{x^{1/3}+y^{1/3}}{x^{1/2}+y^{1/2}}\right]^{1/2}$, then prove that

(i) $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=-\frac{1}{12}\tan u$.

(ii) $x^2u_{xx}+2xyu_{xy}+y^2u_{yy}=\frac{\tan u}{144}\left(13+\tan^2 u\right)$.

Sol.: (i) Here $u=\sin^{-1}\left[\frac{x^{1/3}+y^{1/3}}{x^{1/2}+y^{1/2}}\right]^{1/2}$

$$\Rightarrow \sin u = \left[\frac{x^{1/3}\left(1+\frac{y^{1/3}}{x^{1/3}}\right)}{x^{1/2}\left(1+\frac{y^{1/2}}{x^{1/2}}\right)}\right]^{1/2} = \left[x^{-1/6}\left\{\frac{1+\left(\frac{y}{x}\right)^{1/3}}{1+\left(\frac{y}{x}\right)^{1/2}}\right\}\right]^{1/2} = x^{-1/12}\left[\frac{1+\left(\frac{y}{x}\right)^{1/3}}{1+\left(\frac{y}{x}\right)^{1/2}}\right]^{1/2} = x^{-1/12} f\left(\frac{y}{x}\right).$$

$\Rightarrow \sin u$ is a homogeneous function of x and y of degree $-\frac{1}{12}$.

Then, by Euler's theorem, we have

$$x\frac{\partial(\sin u)}{\partial x} + y\frac{\partial(\sin u)}{\partial y} = n\sin u = -\frac{1}{12}\sin u$$

$$\Rightarrow x\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} = -\frac{1}{12}\sin u$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{1}{12}\tan u \ . \qquad \text{...(i)}$$

Hence the result.

(ii) Differentiating (i) w. r. t. x partially, we get $xu_{xx} + u_x + yu_{xy} = -\frac{1}{12}\sec^2 u(u_x)$

Multiplying by x, we get

$$x^2u_{xx} + xu_x + xyu_{xy} = -\frac{1}{12}\sec^2 u(xu_x). \qquad \text{.....(ii)}$$

Differentiating (i) w. r. t. y partially, we get

$$xu_{xy} + u_y + yu_{yy} = -\frac{1}{12}\sec^2 u(u_y).$$

Multiplying by y, we get

$$yxu_{xy} + yu_y + y^2u_{yy} = -\frac{1}{12}\sec^2 u(yu_y). \qquad \text{......(iii)}$$

Adding (ii) and (iii), we get

$$x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = -\frac{1}{12}\sec^2 u(xu_x + yu_y) - (xu_x + yu_y) = \left(-\frac{1}{12}\sec^2 u - 1\right)(xu_x + yu_y)$$

$$= \left(\frac{1}{12}\sec^2 u + 1\right)\left(\frac{1}{12}\tan u\right) = \frac{1}{144}\sec^2 u\tan u + \frac{1}{12}\tan u$$

$$= \frac{1}{144}\left(1+\tan^2 u\right)\tan u + \frac{1}{12}\tan u = \frac{1}{144}\tan u\left(1+\tan^2 u + 12\right)$$

$$= \frac{1}{144}\tan u\left(13 + \tan^2 u\right).$$

Hence the result.

Q.No.13.: If $u = \frac{\left(x^2 + y^2\right)^m}{2m-1} + x\varphi\left(\frac{y}{x}\right) + \psi\left(\frac{y}{x}\right)$, show that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 2m\left(x^2 + y^2\right)^m .$$

Sol.: Here $u = \frac{\left(x^2 + y^2\right)^m}{2m-1} + x\varphi\left(\frac{y}{x}\right) + \psi\left(\frac{y}{x}\right) = u_1 + u_2 + u_3$ (say)

Then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x\frac{\partial u_1}{\partial x} + y\frac{\partial u_1}{\partial y} + x\frac{\partial u_2}{\partial x} + y\frac{\partial u_2}{\partial y} + x\frac{\partial u_3}{\partial x} + y\frac{\partial u_3}{\partial y}$.

Now since u_1, u_2 and u_3 are homogeneous function of x and y of degree $2m$, 1 and 0 respectively. Then by Euler's theorem, we have

$$x\frac{\partial u_1}{\partial x} + y\frac{\partial u_1}{\partial y} = nu_1 = 2mu_1,\ x\frac{\partial u_2}{\partial x} + y\frac{\partial u_2}{\partial y} = nu_2 = u_2 \text{ and } x\frac{\partial u_3}{\partial x} + y\frac{\partial u_3}{\partial y} = nu_3 = 0.$$

Hence $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x\frac{\partial u_1}{\partial x} + y\frac{\partial u_1}{\partial y} + x\frac{\partial u_2}{\partial x} + y\frac{\partial u_2}{\partial y} + x\frac{\partial u_3}{\partial x} + y\frac{\partial u_3}{\partial y} = 2m.u_1 + 1.u_2$.

Again, since u_1, u_2 and u_3 are homogeneous function of x and y of degree 2m, 1 and 0 respectively.

Then, by extension of Euler's theorem, we have

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u = 2m(2m-1)u_1 + 1(1-1)u_2 = 2m(2m-1)\frac{\left(x^2 + y^2\right)^m}{2m-1}$$

Hence $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 2m\left(x^2 + y^2\right)^m$.

Hence the result.

Q.No.14.: If $u = \sin\left(\sqrt{x} + \sqrt{y}\right)$, show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\left(\sqrt{x} + \sqrt{y}\right)\cos\left(\sqrt{x} + \sqrt{y}\right).$$

Sol.: Here $u = \sin\left(\sqrt{x} + \sqrt{y}\right)$.

$$\Rightarrow \sin^{-1} u(= z, \text{say}) = \left(\sqrt{x} + \sqrt{y}\right) = x^{1/2}\left(1 + \left(\frac{y}{x}\right)^{1/2}\right) = x^{1/2} f\left(\frac{y}{x}\right).$$

$\Rightarrow$ z is a homogeneous function of x and y of degree $\frac{1}{2}$.

Then, by Euler's theorem, we have $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz = \frac{1}{2}z$.

$$\Rightarrow x\frac{1}{\sqrt{1-u^2}}\frac{\partial u}{\partial x} + y\frac{1}{\sqrt{1-u^2}}\frac{\partial u}{\partial y} = \frac{1}{2}\sin^{-1} u = \frac{1}{2}\left(\sqrt{x} + \sqrt{y}\right).$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\left(\sqrt{x} + \sqrt{y}\right)\sqrt{1 - \sin^2\left(\sqrt{x} + \sqrt{y}\right)} = \frac{1}{2}\left(\sqrt{x} + \sqrt{y}\right)\sqrt{\cos^2\left(\sqrt{x} + \sqrt{y}\right)}.$$

Hence $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\left(\sqrt{x} + \sqrt{y}\right)\cos\left(\sqrt{x} + \sqrt{y}\right)$.

Hence the result.

Q.No.15.: If $z = \sin^{-1}\left(\sqrt{x^2 + y^2}\right)$, then show that $x^2 z_{xx} + 2xyz_{xy} + y^2 z_{yy} = \tan^3 z$.

Sol.: Here $z = \sin^{-1}\left(\sqrt{x^2 + y^2}\right) \Rightarrow \sin z (= u, \text{say}) = \sqrt{x^2 + y^2} = x\sqrt{1 + \frac{y^2}{x^2}} = xf\left(\frac{y}{x}\right)$.

$\Rightarrow$ u is a homogeneous function of x and y of degree 1.

Then by Euler's theorem, we have $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu = u$.

$$\Rightarrow x\cos z\frac{\partial z}{\partial x} + y\cos z\frac{\partial z}{\partial y} = \sin z \Rightarrow x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \tan z. \qquad \text{(i)}$$

Differentiating (i) w. r. t. x partially, we get

$$xz_{xx} + z_x + yz_{xy} = \sec^2 z(z_x)$$

Multiplying by x, we get

$$x^2 z_{xx} + xz_x + xyz_{xy} = \sec^2 z(xz_x). \qquad \text{......(ii)}$$

Differentiating (i) w. r. t. y partially, we get

$$xz_{xy} + z_y + yz_{yy} = \sec^2 z(z_y)$$

Multiplying by y, we get

$$yxz_{xy} + yz_y + y^2 z_{yy} = \sec^2 z(yz_y). \qquad \text{......(iii)}$$

Adding (ii) and (iii), we get

$$x^2 z_{xx} + 2xyz_{xy} + y^2 z_{yy} = \sec^2 z(xz_x + yz_y) - (xz_x + yz_y) = \sec^2 z(\tan z) - \tan z.$$

$$= \tan z(\sec^2 z - 1) = \tan z(\tan^2 z) = \tan^3 z.$$

Hence the result**.**

Q.No.16.: If $u + iv = (x \pm iy)^2$, and $w = \frac{u}{v}$, prove that $x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = 0$**.**

Sol.: Here $u + iv = (x \pm iy)^2 \Rightarrow u + iv = x^2 - y^2 \pm 2ixy$.

Thus $$w = \frac{u}{v} = \frac{x^2 - y^2}{2xy} = x^0 \left[\frac{1 - \left(\frac{y}{x}\right)^2}{2\frac{y}{x}} \right] = x^0 f\left(\frac{y}{x}\right).$$

$\Rightarrow$ w is a homogeneous function of x and y of degree 0.

Then, by Euler's theorem, we have

$$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = nw = 0.w = 0.$$

Hence $x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = 0$.

This completes the proof**.**

Q.No.17.: If $u + iv = (ax \pm iby)^3$,show that

(i) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3u$,

(ii) $x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 3v$**,**

(iii) ($x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = 0$ where $w = \frac{u}{v}$**.**

Sol.: (i) Here $u + iv = (ax \pm iby)^3 \Rightarrow u + iv = (a^3x^3 - 3ab^2xy^2) + i(b^3y^3 \pm 3a^2bx^2y)$.

Thus $$u = (a^3x^3 - 3ab^2xy^2) = x^3\left[a^3 - 3ab^2\left(\frac{y}{x}\right)^2\right] = x^3 f\left(\frac{y}{x}\right).$$

$\Rightarrow$ u is a homogeneous function of x and y of degree 3.

Then, by Euler's theorem, we have

$$x\frac{\partial w}{\partial x}+y\frac{\partial w}{\partial y} = nw = 3.w .$$

Hence $x\frac{\partial w}{\partial x}+y\frac{\partial w}{\partial y} = 3w$.

This completes the proof.

(ii) Here $u+iv=(ax\pm iby)^3 \Rightarrow u+iv=(a^3x^3-3ab^2xy^2)+i(b^3y^3\pm 3a^2bx^2y)$.

Thus $v = (b^3y^3 \pm 3a^2bx^2y) = x^3\left[b^3\left(\frac{y}{x}\right)^3 \pm 3a^2b\left(\frac{y}{x}\right)\right] = x^3 f\left(\frac{y}{x}\right)$.

$\Rightarrow$ v is a homogeneous function of x and y of degree 3.

Then, by Euler's theorem, we have

$$x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y} = nv = 3.v .$$

Hence $x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y} = 3v$.

This completes the proof.

(iii) Here $u+iv=(ax\pm iby)^3 \Rightarrow u+iv=(a^3x^3-3ab^2xy^2)+i(b^3y^3\pm 3a^2bx^2y)$.

$$\text{Thus}\quad w = \frac{u}{v} = \frac{a^3x^3-3ab^2xy^2}{b^3y^3\pm 3a^2bx^2y} = x^0\left[\frac{a^3-3ab^2\left(\frac{y}{x}\right)^2}{b^3\left(\frac{y}{x}\right)^3 \pm 3a^2b\left(\frac{y}{x}\right)}\right] = x^0 f\left(\frac{y}{x}\right).$$

$\Rightarrow$ w is a homogeneous function of x and y of degree 0.

Then, by Euler's theorem, we have

$$x\frac{\partial w}{\partial x}+y\frac{\partial w}{\partial y} = nw = 0.w = 0 .$$

Hence $x\frac{\partial w}{\partial x}+y\frac{\partial w}{\partial y} = 0$.

This completes the proof.

Q.No.18.: If $v=\left(x^2+y^2+z^2\right)^{-1/2}$,then show that $x\dfrac{\partial v}{\partial x}+y\dfrac{\partial v}{\partial y}+z\dfrac{\partial v}{\partial z}=-v$.

Sol.: Here $v=\left(x^2+y^2+z^2\right)^{-1/2} \Rightarrow v=x^{-1}\left[1+\left(\dfrac{y}{x}\right)^2+\left(\dfrac{z}{x}\right)^2\right]^{-1/2}=x^{-1}f\left(\dfrac{y}{x},\dfrac{z}{x}\right)$

$\Rightarrow$ v is a homogeneous function of x ,y and z of degree -1.

Hence by Euler's theorem, we have

$x\dfrac{\partial v}{\partial x}+y\dfrac{\partial v}{\partial y}+z\dfrac{\partial v}{\partial z}=nu=(-1).v=-v$.

Hence the result**.**

Q.No.19.: If $u=\sin^{-1}\left(\sqrt{x}+\sqrt{y}\right)$, prove that

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\frac{-\sin u.\cos 2u}{4\cos^3 u}.$$

Sol.: Here $u=\sin^{-1}\left(\sqrt{x}+\sqrt{y}\right)\Rightarrow \sin u=\left(\sqrt{x}+\sqrt{y}\right)=x^{1/2}\left(1+\left(\dfrac{y}{x}\right)^{1/2}\right)=x^{1/2}f\left(\dfrac{y}{x}\right)$.

$\Rightarrow \sin u$ is a homogeneous function of x and y of degree $\dfrac{1}{2}$

$\therefore$ By Euler's theorem, $x\dfrac{\partial(\sin u)}{\partial x}+y\dfrac{\partial(\sin u)}{\partial y}=n\sin u=\dfrac{1}{2}\sin u$

$\Rightarrow x\cos u\dfrac{\partial u}{\partial x}+y\cos u\dfrac{\partial u}{\partial y}=\dfrac{1}{2}\sin u$ $.\Rightarrow x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=\dfrac{1}{2}\tan u$ . ...(i)

Differentiating (i) w. r. t. x partially, we get

$xu_{xx}+u_x+yu_{xy}=\dfrac{1}{2}\sec^2 u\left(u_x\right)$

Multiplying by x , we get

$x^2u_{xx}+xu_x+xyu_{xy}=\dfrac{1}{2}\sec^2 u\left(xu_x\right)$ ......(ii)

Differentiating (i) w. r. t. y partially, we get

$xu_{xy}+u_y+yu_{yy}=\dfrac{1}{2}\sec^2 u\left(u_y\right)$

Multiplying by y , we get

$$yxu_{xy} + yu_y + y^2u_{yy} = \frac{1}{2}\sec^2 u\left(yu_y\right) \qquad \text{......(iii)}$$

Adding (ii) and (iii), we get

$$x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = \frac{1}{2}\sec^2 u\left(xu_x + yu_y\right) - \left(xu_x + yu_y\right) = \left(\frac{1}{2}\sec^2 u - 1\right)\left(xu_x + yu_y\right)$$

$$= \left(\frac{1}{2}\sec^2 u - 1\right)\left(\frac{1}{2}\tan u\right) = \frac{1}{4}\sec^2 u \tan u - \frac{1}{2}\tan u$$

$$= \frac{1}{4}\tan u\left(\sec^2 u - 2\right) = \frac{\tan u}{4}\left(\frac{1}{\cos^2 u} - 2\right) = \frac{\tan u}{4}\frac{\left(1 - 2\cos^2 u\right)}{\cos^2 u}$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{-\sin u.\cos 2u}{4\cos^3 u}.$$

Hence the result**.**

Q.No.20.: If $V = \tan^{-1}\left(\frac{x^3 + y^3}{2x + 3y}\right)$, prove that

$$x^2\frac{\partial^2 V}{\partial x^2} + 2xy\frac{\partial^2 V}{\partial x \partial y} + y^2\frac{\partial^2 V}{\partial y^2} = \sin 4V - \sin 2V\textbf{.}$$

Sol.: Here $V = \tan^{-1}\left(\frac{x^3 + y^3}{2x + 3y}\right) \Rightarrow \tan V = \frac{x^3 + y^3}{2x + 3y} = x^2\left[\frac{1 + \left(\frac{y}{x}\right)^3}{2 + 3\frac{y}{x}}\right] = x^2 f\left(\frac{y}{x}\right) = z$ (say).

$\Rightarrow$ z is a homogeneous function of x and y of degree 2.

Then by Euler's theorem, we have $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz = 2z$.

$$\Rightarrow x\sec^2 V\frac{\partial V}{\partial x} + y\sec^2 V\frac{\partial V}{\partial y} = 2\tan V$$

$$\Rightarrow x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y} = \frac{2\tan V}{\sec^2 V} = 2\frac{\sin V}{\cos V}.\cos^2 V = 2\sin V\cos V = \sin 2V. \qquad \text{....(i)}$$

Differentiating (i) partially w. r. t. x [keeping y as constant], we get

$$x\frac{\partial^2 V}{\partial x^2} + \frac{\partial V}{\partial x} + y\frac{\partial^2 V}{\partial x \partial y} = \cos 2V.2\frac{\partial V}{\partial x}.$$

Multiplying by x, we get

$$x^2\frac{\partial^2 V}{\partial x^2}+x\frac{\partial V}{\partial x}+xy\frac{\partial^2 V}{\partial x\partial y}=2x\cos 2V\frac{\partial V}{\partial x}$$

$$\Rightarrow x^2\frac{\partial^2 V}{\partial x^2}+xy\frac{\partial^2 V}{\partial x\partial y}=(2\cos 2V-1)x\frac{\partial V}{\partial x} \qquad \text{. ..(ii)}$$

Again, Differentiating (i) partially w. r. t. y [keeping x as constant], we get

$$x\frac{\partial^2 V}{\partial y\partial x}+y\frac{\partial^2 V}{\partial y^2}+\frac{\partial V}{\partial y}=\cos 2V.2\frac{\partial V}{\partial y}.$$

Multiplying by y, we get

$$xy\frac{\partial^2 V}{\partial y\partial x}+y^2\frac{\partial^2 V}{\partial y^2}+y\frac{\partial V}{\partial y}=2y\cos 2V\frac{\partial V}{\partial y}$$

$$\Rightarrow xy\frac{\partial^2 V}{\partial y\partial x}+y^2\frac{\partial^2 V}{\partial y^2}=(2\cos 2V-1)y\frac{\partial V}{\partial y}. \qquad \text{.....(iii)}$$

Adding (ii) and (iii), we get

$$x^2\frac{\partial^2 V}{\partial x^2}+2xy\frac{\partial^2 V}{\partial x\partial y}+y^2\frac{\partial^2 V}{\partial y^2}=(2\cos 2V-1)\left[x\frac{\partial V}{\partial x}+y\frac{\partial V}{\partial y}\right]=(2\cos 2V-1)\sin 2V$$

$$=2\sin 2V\cos 2V-\sin 2V=\sin 4V-\sin 2V.$$

Hence $x^2\frac{\partial^2 V}{\partial x^2}+2xy\frac{\partial^2 V}{\partial x\partial y}+y^2\frac{\partial^2 V}{\partial y^2}=\sin 4V-\sin 2V$.

Hence the result**.**

Q.No.21.: If $u=x^n f\left(\frac{y}{x},\frac{z}{x}\right)$, show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=nu$ **.**

Sol.: Here $u=x^n f\left(\frac{y}{x},\frac{z}{x}\right)$.Let $\frac{y}{x}=t_1$, $\frac{z}{x}=t_2$

$\therefore u=x^n f(t_1,t_2)=x^n f$

Differentiating partially w. r. t x , y and z respectively, we get

$$\frac{\partial u}{\partial x}=nx^{n-1}f+x^n\left[\frac{\partial f}{\partial t_1}.\frac{\partial t_1}{\partial x}+\frac{\partial f}{\partial t_2}.\frac{\partial t_2}{\partial x}\right]=nx^{n-1}f+x^n\left[\frac{\partial f}{\partial t_1}\left(\frac{-y}{x^2}\right)+\frac{\partial f}{\partial t_2}\left(\frac{-z}{x^2}\right)\right]$$

$$\Rightarrow \frac{\partial u}{\partial x}=nx^{n-1}f-yx^{n-1}\frac{\partial f}{\partial t_1}-zx^{n-1}\frac{\partial f}{\partial t_2},$$

$$\frac{\partial u}{\partial y}=x^n\left[\frac{\partial f}{\partial t_1}.\frac{\partial t_1}{\partial y}+\frac{\partial f}{\partial t_2}.\frac{\partial t_2}{\partial y}\right]=x^n\left[\frac{\partial f}{\partial t_1}\left(\frac{1}{x}\right)\right]=x^{n-1}\frac{\partial f}{\partial t_1},$$

$$\frac{\partial u}{\partial z}=x^n\left[\frac{\partial f}{\partial t_1}.\frac{\partial t_1}{\partial z}+\frac{\partial f}{\partial t_2}.\frac{\partial t_2}{\partial z}\right]=x^n\left[\frac{\partial f}{\partial t_2}\left(\frac{1}{x}\right)\right]=x^{n-1}\frac{\partial f}{\partial t_2}$$

$$\therefore x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=x\left[nx^{n-1}f-yx^{n-1}\frac{\partial f}{\partial t_1}-zx^{n-1}\frac{\partial f}{\partial t_2}\right]+y\left[x^{n-1}\frac{\partial f}{\partial t_1}\right]+z\left[x^{n-1}\frac{\partial f}{\partial t_2}\right].$$

Hence $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=nx^nf(t_1,t_2)=nu$.

Hence the result.

Q.No.22.: If $u=\frac{1}{x^2}+\frac{1}{xy}+\frac{\log x-\log y}{x^2+y^2}$ show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+2u=0$.

Sol.: Here $u=\frac{1}{x^2}+\frac{1}{xy}+\frac{\log x-\log y}{x^2+y^2}=x^{-2}\left[1+\frac{1}{\frac{y}{x}}-\frac{\log\left(\frac{y}{x}\right)}{1+\frac{y^2}{x^2}}\right]=x^{-2}f\left(\frac{y}{x}\right)$.

$\Rightarrow$ u is a homogeneous function of x and y of degree -2.

Hence by Euler's theorem, we have $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=-2u$.

Hence $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+2u=0$.

Q.No.23.: If $f(x,y,z)=\log\left(\frac{x^5+y^5+z^5}{x+y+z}\right)$, show that $x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z}=4$.

Sol.: Here $f(x,y,z)=\log\left(\frac{x^5+y^5+z^5}{x+y+z}\right)\Rightarrow e^f=\frac{x^5+y^5+z^5}{x+y+z}$ ...(i)

$$\Rightarrow e^f=x^4\left[\frac{1+\left(\frac{y}{x}\right)^5+\left(\frac{z}{x}\right)^5}{1+\frac{y}{x}+\frac{z}{x}}\right]=x^4f\left(\frac{y}{x},\frac{z}{x}\right).$$

$\Rightarrow$ e^f is a homogeneous function of x, y and z of degree 4.

Hence by Euler's theorem, we have

$$x\frac{\partial\left(e^f\right)}{\partial x}+y\frac{\partial\left(e^f\right)}{\partial y}+z\frac{\partial\left(e^f\right)}{\partial z}=4e^f.$$

$$\Rightarrow xe^f\frac{\partial f}{\partial x}+ye^f\frac{\partial f}{\partial y}+ze^f\frac{\partial f}{\partial z}=4e^f \Rightarrow x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z}=4.$$

Hence the result**.**

Q.No.24.: If $u=\sin\left(\frac{x^2-y^2+z^2}{xy-yz-zx}\right)$, show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=0$.

Sol.: Here $u=\sin\left(\frac{x^2-y^2+z^2}{xy-yz-zx}\right)=x^0\sin\left[\frac{1-\left(\frac{y}{x}\right)^2+\left(\frac{z}{x}\right)^2}{\frac{y}{x}-\frac{y}{x}\cdot\frac{z}{x}-\frac{z}{x}}\right]=x^0 f\left(\frac{y}{x},\frac{z}{x}\right)$.

$\Rightarrow$ u is a homogeneous function of x ,y and z of degree 0.

Then by Euler's theorem, we have

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=nu=0.u=0.$$

Hence the result**.**

Q.No.25: Given that F(u) = V(x , y , z), where V is a homogeneous function of x , y , z of degree n , then prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=n\frac{F(u)}{F'(u)}$.

Sol.: Here V(x , y , z) is a homogeneous function of x , y , z of degree n , then by Euler's theorem $x\frac{\partial V}{\partial x}+y\frac{\partial V}{\partial y}+z\frac{\partial V}{\partial z}=nV \Rightarrow x\frac{\partial F(u)}{\partial x}+y\frac{\partial F(u)}{\partial y}+z\frac{\partial F(u)}{\partial z}=nF(u)$

$$\Rightarrow xF'(u)\frac{\partial u}{\partial x}+yF'(u)\frac{\partial u}{\partial y}+zF'(u)\frac{\partial u}{\partial z}=nF(u)$$

$$\Rightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=n\frac{F(u)}{F'(u)}.$$

Hence the result**.**

Thank you

NEXT TOPIC

Total Differentials,

Explicit Function, Implicit Functions

and

Total Differential Coefficient

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Differential Calculus

Indeterminate Forms

Cauchy's Rule or L'Hospital's Rule

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(ii) Indeterminate forms-Problems of $\frac{0}{0}, \frac{\infty}{\infty}, 0\times\infty$.

Cauchy's Rule or L'Hospital's Rule:

Suppose we are interested to find the value of

$\left[\frac{f(x)}{\phi(x)}\right]$at $x = a$, where $[f(x)]_{x=a} = f(a) = 0$ (i)

$[\phi(x)]_{x=a} = \phi(a) = 0$. (ii)

Then $\left[\frac{f(x)}{\phi(x)}\right]_{x=a}$ is of the form $\frac{0}{0}$.

Then by **L'Hospital's Rule, "**we differentiate the numerator and denominator w.r.t. x separately. If once again, we find indeterminate form $\frac{0}{0}$, we have further repetition of the process till we get some definite result".

Proof: The limiting value of $\left[\frac{f(x)}{\phi(x)}\right]_{x=a} = \underset{x\to a}{\mathrm{Lt}}\, \frac{f(x)}{\phi(x)}$

Putting $x = a + h$ in $\underset{x\to a}{\mathrm{Lt}} \left[\frac{f(x)}{\phi(x)}\right]$, we have when $x \to a$ then $h \to 0$

$$\therefore \underset{x\to a}{\mathrm{Lt}}\left[\frac{f(x)}{\phi(x)}\right]=\underset{h\to 0}{\mathrm{Lt}}\frac{f(a+h)}{\phi(a+h)}$$

Using Taylor's Theorem, we get

$$=\underset{h\to 0}{\mathrm{Lt}}\frac{f(a+h)}{\phi(a+h)}=\underset{h\to 0}{\mathrm{Lt}}\frac{f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+........}{\phi(a)+h\phi'(a)+\frac{h^2}{2!}\phi''(a)+.....}$$

$$=\underset{h\to 0}{\mathrm{Lt}}\frac{hf'(a)+\frac{h^2}{2!}f''(a)+........}{h\phi'(a)+\frac{h^2}{2!}\phi''(a)+.........}\left[\begin{array}{l}\because f(a)=0 \text{ and}\\ \phi(a)=0 \text{ from (i) and (ii)}\end{array}\right]$$

As $h\neq 0$, we have

$$\underset{h\to 0}{\mathrm{Lt}}\frac{f'(a)+\frac{h}{2!}f''(a)+........}{\phi'(a)+\frac{h}{2!}\phi''(a)+.........}=\frac{f'(a)}{\phi'(a)}=\underset{x\to a}{\mathrm{Lt}}\frac{f'(x)}{\phi'(x)}$$

In case both $f'(a)$ and $\phi'(a)$ are zero, the above process can be repeated and we shall get $\underset{x\to a}{\mathrm{Lt}}\frac{f(x)}{\phi(x)}=\frac{f''(a)}{\phi''(a)}=\underset{x\to a}{\mathrm{Lt}}\frac{f''(x)}{\phi''(x)}$ and like this we can have further repetition of the process till we get some definite results.

Note: Cauchy's rule is also be applicable to $\frac{\infty}{\infty}$ form**.**

Q.No.1: Evaluate $\underset{x\to 0}{\mathrm{Lim}}\frac{x\cos x-\log(x+1)}{x^2}$**.**

Sol.: $\underset{x\to 0}{\mathrm{Lim}}\frac{x\cos x-\log(1+x)}{x^2}$. $\left[\frac{0}{0}\text{ form}\right]$

Apply Cauchy's Rule (i.e. differentiate the numerator and denominator w.r.t. to x separately), we get

$$=\underset{x\to 0}{\mathrm{Lim}}\frac{\cos x-x\sin x-\frac{1}{1+x}}{2x}\left[\frac{0}{0}\text{ form}\right]$$

Again apply Cauchy's Rule, we get

$$= \lim_{x\to 0}\frac{-\sin x - \sin x - x\cos x + \frac{1}{(1+x)^2}}{2} = \frac{1}{2}. \text{ Ans.}$$

Q.No.2: Evaluate $\lim\limits_{x\to 0}\frac{e^x + e^{-x} - 2\cos x}{x\sin x}$.

Sol.: $\lim\limits_{x\to 0}\frac{e^x + e^{-x} - 2\cos x}{x\sin x}\ \left[\frac{0}{0}\text{ form}\right]$

$$= \lim_{x\to 0}\frac{e^x + e^{-x} - 2\cos x}{x^2}\times\frac{x}{\sin x}$$

$$= \lim_{x\to 0}\frac{e^x + e^{-x} - 2\cos x}{x^2}\ \left[\because \lim_{x\to 0}\frac{x}{\sin x} = 1\right]$$

$$= \lim_{x\to 0}\frac{e^x - e^{-x} + 2\sin x}{2x}\ \left[\frac{0}{0}\text{ form}\right]$$

$$= \lim_{x\to 0}\frac{e^x + e^{-x} + 2\cos x}{2} = \frac{1+1+2}{2} = 2. \text{ Ans.}$$

Q.No.3: Evaluate $\lim\limits_{x\mapsto 1}\frac{x^x - x}{1 - x + \log x}$.

Sol. $\lim\limits_{x\mapsto 1}\frac{x^x - x}{1 - x + \log x}\ \left[\frac{0}{0}\text{ form}\right]$

Apply Cauchy's Rule, we get

$$= \lim_{x\mapsto 1}\frac{x^x(1+\log x) - 1}{0 - 1 + \frac{1}{x}}\ \left[\frac{0}{0}\text{ form}\right]$$

$$= \lim_{x\to 1}\frac{x^x(1/x) + (1+\log x)x^x(1+\log x)}{0 - 0 - \frac{1}{x^2}}\ \left[\begin{array}{l}\therefore \text{Let } y = x^x\\ \log y = \log x^x = \log x\\ \text{Differentiate w.r.t. to x}\\ \frac{dy}{dx} = y(1+\log x) = x^x(1+\log x)\end{array}\right]$$

$= -2$ **Ans.**

Q.No.4: Find the values of a, b and c so that $\lim\limits_{x\to 0}\frac{ae^x - b\cos x + ce^{-x}}{x\sin x} = 2$.

Sol.: $\lim\limits_{x\to 0}\dfrac{ae^{x}-b\cos x+ce^{-x}}{x\sin x}$.

This is of $\frac{0}{0}$ form, if $a-b+c=0$. (i)

Apply Cauchy's Rule, we get

$$\lim_{x\to 0}\frac{ae^{x}+b\sin x-ce^{-x}}{x\cos x+\sin x}$$

This is of $\frac{0}{0}$ form, if $a-c=0$. (ii)

$$=\lim_{x\to 0}\frac{ae^{x}+b\cos x+ce^{-x}}{x(-\sin x)+\cos x+\cos x}=\lim_{x\to 0}\frac{ae^{x}+b\cos x+ce^{-x}}{-x\sin x+2\cos x}=\frac{a+b+c}{2}=2\,\text{(given)}$$

$\Rightarrow a+b+c=4$.(iii)

Solving (i), (ii) and (iii), for a, b, c, we get

a = 1, b = 2, and c = 1. Ans**.**

Q.No.5: Evaluate $\lim\limits_{x\to 0}\log_x \sin x$**.**

Sol.: $\mathop{\mathrm{Lt}}\limits_{x\to 0}\dfrac{\log_e \sin x}{\log_e x}$**.** $\left[\frac{\infty}{\infty}\text{form}\right]$

Applying Cauchy's Rule, we get

$$=\mathop{\mathrm{Lt}}_{x\to 0}\frac{\frac{1}{\sin x}\times\cos x}{\frac{1}{x}}=\mathop{\mathrm{Lt}}_{x\to 0}x\cot x=\mathop{\mathrm{Lt}}_{x\to 0}\frac{x}{\tan x}=1.\ \text{Ans.}\qquad\left[\because \mathop{\mathrm{Lt}}_{x\to 0}\frac{x}{\tan x}=1\right]$$

Q.No.6: Evaluate $\mathop{\mathrm{Lt}}\limits_{x\to 1}\sec\dfrac{\pi}{2x}.\log x$**.**

Sol.: $\mathop{\mathrm{Lt}}\limits_{x\to 1}\sec\dfrac{\pi}{2x}.\log x$ $\quad(\infty\times 0)$form

$$=\mathop{\mathrm{Lt}}_{x\to 1}\frac{\log x}{\cos\left(\frac{\pi}{2x}\right)}.\left[\frac{0}{0}\ \text{form}\right]$$

Applying Cauchy's Rule, we get

$$= \underset{x\to 1}{\mathrm{Lt}} \frac{\frac{1}{x}}{-\sin\left(\frac{\pi}{2x}\right)\times\frac{\pi}{2}\times\left(-\frac{1}{x^2}\right)} = \underset{x\to 1}{\mathrm{Lt}} \frac{2\times 1\times x^2}{\pi\times x\times \sin\left(\frac{x}{2x}\right)} = \underset{x\to 1}{\mathrm{Lt}} \frac{2\times x}{\pi\times\sin\left(\frac{\pi}{2x}\right)} = \frac{2}{\pi}. \text{ Ans.}$$

Q.No.7: Evaluate $\underset{x\to 0}{\mathrm{Lt}}\left[\frac{1}{x^2}-\cot^2 x\right]$.

Sol.: We know that $\tan x = x+\frac{x^3}{3}+\frac{2x^5}{15}+\ldots\ldots$

$$\therefore \underset{x\to 0}{\mathrm{Lt}}\left[\frac{1}{x^2}-\frac{1}{\tan^2 x}\right] = \underset{x\to 0}{\mathrm{Lt}}\left[\frac{1}{x^2}-\left(x+\frac{x^3}{3}+\frac{2x^5}{15}+\ldots\ldots\right)^{-2}\right]$$

$$= \underset{x\to 0}{\mathrm{Lt}}\left[\frac{1}{x^2}-x^{-2}\left(1+\frac{x^2}{3}+\frac{2}{15}x^4+\ldots\ldots\right)^{-2}\right]$$

$$= \underset{x\to 0}{\mathrm{Lt}}\left[\frac{1}{x^2}-\frac{1}{x^2}\left\{1-2\left(\frac{x^2}{3}+\frac{2}{15}x^4+\ldots..\right)+\text{terms of higher powers of } x\right\}\right]$$

$$= \underset{x\to 0}{\mathrm{Lt}}\left[\frac{1}{x^2}-\frac{1}{x^2}+\frac{2}{3}+\text{terms containing } x\right] = \frac{2}{3}. \text{ Ans.}$$

Similar Problem: Evaluate $\underset{x\to 0}{\mathrm{Lt}}\left(\cot^2 x-\frac{1}{x^2}\right)$ $(\infty-\infty)$ form.

Sol.: $\underset{x\to 0}{\mathrm{Lt}}\left(\cot^2 x-\frac{1}{x^2}\right) = \underset{x\to 0}{\mathrm{Lt}}\left(\frac{1}{\tan^2 x}-\frac{1}{x^2}\right) = \underset{x\to 0}{\mathrm{Lt}}\frac{x^2-\tan^2 x}{x^2\tan^2 x}$

$$= \underset{x\to 0}{\mathrm{Lt}}\frac{x^2-\tan^2 x}{x^4}\left(\frac{x}{\tan x}\right)^2$$

$$= \underset{x\to 0}{\mathrm{Lt}}\frac{x^2-\tan^2 x}{x^4}(1)^2 \quad \left[\because \underset{x\to 0}{\mathrm{Lt}}\frac{x}{\tan x}=1\right]$$

$$= \underset{x\to 0}{\mathrm{Lt}}\frac{x^2-\tan^2 x}{x^4} \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \underset{x\to 0}{\mathrm{Lt}}\frac{2x-2\tan x\sec^2 x}{4x^3} = \underset{x\to 0}{\mathrm{Lt}}\frac{2x-2\tan x\left(1+\tan^2 x\right)}{2x^3}$$

$$= \underset{x\to 0}{\mathrm{Lt}}\frac{x-\tan x-\tan^2 x}{2x^3} \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \underset{x\to 0}{\mathrm{Lt}} \frac{1-\sec^2 x - 3\tan^2 x \sec^2 x}{6x^2}$$

$$= \underset{x\to 0}{\mathrm{Lt}} \frac{1-\left(1+\tan^2 x\right) - 3\tan^2 x\left(1+\tan^2 x\right)}{6x^2}$$

$$= \underset{x\to 0}{\mathrm{Lt}} \frac{1-1-\tan^2 x - 3\tan^2 x - 3\tan^4 x}{6x^2}$$

$$= \underset{x\to 0}{\mathrm{Lt}} \frac{-4\tan^4 x - 3\tan^4 x}{6x^2}$$

$$= -\underset{x\to 0}{\mathrm{Lt}} \frac{4+3\tan^2 x}{6}\left(\frac{\tan x}{x}\right)^2$$

$$= \frac{-4+0}{6}(1)^3 = \frac{-4}{6} = \frac{-2}{3}. \text{ Ans.}$$

Q.No.8: Find the value of $\underset{x\to 0}{\mathrm{Lt}} \dfrac{x^3.e^{x^4/4} - \sin^{3/2}\left(x^2\right)}{x^7}$.

Sol.: As $x \to 0$, the required limit takes the indeterminate form $\frac{0}{0}$. The denominator here is x^7 and the application of Cauchy's Rule will required us to differentiate the nominator and denominator at least seven times to come to the true value of the limit, which will be cumbersome.

We therefore, use the method of expansion by Macaulurin's Theory, which is very convenient.

Thus, using the expansion e^x

$$e^{\frac{x^4}{4}} = 1+\frac{x^4}{4}+\frac{1}{2!}\left(\frac{x^4}{4}\right)^2 .... = 1+\frac{x^4}{4}+\frac{x^8}{32}+..........$$

And using the series for $\sin x$

$$\left(\sin x^2\right)^{3/2} = \left[x^2 - \frac{\left(x^2\right)^3}{3!} + \frac{\left(x^2\right)^5}{5!} - ......\right]^{3/2} = x^3\left[1-\frac{x^4}{6}+........\right]^{3/2},$$

Now using Binomial Theorem, we get

$$\left(\sin x^2\right)^{3/2} = x^3\left[1-\frac{3}{2}\left(\frac{x^4}{6}-.......\right)+.....\right] = x^3\left[1-\frac{x^4}{4}+........\right]$$

$$\therefore \underset{x\to 0}{\mathrm{Lt}}\frac{x^3.e^{x^4/4}-\sin^{3/2}\left(x^2\right)}{x^7} = \underset{x\to 0}{\mathrm{Lt}}\frac{x^3\left[1+\frac{x^4}{4}+\frac{x^8}{32}+.......\right]-x^3\left[1-\frac{x^4}{4}+........\right]}{x^7}$$

$$= \underset{x\to 0}{\mathrm{Lt}}\left[\frac{1}{2}+\text{terms containing } x\right] = \frac{1}{2}. \text{ Ans.}$$

Q.No.9.: Evaluate $\underset{x\to 0}{\mathrm{Lim}}\frac{e^x-1-\sin x}{x^2}$.

Sol.: $\underset{x\to 0}{\mathrm{Lim}}\frac{e^x-1-\sin x}{x^2}$. $\left[\frac{0}{0}\text{form}\right]$

$\therefore$ Using L'Hospital Rule, we get

$= \underset{x\to 0}{\mathrm{Lim}}\frac{e^x-\cos x}{2x}$. $\left[\frac{0}{0}\text{form}\right]$

$\therefore$ Again using L'Hospital Rule, we get

$= \underset{x\to 0}{\mathrm{Lim}}\frac{e^x+\sin x}{2} = \frac{1}{2}$ .Ans.

Q.No.10.: Evaluate **(a)** $\underset{x\to 0}{\mathrm{Lim}}\frac{e^{ax}-e^{-ax}}{\log(1+bx)}$,

(b) $\underset{x\to 0}{\mathrm{Lim}}\frac{e^x--e^{-x}-2x}{\tan x-x}$.

Sol.:(a) $\underset{x\to 0}{\mathrm{Lim}}\frac{e^{ax}-e^{-ax}}{\log(1+bx)}$. $\left[\frac{0}{0}\text{form}\right]$

$\therefore$ Using L'Hospital Rule, we get

$$= \underset{x\to 0}{\mathrm{Lim}}\frac{e^{ax}(a)-e^{-ax}(-a)}{\frac{1}{1+bx}.b} = \underset{x\to 0}{\mathrm{Lim}}\frac{a\left(e^{ax}+e^{-ax}\right)(1+bx)}{b} = \frac{2a}{b}\text{.Ans.}$$

(b) $\underset{x\to 0}{\mathrm{Lim}}\frac{e^x--e^{-x}-2x}{\tan x-x}$. $\left[\frac{0}{0}\text{form}\right]$

$\therefore$ Using L'Hospital Rule, we get

$$= \lim_{x\to 0}\frac{e^x - e^{-x}(-1) - 2}{\sec^2 x - 1} = \lim_{x\to 0}\frac{e^x + e^{-x} - 2}{\sec^2 x - 1}. \left[\frac{0}{0}\text{form}\right]$$

$\therefore$ Using L'Hospital Rule, we get

$$= \lim_{x\to 0}\frac{e^x + e^{-x}(-1)}{2\sec x.\sec x\tan x} = \lim_{x\to 0}\frac{e^x - e^{-x}}{2\sec^2 x\tan x}. \left[\frac{0}{0}\text{form}\right]$$

$\therefore$ Using L'Hospital Rule, we get

$$= \lim_{x\to 0}\frac{e^x - e^{-x}(-1)}{2\left(\sec^2 x\sec^2 x + \tan x.2\sec x\sec x\tan x\right)}.$$

$$= \lim_{x\to 0}\frac{e^x + e^{-x}}{2\left(\sec^4 x + 2\sec^2 x\tan^2 x\right)} = \frac{1+1}{2(1+0)} = 1. \text{ Ans.}$$

Q.No.11.: Evaluate $\lim_{x\to 0}\frac{x - \tan x}{x^3}$.

Sol.: $\lim_{x\to 0}\frac{x - \tan x}{x^3}$ $\left[\frac{0}{0}\text{form}\right]$

$\therefore$ Using L'Hospital Rule, we get

$$= \lim_{x\to 0}\frac{1 - \sec^2 x}{3x^2} \left[\frac{0}{0}\text{form}\right]$$

$\therefore$ Again using L'Hospital Rule, we get

$$= \lim_{x\to 0}\frac{-2\sec x.\sec x\tan x}{6x} = \lim_{x\to 0}\frac{-\sec^2 x\tan x}{3x} \left[\frac{0}{0}\text{form}\right]$$

$\therefore$ Using L'Hospital Rule, we get

$$= \lim_{x\to 0} - \frac{\sec^2 x.\sec^2 x + \tan x.2\sec x.\sec x\tan x}{3}$$

$$= \lim_{x\to 0} - \frac{\sec^4 x + 2\sec^2 x\tan^2 x}{3} = -\frac{1+0}{3} = -\frac{1}{3}. \text{ Ans.}$$

Q.No.12.: Evaluate $\lim_{x\to 0}\frac{xe^x - \log(1+x)}{x^2}$.

Sol.: $\lim_{x\to 0}\frac{xe^x - \log(1+x)}{x^2}$ $\left[\frac{0}{0}\text{form}\right]$

$\therefore$ Using L'Hospital Rule, we get

$$= \lim_{x\to 0}\frac{x.e^x + e^x.1 - \frac{1}{x+1}}{2x}\left[\frac{0}{0}\text{form}\right]$$

$\therefore$ Again using L'Hospital Rule, we get

$$= \lim_{x\to 0}\frac{x.e^x + e^x.1 + e^x + \frac{1}{(x+1)^2}}{2} = \frac{0.e^0 + e^0.1 + e^0 + \frac{1}{(1+0)^2}}{2} = \frac{3}{2}\text{ . Ans.}$$

Q.No.13.: Evaluate $\lim_{x\to 0}\frac{\sin x - x + \frac{x^3}{6}}{x^5}$.

Sol.: $\lim_{x\to 0}\frac{\sin x - x + \frac{x^3}{6}}{x^5}\left[\frac{0}{0}\text{form}\right]$

$\therefore$ Using L'Hospital Rule, we get

$$= \lim_{x\to 0}\frac{\cos x - 1 + \frac{3x^2}{6}}{5x^4}\left[\frac{0}{0}\text{form}\right]$$

$\therefore$ Again using L'Hospital Rule, we get

$$= \lim_{x\to 0}\frac{-\sin x + x}{20x^3}\left[\frac{0}{0}\text{form}\right]$$

$\therefore$ Again using L'Hospital Rule, we get

$$= \lim_{x\to 0}\frac{-\cos x + 1}{60x^2}\left[\frac{0}{0}\text{form}\right]$$

$\therefore$ Again using L'Hospital Rule, we get

$$= \lim_{x\to 0}\frac{\sin x}{120x}\left[\frac{0}{0}\text{form}\right]$$

$\therefore$ Again using L'Hospital Rule, we get

$$= \lim_{x\to 0}\frac{\cos x}{120} = \frac{1}{120}\text{ . Ans.}$$

Q.No.14.: Evaluate $\lim_{x\to 0}\frac{\sin 2x + 2\sin^2 x - 2\sin x}{\cos x - \cos^2 x}$.

Sol.: $\lim_{x\to 0}\frac{\sin 2x + 2\sin^2 x - 2\sin x}{\cos x - \cos^2 x}\left[\frac{0}{0}\text{form}\right]$

$\therefore$ Using L'Hospital Rule, we get

$$= \operatorname*{Lim}_{x\to 0}\frac{2\cos 2x + 4\sin x\cos x - 2\cos x}{-\sin x + \cos x\sin x}$$

$$= \operatorname*{Lim}_{x\to 0}\frac{2\cos 2x + 2\sin 2x - 2\cos x}{-\sin x + \sin 2x}\left[\frac{0}{0}\text{form}\right]$$

$\therefore$ Again using L'Hospital Rule, we get

$$= \operatorname*{Lim}_{x\to 0}\frac{-4\sin 2x + 4\cos 2x + 2\sin x}{-\cos x + 2\cos 2x} = \frac{-0+4.1+0}{-1+2} = 4\text{ . Ans.}$$

Q.No.15.: Evaluate $\operatorname*{Lim}_{x\to\infty}\frac{\log x}{x^n}\ (n>0)$.

Sol.: $\operatorname*{Lim}_{x\to\infty}\frac{\log x}{x^n}\left[\frac{\infty}{\infty}\text{form}\right]$

$\therefore$ Using L'Hospital Rule, we get

$$= \operatorname*{Lim}_{x\to\infty}\frac{\frac{1}{x}}{nx^{n-1}} = \operatorname*{Lim}_{x\to\infty}\frac{1}{nx^n} = \frac{1}{\infty} = 0\text{ . Ans.}$$

Q.No.16.: Evaluate $\operatorname*{Lt}_{x\to 0}\frac{x-\sin x}{\tan^3 x}$.

Sol.: $\operatorname*{Lt}_{x\to 0}\frac{x-\sin x}{\tan^3 x}$

Applying Cauchy's rule, we get

$$\operatorname*{Lt}_{x\to 0}\frac{1-\cos x}{3\tan^2 x.\sec^2 x}$$

Again applying Cauchy's rule, we get

$$\operatorname*{Lim}_{x\to 0}\frac{\sin x}{3\left[\left(\sec^4 x.2\tan x\right)+\left(\tan^3 x 2\sec^2 x\right)\right]} = \operatorname*{Lim}_{x\to 0}\frac{\sin x}{6\tan x\sec^2 x\left(\tan^2 x + \sec^2 x\right)}$$

$$= \operatorname*{Lim}_{x\to 0}\frac{1}{6\sec^3 x\left(\tan^2 x + \sec^2 x\right)} = \frac{1}{6}\text{ . Ans.}$$

Q.No.17.: Evaluate $\operatorname*{Lt}_{x\to 0}\left(\frac{1}{x^2} - \operatorname{cosec}^2 x\right)$.

Sol.: $\operatorname*{Lt}_{x\to 0}\left(\frac{1}{x^2} - \operatorname{cosec}^2 x\right) = \operatorname*{Lt}_{x\to 0}\left(\frac{1}{x^2} - \sin^{-2} x\right)$

$$= \lim_{x\to 0}\left(\frac{1}{x^2}-\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}+\ldots\ldots\ldots\ldots\right)^{-2}\right) = \lim_{x\to 0}\left(\frac{1}{x^2}-\frac{1}{x^2}\left(1-\frac{x^2}{3!}+\frac{x^4}{5!}+\ldots\ldots\ldots\ldots\right)^{-2}\right)$$

$$= \lim_{x\to 0}\left[\frac{1}{x^2}-\frac{1}{x^2}\left(1+2\left(\frac{x^2}{3!}-\frac{x^4}{5!}+\ldots\ldots\ldots\ldots\right)\right)\right]$$

$$= \lim_{x\to 0}\left(-\frac{2}{3!}+\frac{2x^4}{5!}\ldots\ldots\ldots\ldots\right) = -\frac{2}{3!} = -\frac{1}{3}. \text{ Ans.}$$

Q.No.18.: Evaluate $\mathop{\mathrm{Lt}}_{x\to 0}\left(\frac{1}{x}-\cot x\right)$.

Sol.: $\mathop{\mathrm{Lt}}_{x\to 0}\left(\frac{1}{x}-\cot x\right) = \mathop{\mathrm{Lt}}_{x\to 0}\left(\frac{1}{x}-\frac{\cos x}{\sin x}\right) = \mathop{\mathrm{Lt}}_{x\to 0}\frac{\sin x - x\cos x}{x\sin x} = \mathop{\mathrm{Lt}}_{x\to 0}\frac{\sin x - x\cos x}{x^2\left(\frac{\sin x}{x}\right)}$

$$= \mathop{\mathrm{Lt}}_{x\to 0}\frac{\sin x - x\cos x}{x^2}$$

Applying Cauchy's rule, $\therefore$ above equation is $\left[\frac{0}{0}\right]$ form, we get

$$= \mathop{\mathrm{Lt}}_{x\to 0}\frac{\cos x-\cos x + x\sin x}{2x} = \mathop{\mathrm{Lt}}_{x\to 0}\frac{\sin x}{2} = 0. \text{ Ans}$$

Q.No.19.: Evaluate $\mathop{\mathrm{Lt}}_{x\to 0}\left(\frac{a}{x}-\cot\frac{x}{a}\right)$.

Sol.: $\mathop{\mathrm{Lt}}_{x\to 0}\left(\frac{a}{x}-\cot\frac{x}{a}\right) = \mathop{\mathrm{Lt}}_{x\to 0}\left(\frac{a}{x}-\frac{\cos\frac{x}{a}}{\sin\frac{x}{a}}\right) = \lim_{x\to 0}\frac{a\sin\frac{x}{a}-x\cos\frac{x}{a}}{x\sin\frac{x}{a}}$

$$= \lim_{x\to 0}\frac{a\sin\frac{x}{a}-x\cos\frac{x}{a}}{\frac{x^2}{a}\cdot\frac{\sin\frac{x}{a}}{\frac{x}{a}}} = \lim_{x\to 0}\frac{a\sin\frac{x}{a}-x\cos\frac{x}{a}}{\frac{x^2}{a}} = \lim_{x\to 0}\frac{a^2\sin\frac{x}{a}-ax\cos\frac{x}{a}}{x^2}$$

Applying L hospital's rule, $\therefore$ above equation is $\left[\frac{0}{0}\right]$ form, we get

$$= \lim_{x\to 0}\frac{a\cos\frac{x}{a} - a\cos\frac{x}{a} + x\sin\frac{x}{a}}{2x} = \lim_{x\to 0}\frac{x\sin\frac{x}{a}}{2x} = 0 \text{. Ans.}$$

Q.No.20.: Evaluate **(a)** $\lim_{x\to 0}\frac{\cot x - \frac{1}{x}}{x}$, **(b)** $\lim_{x\to 1}(x-1)\tan\frac{\pi x}{2}$**.**

Sol.: (a) $\lim_{x\to 0}\frac{\cot x - \frac{1}{x}}{x}\left[\frac{-\infty}{0}\text{form}\right]$

$$= \lim_{x\to 0}\frac{\frac{\cos x}{\sin x} - \frac{1}{x}}{x} = \lim_{x\to 0}\frac{x\cos x - \sin x}{x^2\sin x} = \lim_{x\to 0}\left[\frac{x\cos x - \sin x}{x^3}\right]\cdot\left[\frac{x}{\sin x}\right]$$

$$= \lim_{x\to 0}\left[\frac{x\cos x - \sin x}{x^3}\right]. \because \lim_{x\to 0}\left[\frac{x}{\sin x}\right] = 1 . \left[\frac{0}{0}\text{form}\right]$$

$\therefore$ Using L'Hospital Rule, we get

$$= \lim_{x\to 0}\left[\frac{-x\sin x + \cos x - \cos x}{3x^2}\right] = \lim_{x\to 0}\left[-\frac{\sin x}{3x}\right] = -\frac{1}{3}\text{ .Ans.}$$

(b) $\lim_{x\to 1}(x-1)\tan\frac{\pi x}{2}$ $[(0\times\infty)\text{form}]$

$$= \lim_{x\to 1}\frac{(x-1)}{\cot\frac{\pi x}{2}}\left[\frac{0}{0}\text{form}\right]$$

$\therefore$ Using L'Hospital Rule, we get

$$= \lim_{x\to 1}\frac{1}{-\frac{\pi}{2}\operatorname{cosec}^2\frac{\pi x}{2}} = \frac{1}{-\frac{\pi}{2}} = -\frac{2}{\pi}\text{ . Ans.}$$

Q.No.21.: Evaluate $\underset{y\to\infty}{\text{Lt}}\ y^2\left(1 - e^{-2gx/y^2}\right)$**.**

Sol.: $\underset{y\to\infty}{\text{Lt}}\ y^2\left(1 - e^{-2gx/y^2}\right)$

Substituting $\lim_{y\to\infty} y^2 = \lim_{n\to 0}\frac{1}{n^2}$

$$\therefore \lim_{n\to 0}\frac{\left(1 - e^{-1gxn^2}\right)}{n^2}$$

$\therefore$ Using L'Hospital Rule, we get

$$\lim_{n\to 0}\frac{2gx.2n.e^{-2gxn^2}}{2n}=2gx\text{ . Ans.}$$

Q.No.22.: Evaluate $\lim\limits_{x\to a}\log\left(2-\frac{x}{a}\right)\cot(x-a)$.

Sol.: $\lim\limits_{x\to a}\log\left(2-\frac{x}{a}\right)\cot(x-a)\ [(0\times\infty)\text{form}]$

$$=\lim_{x\to a}\frac{\log\left(2-\frac{x}{a}\right)}{\tan(x-a)}\left[\frac{0}{0}\text{form}\right]$$

$\therefore$ Using L'Hospital Rule, we get

$$=\lim_{x\to a}\frac{\frac{1}{\left(2-\frac{x}{a}\right)}\times\left(-\frac{1}{a}\right)}{\mathrm{sac}^2(x-a)}=-\frac{1}{a}\text{. Ans.}$$

Q.No.23.: Evaluate $\mathop{\mathrm{Lt}}\limits_{x\to 0}\frac{x^{1/2}\tan x}{\left(e^x-1\right)^{3/2}}$.

Sol.: $\mathop{\mathrm{Lt}}\limits_{x\to 0}\frac{x^{1/2}\tan x}{\left(e^x-1\right)^{3/2}}=\mathop{\mathrm{Lt}}\limits_{x\to 0}\frac{x^{3/2}\frac{\tan x}{x}}{\left(e^x-1\right)^{3/2}}=\lim\limits_{x\to 0}\frac{x^{1/2}\tan x}{\left(x+\frac{x^2}{2!}+\frac{x^3}{3!}+..........\right)^{3/2}}$

$$=\lim_{x\to 0}\frac{x^{1/2}\tan x}{x.x^{1/2}\left(1+\frac{x}{2!}+\frac{x^2}{3!}+..........\right)}=\lim_{x\to 0}\frac{1}{\left(1+\frac{x}{2!}+\frac{x^2}{3!}+..........\right)}=1\text{. Ans.}$$

Q.No.24.: Prove that $\lim\limits_{n\to\infty}\frac{e^{1/n}+e^{2/n}+^{3/n}+............+e^{n/n}}{n}=e-1$

Sol.: TakingL.H.S. $=\lim\limits_{n\to\infty}\frac{e^{1/n}+e^{2/n}+^{3/n}+............+e^{n/n}}{n}$

Here, the series given in numerator is in geometric progression,

where, first term, $a=e^{1/n}$,

common ratio, $r=e^{1/n}>1$,

number of terms = n.

The sum of series given in numerator is, δ_n

$$\therefore \delta_n = \frac{a\left(r^n - 1\right)}{r-1} = \frac{e^{1/n}\left\{\left(e^{1/n}\right)^n - 1\right\}}{e^{1/n} - 1} = \frac{e^{1/n}\{e-1\}}{e^{1/n} - 1}$$

So, L.H.S.= $\lim\limits_{n\to\infty} \frac{e^{1/n}\{e-1\}}{\left(e^{1/n}-1\right)n} = \lim\limits_{n\to\infty} \frac{e^{1/n}\{e-1\}}{\left\{\left(1+\frac{1}{1!n}+\frac{1}{2!n^2}+\ldots\ldots\ldots+\frac{1}{n!n^n}\right)-1\right\}.n}$

$$= \lim_{n\to\infty} \frac{e^{1/n}\{e-1\}}{\left\{\frac{1}{n}+\frac{1}{2!n^2}+\ldots\ldots\ldots+\frac{1}{n!n^n}\right\}.n} = \lim_{n\to\infty} \frac{e^{1/n}\{e-1\}}{\left\{1+\frac{1}{2!n}+\ldots\ldots\ldots+\frac{1}{n!n^{n-1}}\right\}}$$

$$= \lim_{n\to\infty} \frac{e^0\{e-1\}}{(1+0+\ldots\ldots\ldots\ldots\ldots\ldots)} = (e-1). = \text{R.H.S.}$$

Hence this completes the proof.

Q.No.25.: Prove that $\lim\limits_{x\to\infty} \frac{1^2+2^2+3^2+\ldots\ldots+x^2}{x^3} = \frac{1}{3}$**.**

Sol.: $\lim\limits_{x\to\infty} \frac{1^2+2^2+3^2+\ldots\ldots+x^2}{x^3} = \lim\limits_{x\to\infty} \frac{\sum x^2}{x^3} = \lim\limits_{x\to\infty} \frac{x(x+1)(2x+1)}{x^3}$

$$= \lim_{x\to\infty} \frac{x^3\left(1+\frac{1}{x}\right)\left(2+\frac{1}{x}\right)}{x^3} = \frac{2}{6} = \frac{1}{3}.\text{Ans.}$$

Q.No.26.: Prove that $\mathop{\mathrm{Lt}}\limits_{x\to a} \sin^{-1}\sqrt{\frac{a-x}{a+x}}\operatorname{cosec}\sqrt{a^2-x^2} = \frac{1}{2a}$**.**

Sol.: TakingL.H.S.= $\mathop{\mathrm{Lt}}\limits_{x\to a} \sin^{-1}\sqrt{\frac{a-x}{a+x}}\operatorname{cosec}\sqrt{a^2-x^2} = \lim\limits_{x\to a} \frac{\sin^{-1}\sqrt{\frac{a-x}{a+x}}}{\sin\sqrt{a^2-x^2}}$

The given equation is in the form $\left[\frac{0}{0}\right]$. So, apply "Cauchy's Rule" (i. e. differentiate numerator and denominator w. r. t. x separately)

$$\text{L.H.S.} = \lim_{n\to} \frac{\dfrac{1}{\sqrt{1-\left(\sqrt{\dfrac{a-x}{a+x}}\right)^2}}\cdot\dfrac{1}{2\sqrt{\dfrac{a-x}{a+x}}}\cdot\dfrac{(a+x)(-1)-(a-x)(1)}{(a+x)^2}}{\cos\sqrt{a^2-x^2}\cdot\dfrac{1}{2\sqrt{a^2-x^2}}\cdot(-2x)}$$

$$= \lim_{x\to a}\frac{\dfrac{\sqrt{a+x}}{\sqrt{2x}}\cdot\dfrac{\sqrt{a+x}}{2\sqrt{a-x}}\cdot\dfrac{(-2a)}{(a+x)^2}}{\cos\sqrt{a^2-x^2}\cdot\dfrac{(-x)}{\sqrt{(a-x)(a+x)}}}$$

$$= \lim_{x\to a}\frac{(x+a).(2a).\sqrt{a-x}.\sqrt{a+x}}{x.\sqrt{2x}.2\sqrt{a-x}.(a+x)^2.\cos\sqrt{a^2-x^2}} = \lim_{x\to a}\frac{a.\sqrt{a+x}}{x.\sqrt{2x}.(a+x).\cos\sqrt{a^2-x^2}}$$

$$= \frac{a.\sqrt{2a}}{a.\sqrt{2a}.(2a).\cos 0} = \frac{1}{2a} = \text{R.H.S.}$$

Hence this completes the proof.

Q.No.27.: Evaluate $\lim_{x\to y}\dfrac{x^y - y^x}{x^x - y^y}$**.**

Sol.: $\lim_{x\to y}\dfrac{x^y - y^x}{x^x - y^y}\left[\dfrac{0}{0}\text{form}\right]$

$\therefore$ Using L'Hospital Rule, we get

$$= \lim_{x\to y}\frac{x^y\left(\dfrac{y}{x}\right) - y^x(\log y)}{x^x(1+\log x) - 0} = \frac{y^y\left(\dfrac{y}{y}\right) - y^y(y\log y)}{y^y(1+\log y)} = \frac{y^y(1-\log y)}{y^y(1+\log y)} = \frac{1-\log y}{1+\log y}.\ \text{Ans.}$$

Q.No.28.: Determine a, b, c such that

$$\underset{\theta\to 0}{\text{Lt}}\ \frac{\theta(a+b\cos\theta) - c\sin\theta}{\theta^5} = 1.$$

Sol.: $\underset{\theta\to 0}{\text{Lt}}\ \dfrac{\theta(a+b\cos\theta) - c\sin\theta}{\theta^5} = 1$

The given equation is in the form $\left[\dfrac{0}{0}\right]$. So, apply "Cauchy's Rule" , we get

$$\underset{\theta\to 0}{\text{Lt}}\ \frac{a + b\cos\theta + \theta(0 - b\sin\theta) - c\cos\theta}{5\theta^4}$$

$\therefore a+b-c=0 \Rightarrow a+b=c$. (i)

Again apply "Cauchy's Rule" , we get

$$\underset{\theta\to 0}{\mathrm{Lt}} \frac{0-b\sin\theta-b\sin\theta-\theta b\cos\theta+c\sin\theta}{20.\theta^3}$$

The above equation is in the form $\left[\frac{0}{0}\right]$. So, apply "Cauchy's Rule" , we get

$$\underset{\theta\to 0}{\mathrm{Lt}} \frac{-2b\cos\theta-b\cos\theta+\theta b\sin\theta+c\cos\theta}{60.\theta^2}$$

$\therefore -3b+c=0 \Rightarrow c=3b$.(ii)

The above equation is in the form $\left[\frac{0}{0}\right]$. So, apply "Cauchy's Rule" , we get

$$\underset{\theta\to 0}{\mathrm{Lt}} \frac{2b\sin\theta+b\sin\theta+b\sin\theta+\theta b\cos\theta-c\sin\theta}{120.\theta}$$

$$\Rightarrow \left\{\frac{4b}{120}\frac{\sin\theta}{\theta}+\frac{\theta b\cos\theta}{120.\theta}-\frac{c\sin\theta}{120.\theta}\right\}=\frac{4b}{120}+\frac{b}{120}-\frac{c}{120}=1 \text{ (given)}$$

$\therefore 5b-c=120$. (iii)

$\therefore$ From (ii) and (iii), we get

$b=60, \quad c=180, \quad a=120$. Ans.

Q.No.29.: Find the values of a and b such that $\lim_{x\to 0}\frac{x(1+a\cos x)-b\sin x}{x^3}=1$.

Sol.: $$\lim_{x\to 0}\frac{x(1+a\cos x)-b\sin x}{x^3}=\lim_{x\to 0}\frac{x\left\{1+a\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\ldots\ldots\right)\right\}-b\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots\right)}{x^3}$$

$$=\lim_{x\to 0}\frac{(1+a-b)x+\left(-\frac{a}{2}+\frac{b}{6}\right)x^3+\ldots}{x^3}\quad\left[\frac{0}{0}\text{form}\right]$$

Since the given limit is equal to 1, we must have

$1+a-b=0$ (i)

and $-\frac{a}{2}+\frac{b}{6}=1$. (ii)

Solving (i) and (ii), we get

$a = -\frac{5}{2}$, $b = -\frac{3}{2}$. Ans.

Q.No.30.: Evaluate **(a)** $\lim\limits_{\theta \to \frac{\pi}{2}} \frac{\log\left(\theta - \frac{\pi}{2}\right)}{\tan\theta}$, **(b)** $\lim\limits_{\theta \to \frac{\pi}{3}} \frac{1 - 2\cos x}{\sin\left(x - \frac{\pi}{3}\right)}$.

Sol.: (a) $\lim\limits_{\theta \to \frac{\pi}{2}} \frac{\log\left(\theta - \frac{\pi}{2}\right)}{\tan\theta} . \left[\frac{\infty}{\infty} \text{form}\right]$

$\therefore$ Using L'Hospital Rule, we get

$$= \lim_{\theta \to \frac{\pi}{2}} \frac{\frac{1}{x - \frac{\pi}{2}}}{\sec^2\theta} = \lim_{\theta \to \frac{\pi}{2}} \frac{\cos^2 x}{x - \frac{\pi}{2}} \left[\frac{0}{0} \text{form}\right]$$

$\therefore$ Using L'Hospital Rule, we get

$$= \lim_{\theta \to \frac{\pi}{2}} \frac{2\cos x(-\sin x)}{1} = \lim_{\theta \to \frac{\pi}{2}} (-\sin 2x) = 0 \text{ . Ans.}$$

(b) $\lim\limits_{\theta \to \frac{\pi}{3}} \frac{1 - 2\cos x}{\sin\left(x - \frac{\pi}{3}\right)} = \lim\limits_{\theta \to \frac{\pi}{3}} \frac{2\sin x}{\cos\left(x - \frac{\pi}{3}\right)}$

$\therefore$ Using L'Hospital'sRule, we get

$$= \frac{2.\left(\frac{\sqrt{3}}{2}\right)}{\cos 0} = \sqrt{3} \text{ . Ans.}$$

Q.No.31.: Evaluate $\underset{x \to a}{\mathrm{Lt}} \frac{\log(x - a)}{\log\left(e^x - e^a\right)}$.

Sol.: $\underset{x \to a}{\mathrm{Lt}} \frac{\log(x - a)}{\log\left(e^x - e^a\right)} = \frac{\left(\frac{1}{x - a}\right)}{\left(\frac{e^x}{e^x - e^a}\right)} = \lim\limits_{x \to a} \frac{e^x - e^a}{e^x (x - a)}$

$\therefore$ Using L'Hospital Rule, we get

$$= \lim_{x\to a}\frac{e^x}{e^x(x-a)+e^x} = \frac{e^a}{0+e^a} = 1. \text{ Ans.}$$

Q.No.32.: Evaluate $\underset{x\to 0}{\text{Lt}}\ \frac{\log\sin 2x}{\sin x}$.

Sol.: $\underset{x\to 0}{\text{Lt}}\ \frac{\log\sin 2x}{\sin x} = \underset{x\to 0}{\text{Lt}}\ \frac{\log(2\sin x\cos x)}{\sin x} = \underset{x\to 0}{\text{Lt}}\ \frac{\log(2\sin x)+\log\cos x}{\sin x}$

$$= \lim_{x\to 0}\left[\frac{\log(2\sin x)}{\sin x}+\frac{\log\cos x}{\sin x}\right] = \lim_{x\to 0}\frac{\log(2\sin x)}{\sin x}+\lim_{x\to 0}\frac{\log\cos x}{\sin x}$$

The second limit is of $\frac{0}{0}$ form and can be evaluated with the L' Hospital's rule

$$\therefore \lim_{x\to 0}\frac{\log\cos x}{\sin x} = \lim_{x\to 0}\frac{-\tan x}{\cos x} = 0. \text{ Ans.}$$

Q.No.33.: Evaluate $\lim_{x\to 0} x\log\sin x$.

Sol.: $\lim_{x\to 0} x\log\sin x$ $[0\times\infty \text{ form}]$ $\left[\because \lim_{x\to 0}\log x \to -\infty\right]$

$$= \lim_{x\to 0}\frac{\log\sin x}{\frac{1}{x}} \left[\frac{\infty}{\infty}\text{form}\right]$$

Using L'Hospital Rule, we get

$$= \lim_{x\to 0}\frac{\frac{\cos x}{\sin x}}{-\frac{1}{x^2}} = -\lim_{x\to 0} x^2\cot x = -\lim_{x\to 0}\frac{x^2}{\tan x} \left[\frac{0}{0}\text{form}\right]$$

$\therefore$ Using L'Hospital Rule, we get

$$= \lim_{x\to 0}\frac{2x}{\sec^2 x} = 0. \text{ Ans.}$$

Q.No.34.: Evaluate $\lim_{x\to 0} x\log x$.

Sol.: $\lim_{x\to 0} x\log x$ $[0\times\infty \text{ form}]$

$$= \lim_{x\to 0}\frac{\log x}{\frac{1}{x}} \left[\frac{\infty}{\infty}\text{form}\right]$$

$\therefore$ Using L'Hospital Rule, we get

$$= \lim_{x\to 0} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0}(-x) = 0 \text{. Ans.}$$

Q.No.35.: Evaluate $\mathop{Lt}\limits_{x\to\infty} x\tan\left(\frac{1}{x}\right)$.

Sol.: $\mathop{Lt}\limits_{x\to\infty} x\tan\left(\frac{1}{x}\right) = \mathop{Lt}\limits_{x\to\infty} x\frac{\sin\left(\frac{1}{x}\right)}{\cos\left(\frac{1}{x}\right)} = \left[\mathop{Lt}\limits_{\frac{1}{x}\to 0} \frac{\sin\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)}\right]\left[\mathop{Lt}\limits_{x\to\infty} \frac{1}{\cos\frac{1}{x}}\right] = 1$. Ans. $\left[\begin{matrix} x\to\infty \\ \frac{1}{x}\to 0\end{matrix}\right]$

Q.No.36.: Evaluate $\mathop{Lt}\limits_{x\to\infty}\left(a^{\frac{1}{x}}-1\right)x$.

Sol.: $\mathop{Lt}\limits_{x\to\infty}\left(a^{\frac{1}{x}}-1\right)x$

Let $\frac{1}{x} = y \quad \therefore x\to\infty, \quad y\to 0$.

Then $\mathop{Lt}\limits_{x\to\infty}\left(a^{\frac{1}{x}}-1\right)x = \mathop{Lt}\limits_{y\to 0}\frac{\left(a^y-1\right)}{y} = \mathop{Lt}\limits_{y\to 0}\frac{\left(a^y\log a\right)}{1} = \log a$. Ans.

Q.No.37.: Evaluate $\mathop{Lt}\limits_{x\to 0}\frac{A}{x^2}\left[\frac{\sin kx}{\sin \ell x} - \frac{k}{\ell}\right]$.

Sol.: $\mathop{Lt}\limits_{x\to 0}\frac{A}{x^2}\left[\frac{\sin kx}{\sin \ell x} - \frac{k}{\ell}\right] = \mathop{Lt}\limits_{x\to 0}\frac{A}{x^2}\left[\frac{\ell\sin kx - k\sin\ell x}{\ell\sin\ell x}\right]$

$$= \mathop{Lt}\limits_{x\to 0}\frac{A}{x^2}\left[\frac{\left(\ell kx - \frac{\ell k^3x^3}{3!} + \frac{\ell k^5x^5}{5!} - \ldots\ldots..\infty\right) - \left(k\ell x - \frac{k\ell^3x^3}{3!} + \frac{k\ell^5x^5}{5!} - \ldots\ldots..\infty\right)}{\left(\ell^2x - \frac{\ell^4x^3}{3!} + \frac{\ell^6x^5}{5!}\ldots\ldots\ldots\ldots\ldots\ldots\right)}\right]$$

$$= \mathop{Lt}\limits_{x\to 0}\frac{Ak\ell}{\ell^2}\left[\frac{\left(-\frac{k^2x}{3!} + \frac{k^4x^3}{5!} - \ldots\ldots..\infty\right) - \left(-\frac{\ell^2x}{3!} + \frac{k\ell^4x^5}{5!} - \ldots\ldots..\infty\right)}{\left(\ell^2x - \frac{\ell^4x^3}{3!} + \frac{\ell^6x^5}{5!}\ldots\ldots\ldots\ldots\ldots\ldots\right)}\right]$$

Using L'Hospital Rule, we get

$$= \underset{x\to 0}{\mathrm{Lt}} \frac{Ak}{\ell}\left[\frac{\frac{-k^2}{3!}+\frac{3k^4x^2}{5!}-...........\infty+\frac{\ell^2}{3!}-\frac{5\ell^4x^4}{5!}-...........\infty}{1-\frac{3\ell^2x^2}{3!}+\frac{5\ell^4x^4}{5!}............\infty}\right]$$

$$=\frac{Ak}{\ell}\left[\frac{\ell^2}{6}-\frac{k^2}{6}\right]=\frac{Ak}{6!}\left(\ell^2-k^2\right). \text{ Ans.}$$

Q.No.38.: Evaluate $\underset{x\to 1}{\mathrm{Lim}}\left[\frac{x}{x-1}-\frac{1}{\log x}\right]$**.**

Sol.: $\underset{x\to 1}{\mathrm{Lim}}\left[\frac{x}{x-1}-\frac{1}{\log x}\right]=\underset{x\to 1}{\mathrm{Lim}}\left[\frac{x\log x-(x-1)}{(x-1)\log x}\right]$

$\therefore$ Using L'Hospital Rule, we get

$$\underset{x\to 1}{\mathrm{Lim}}\left[\frac{1+\log x-1}{\left(\frac{x-1}{x}\right)+\log x}\right]=\underset{x\to 1}{\mathrm{Lim}}\left(\frac{\frac{1}{x}}{\frac{1}{x^2}+\frac{1}{x}}\right)=\frac{1}{2}\textbf{.} \text{ Ans}\textbf{.}$$

Q.No.39.: Find $\underset{x\to a}{\mathrm{Lt}}\left[\frac{f'(x)}{f(x)-f(a)}-\frac{1}{x-a}\right]$**.**

Sol.: $\underset{x\to a}{\mathrm{Lt}}\left[\frac{f'(x)}{f(x)-f(a)}-\frac{1}{x-a}\right]=\underset{x\to a}{\mathrm{Lt}}\left[\frac{f'(x)(x-a)-f(x)+f(a)}{f(x)-f(a)(x-a)}\right]$

$$=\underset{x\to a}{\mathrm{Lt}}\left[\frac{f'(x).x-f'(x).a-f(x)+f(a)}{f'(x).x-f(a)x-f(x)a+f(a).a}\right]$$

$$=\underset{x\to a}{\mathrm{Lt}}\left[\frac{x.f''(x)+f'(x)-af''(x)-f'(x)+f'(a)}{x.f'(x)-f(x)-xf'(a)+f(a)-af'(x)+af'(a)}\right]$$

$$=\underset{x\to a}{\mathrm{Lt}}\left[\frac{x.f'''(x)+f''(x)-af'''(x)-f''(a)}{x.f''(x)-f'(x)+f'(x)-xf''(a)+f'(a)+f'(a)-af''(x)+af''(a)}\right]$$

Applying the limits, we get

$$=\left[\frac{a.f'''(a)+f''(a)-af'''(a)-f''(a)}{a.f''(a)-f'(a)+f'(a)-af''(a)+f'(a)+f'(a)-af''(a)+af''(a)}\right]$$

$$=\frac{2f''(a)}{4f'(a)}=\frac{f''(a)}{2f'(a)}.\text{Ans}\textbf{.}$$

Q.No.40.: Evaluate $\lim\limits_{x\to 0}\left[\frac{1}{x}-\frac{1}{e^x-1}\right]$.

Sol.: $\lim\limits_{x\to 0}\left[\frac{1}{x}-\frac{1}{e^x-1}\right]$ $[\infty\times\infty \text{ form}]$

$$=\lim_{x\to 0}\left[\frac{(e^x-1)-x}{x(e^x-1)}\right]\left[\frac{0}{0}\text{form}\right]$$

$\therefore$ Using L'Hospital Rule, we get

$$=\lim_{x\to 0}\left[\frac{e^x-1}{x.e^x+(e^x-1).1}\right]=\lim_{x\to 0}\frac{e^x-1}{(x+1)e^x-1}\left[\frac{0}{0}\text{form}\right]$$

$\therefore$ Using L'Hospital Rule, we get

$$=\lim_{x\to 0}\frac{e^x}{(x+1).e^x+e^x}=\frac{1}{1+1}=\frac{1}{2}.\text{Ans.}$$

Q.No.41.: Evaluate $\lim\limits_{x\to 0}\left[\frac{1}{x}-\frac{1}{x^2}\log(1+x)\right]$.

Sol.: $\lim\limits_{x\to 0}\left[\frac{1}{x}-\frac{1}{x^2}\log(1+x)\right]$ $[(\infty-\infty)\text{form}]$

$$=\lim_{x\to 0}\left[\frac{1}{x}-\frac{1}{x^2}\left(x-\frac{x^2}{2}+\frac{x^3}{3}-......\right)\right]=\lim_{x\to 0}\left[\frac{1}{2}-\frac{1}{3}x+...\right]=\frac{1}{2}.\text{ Ans.}$$

Q.No.42.: Prove that $\lim\limits_{x\to 0}\frac{(1+x)^{1/x}-e}{x}=-\frac{e}{2}$.

Sol.: $\lim\limits_{x\to 0}\frac{(1+x)^{1/x}-e}{x}$. $\left[\frac{0}{0}\text{form}\right]\left[\because \lim\limits_{x\to 0}(1+x)^{1/x}=e\right]$

We first evaluate $(1+x)^{1/x}$.

Let $y=(1+x)^{1/x}$. $\therefore$

$$\log y=\frac{1}{x}\log(1+x)=\frac{1}{x}\left(x-\frac{x^2}{2}+\frac{x^3}{3}-..\right)=1+\left(-\frac{x}{2}+\frac{x^2}{3}-..\right)=1+z,$$

where $z=-\frac{x}{2}+\frac{x^2}{3}-...$.

$$\therefore\ y = e^{1+z} = e.e^{z} = e\left[1 + z + \frac{z^2}{2!} + ...\right]$$

$$= e\left[1 + \left(-\frac{x}{2} + \frac{x^2}{3} - ...\right) + \frac{1}{2!}\left(-\frac{x}{2} + \frac{x^2}{3} - ...\right)^2 + ...\right]$$

$$= e\left[1 - \frac{x}{2} + \frac{11}{24}x^2 - ......\right].$$

$$\therefore \lim_{x\to 0}\frac{(1+x)^{1/x} - e}{x} = \lim_{x\to 0}\frac{e\left(1 - \frac{x}{2} + \frac{11}{24}x^2 - ...\right) - e}{x} = \lim_{x\to 0} e\left(-\frac{1}{2} + \frac{11}{24}x + ...\right) = -\frac{1}{2}e \text{ .Ans.}$$

This completes the proof.

Q.No.43.: Evaluate $\underset{x\to 0}{\text{Lt}}\ \frac{e^x \sin x - x - x^2}{x^2 + x\log(1-x)}$.

Sol.: $\underset{x\to 0}{\text{Lt}}\ \frac{e^x \sin x - x - x^2}{x^2 + x\log(1-x)}$

Using the expansion of $e^x \sin x$ and $\log(1-x)$, we get

$$= \underset{x\to 0}{\text{Lt}}\ \frac{\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...\right)\left(x - \frac{x^3}{3!} + ...\right) - x - x^2}{x^2 + x\left(-x - \frac{x^2}{2} - \frac{x^3}{3} - ....\right)}$$

$$= \underset{x\to 0}{\text{Lt}}\ \frac{\left(x + x^2 + \frac{x^3}{3} - x^4 + ...\right) - x - x^2}{x^2 - \left(x^2 - \frac{x^3}{2} - \frac{x^4}{3} + ...\right)} = \underset{x\to 0}{\text{Lt}}\ \frac{\left(\frac{x^3}{3} - x^4 + ...\right)}{\left(-\frac{1}{2} - \frac{x}{3} + ...\right)} = \frac{-2}{3} \text{. Ans.}$$

Q.No.44.: Prove that $\lim_{x\to 0}\frac{(1+x)^{1/x} - e + \frac{ex}{2}}{x^2} = \frac{11e}{24}$.

Sol.: $\lim_{x\to 0}\frac{(1+x)^{1/x} - e + \frac{ex}{2}}{x^2}$. $\left[\frac{0}{0}\text{form}\right]\left[\because \lim_{x\to 0}(1+x)^{1/x} = e\right]$

We first evaluate $(1+x)^{1/x}$.

Let $y=(1+x)^{1/x}$.

$$\therefore\ \log y=\frac{1}{x}\log(1+x)=\frac{1}{x}\left(x-\frac{x^2}{2}+\frac{x^3}{3}-..\right)=1+\left(-\frac{x}{2}+\frac{x^2}{3}-..\right)=1+z,$$

where $z=-\frac{x}{2}+\frac{x^2}{3}-\ldots$.

$$\therefore\ y=e^{1+z}=e.e^z=e\left[1+z+\frac{z^2}{2!}+.......\right]$$

$$=e\left[1+\left(-\frac{x}{2}+\frac{x^2}{3}-\ldots\right)+\frac{1}{2!}\left(-\frac{x}{2}+\frac{x^2}{3}-\ldots\right)^2+......\right]$$

$$=e\left[1-\frac{x}{2}+\frac{11}{24}x^2-......\right].$$

$$\therefore\ \lim_{x\to 0}\frac{(1+x)^{1/x}-e+\frac{ex}{2}}{x^2}=\lim_{x\to 0}\frac{e\left(1-\frac{x}{2}+\frac{11}{24}x^2-\ldots\right)-e+\frac{ex}{2}}{x^2}$$

$$=\lim_{x\to 0}\left[\frac{11}{24}e+\text{terms containing powers of } x\right]=\frac{11}{24}e.$$

This completes the proof**.**

Q.No.45.: Evaluate $\lim_{x\to 0}\frac{\tanh x-2\sin x+x}{x^5}$**.**

Sol.: $\lim_{x\to 0}\frac{\tanh x-2\sin x+x}{x^5}$

$$=\operatorname*{Lt}_{x\to 0}\frac{\dfrac{\left(x+\frac{x^3}{3!}+\frac{x^5}{5!}+.........\right)}{\left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+.........\right)}-2\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-.........\right)+x}{x^5}$$

$$= \underset{x\to 0}{\text{Lt}}\frac{\left(x+\frac{x^3}{3!}+\frac{x^5}{5!}+.........\right)-2\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-.........\right)\left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+.........\right)+x\left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+.........\right)}{x^5\left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+.........\right)}$$

$$= \underset{x\to 0}{\text{Lt}}\frac{(1-2+1)x+\left(\frac{1}{6}-1+\frac{1}{3}+\frac{1}{2}\right)x^3+\left(\frac{1}{120}-\frac{2}{24}+\frac{2}{12}-\frac{2}{120}+\frac{1}{24}\right)x^5+(...)x^6+............}{x^5\left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+.........\right)}$$

$$= \underset{x\to 0}{\text{Lt}}\frac{\frac{14}{120}x^5+(....)x^6+.............}{x^5}=\frac{14}{120}=\frac{7}{60}\text{. Ans.}$$

Q.No.46.: Evaluate $\underset{x\to 0}{\text{Lim}}\frac{x\sin(\sin x)-\sin^2 x}{x^6}$**.**

Sol.: We make use of one standard series to obtain this limit

$$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-.........$$

Now $\underset{x\to 0}{\text{Lim}}\frac{x\sin(\sin x)-\sin^2 x}{x^6}=\underset{x\to 0}{\text{Lim}}\frac{x\sin\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-......\right)-\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-......\right)^2}{x^6}$

$$= \underset{x\to 0}{\text{Lim}}\frac{x\left\{\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-......\right)-\frac{1}{3!}\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-......\right)^3+\frac{1}{5!}\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-......\right)^5+....\right\}-\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-......\right)^2}{x^6}$$

$$= \underset{x\to 0}{\text{Lim}}\frac{\left\{\left(x^2-\frac{x^4}{3!}+\frac{x^6}{5!}-......\right)-\frac{x^4}{3!}\left(1-\frac{x^2}{3!}+\frac{x^4}{5!}-......\right)^3+\frac{x^6}{5!}\left(1-\frac{x^2}{3!}+\frac{x^4}{5!}-......\right)^5+....\right\}-\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-......\right)^2}{x^6}$$

Expanding by Binomial expansion, we get

$$\left\{\left(x^2-\frac{x^4}{3!}+\frac{x^6}{5!}-....\right)-\frac{x^4}{3!}\left\{1-3\left(\frac{x^2}{3!}-\frac{x^4}{5!}+......\right)+...\right\}+\frac{x^6}{5!}\left\{1-5\left(\frac{x^2}{3!}-\frac{x^4}{5!}+....\right)+.....\right\}+..\right\}$$

$$=\lim_{x\to 0}\frac{-x^2\left\{1-2\left(\frac{x^2}{3!}-\frac{x^4}{5!}+......\right)+\left(\frac{x^2}{3!}-\frac{x^4}{5!}+......\right)^2\right\}}{x^6}$$

$$=\lim_{x\to 0}\frac{x^6\left(\frac{1}{120}+\frac{1}{12}+\frac{1}{120}-\frac{2}{120}-\frac{1}{36}\right)+(........)x^7+....}{x^6}$$

$$=\lim_{x\to 0}\frac{\frac{1}{18}x^6+(........)x^7+....}{x^6}=\frac{1}{18}.\ \text{Ans.}$$

Q.No.47.: Evaluate $\lim\limits_{x\to 0}\dfrac{e^{x\sin x}-\cosh\left(x\sqrt{2}\right)}{x^4}$.

Sol.: $\lim\limits_{x\to 0}\dfrac{e^{x\sin x}-\cosh\left(x\sqrt{2}\right)}{x^4}$

We make use of two standard series to obtain this limit

$$e^{x\sin x}=1+x\sin x+\frac{(x\sin x)^2}{2!}+\frac{(x\sin x)^3}{3!}+\frac{(x\sin x)^4}{4!}+............$$

$$=1+x\sin x+\frac{x^2}{2!}(\sin x)^2+\frac{x^3}{3!}(\sin x)^3+\frac{x^4}{4!}(\sin x)^4+............$$

Now using expansion of $\sin x$

$$e^{x\sin x}=1+x\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}+...........\right)+\frac{x^2}{2!}\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}+...........\right)^2$$

$$+\frac{x^3}{3!}\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}+...........\right)^3+\frac{x^4}{4!}\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}+...........\right)^4$$

$$=1+x^2\left(1-\frac{x^2}{6}+\frac{x^4}{120}-...........\right)+\frac{x^4}{2}\left(1-\frac{x^2}{6}+\frac{x^5}{120}-...........\right)^2$$

$$+\frac{x^6}{6}\left(1-\frac{x^2}{6}+\frac{x^4}{120}-...........\right)^3+\frac{x^8}{24}\left(1-\frac{x^2}{6}+\frac{x^4}{120}-...........\right)^4$$

Now expanding by Binomial theorem,

$$e^{x\sin x} = 1 + x^2\left(1 - \frac{x^2}{6} + \frac{x^4}{120} + \ldots\ldots\ldots\ldots\right) + \frac{x^4}{2}\left[1 - 2\left(\frac{x^2}{6} + \frac{x^5}{120} + \ldots\ldots\ldots\right)\right]$$

$$+ \frac{x^6}{6}\left[1 - 3\left(\frac{x^2}{6} + \frac{x^4}{120} + \ldots\ldots\ldots\right)\right] + \ldots\ldots\ldots\ldots\ldots$$

Collecting terms of same type

$$e^{x\sin x} = 1 + x^2 + \frac{x^4}{3} + \frac{x^6}{120}\ldots\ldots\ldots\ldots \qquad \text{(i)}$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots\ldots\ldots\ldots\ldots$$

$$\cosh\left(\sqrt{2}x\right) = 1 + \frac{2x^2}{2!} + \frac{4x^4}{4!} + \frac{8x^6}{6!} + \ldots\ldots\ldots\ldots\ldots$$

$$= 1 + x^2 + \frac{x^4}{6} + \frac{x^6}{90} + \ldots\ldots\ldots\ldots\ldots \qquad \text{(ii)}$$

$$\underset{x\to 0}{\text{Lt}} \frac{e^{x\sin x} - \cosh\left(\sqrt{2}x\right)}{x^4}$$

$$= \underset{x\to 0}{\text{Lt}} \frac{\left(1 + x^2 + \frac{x^4}{3} + \frac{x^6}{120} + \ldots\ldots\right) - \left(1 + x^2 + \frac{x^4}{6} + \frac{x^6}{90} + \ldots\ldots\right)}{x^4}$$

$$= \underset{x\to 0}{\text{Lt}} \frac{\left(\frac{x^4}{3} + \frac{x^6}{120} + \ldots\ldots\right) - \left(\frac{x^4}{6} + \frac{x^6}{90} + \ldots\ldots\right)}{x^4}$$

Neglecting terms having powers more than 4

$$= \underset{x\to 0}{\text{Lt}} \frac{x^4\left(\frac{1}{3} - \frac{1}{6}\right)}{x^4} = \frac{1}{6}.\ \text{Ans.}$$

Q.No.48.: Prove that $\underset{x\to 0}{\text{Lim}} \frac{2x^2 - 2e^{x^2} + 2\cos x^{3/2} + \sin^3 x}{x^4} = -1$.

Sol.: $\underset{x\to 0}{\text{Lim}} \frac{2x^2 - 2e^{x^2} + 2\cos x^{3/2} + \sin^3 x}{x^4} = -1$

We use $\sin x$ series for expansion of $\sin^3 x$

$$\sin^3 x = \left[x - \frac{(x)^3}{3!} + \frac{(x)^5}{5!} - ....\right]^3 = x^3\left[1 - \frac{(x)^3}{6} + \frac{(x)^5}{120} - ....\right]^3$$

Using Binomial Theorem

$$= x^3\left[1 - \frac{(x)^3}{2} + \frac{(x)^5}{40} - ....\right]$$

Similarly

$$\cos x^{3/2} = 1 - \frac{\left(x^{3/2}\right)^2}{2!} + \frac{\left(x^{3/2}\right)^4}{4!} - \frac{\left(x^{3/2}\right)^6}{6!} + ... = 1 - \frac{x^3}{2} + \frac{x^6}{24} - ....$$

$$\therefore \lim_{x\to 0} \frac{2x^2 - 2e^{x^2} + 2\cos x^{3/2} + \sin^3 x}{x^4}$$

$$= \lim_{x\to 0} \frac{2x^2 - 2\left(1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + ...\right) + 2\left(1 - \frac{x^3}{2} + \frac{x^6}{24} - ....\right) + x^3\left(1 - \frac{x^2}{2} + \frac{x^4}{40} - ....\right)}{x^4}$$

$$= \lim_{x\to 0} \frac{-\left(x^4 + \frac{x^6}{3} + ...\right) + \left(\frac{x^6}{24} - ....\right) + x^4\left(\frac{x}{2} + \frac{x^3}{40} - ....\right)}{x^4} = -1. \text{ Ans.}$$

Q.No.49.: Evaluate $\lim_{x\to 0} \frac{1 + x\cos x - \cosh x - \log(1+x)}{\tan x - x}$.

Sol.: $\lim_{x\to 0} \frac{1 + x\cos x - \cosh x - \log(1+x)}{\tan x - x}$

$$= \lim_{x\to 0} \frac{1 + x\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - ......\right) - \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + ...\right) - \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ...\right)}{x + \frac{x^3}{3} + \frac{2x^5}{15} - x........}$$

$$= \lim_{x\to 0} \frac{x\left(-\frac{x^2}{2!} + \frac{x^4}{4!} - ......\right) - \left(\frac{x^4}{4!} + \frac{x^6}{6!} + ...\right) - \left(\frac{x^3}{3} - \frac{x^4}{4} + ...\right)}{\frac{x^3}{3} + \frac{2x^5}{15} - x......}$$

Neglecting terms greater than x^3

$$= \lim_{x\to 0}\frac{\left(-\frac{x^3}{2}-\frac{x^3}{3}\right)}{\frac{x^3}{3}} = -\frac{5}{2}. \text{ Ans.}$$

Q.No.50.: The current i in a circuit containing an inductance L, a capacitance C and an alternator of angular frequency ω and maximum e.m.f. E., is given by

$i = \frac{\omega E}{L\left(n^2-\omega^2\right)}\left(\cos\omega t - \cos nt\right)$ where $n = \frac{1}{\sqrt{LC}}$. Find the limiting form of the expression for i, when $\omega \to n$.

Sol.: Since $\lim\limits_{\omega\to n}\frac{\omega E}{L\left(n^2-\omega^2\right)}\left(\cos\omega t - \cos nt\right)\left[\frac{0}{0}\text{form}\right]$

$\therefore$ Using L'Hospital Rule, we get

$$= \lim_{\omega\to n}\frac{E\left(\cos\omega t - \cos nt\right) + E\omega t\left(-\sin\omega t\right)}{-2\omega L}$$

$$= \frac{E\left(\cos nt - \cos nt\right) + Ent\left(-\sin nt\right)}{-2nL}$$

$$= \frac{E\times 0 - Ent\left(\sin nt\right)}{-2nL} = \frac{Et}{2L}\sin nt. \text{ Ans.}$$

Q.No.51.: A column of length ℓ has a vertical load P and horizontal load F at the top, and the transverse deflection is given by

$D = \frac{F\ell}{P}\left[\frac{\tan m\ell}{m\ell} - 1\right]$, where $m^2 = \frac{P}{EI}$. Show that as $P \to 0$, $D \to \frac{F\ell^3}{3EI}$.

Sol.: Given $D = \frac{F\ell}{P}\left[\frac{\tan m\ell}{m\ell} - 1\right]$, where $m^2 = \frac{P}{EI}$.

$$\text{Now} \lim_{P\to 0} D = \lim_{P\to 0}\frac{F\ell}{P}\left[\frac{\tan m\ell}{m\ell} - 1\right] = \lim_{P\to 0}\frac{F\ell}{P}\left[\frac{\left(m\ell + \frac{m^3\ell^3}{3} + \frac{2}{15}m^5\ell^5 + \ldots\ldots\right)}{m\ell} - 1\right]$$

$$= \lim_{P\to 0}\frac{F\ell}{P}\left[\left(1 + \frac{m^2\ell^2}{3} + \frac{2}{15}m^4\ell^4 + \ldots\ldots\right) - 1\right] = \lim_{P\to 0}\frac{F\ell}{P}\left[\left(\frac{m^2\ell^2}{3} + \frac{2}{15}m^4\ell^4 + \ldots\ldots\right)\right]$$

$$= \underset{P\to 0}{\text{Lim}}\frac{F\ell}{P}\left[\frac{1}{3}\frac{P\ell^2}{EI}+\frac{2}{15}\left(\frac{P}{EI}\right)^2\ell^4+......\right] = \underset{P\to 0}{\text{Lim}}\frac{F\ell}{P}\times\frac{P\ell^2}{3EI}\left[1+\frac{2}{05}\frac{P}{EI}\ell^2+......\right]$$

$$= \underset{P\to 0}{\text{Lim}}\frac{F\ell^3}{3EI}\left[1+\frac{2}{05}\frac{P}{EI}\ell^2+......\right] = \frac{F\ell^3}{3EI}.$$

Thus as $P\to 0$, $D\to\frac{F\ell^3}{3EI}$.

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Differential Calculus

Partial Differentiation

(Total Differentials, Explicit Function, Implicit Functions and Total Differential Coefficient)

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Explicit Function:

A function, where the dependent variable say y, is **expressed** in terms of the independent variable say x, then that function is called ***explicit function***.

Example: $y = 4x^3 + 3x^2 + 5x + 9$.

Implicit Functions:

A function, where one of the various variables **cannot be expressed** explicitly in terms of the other variables, then that function is called ***implicit function***.

Example: Consider the relation $x^3 + y^3 + 3axy = 0$.

In this case, we obtain $\frac{dy}{dx}$ by differentiating throughout w.r.t. x.

Total Differentials:

Let u be a function of x and y i.e. u = f(x , y).

Then the total differential of u is defined and written as

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy.$$

Similarly, if u be a function of x , y and z i.e. u = f(x, y, z).

Then the total differential of u is defined and written as

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz.$$

Evaluation of $\frac{dy}{dx}$ for an implicit function:

Let u = f(x, y) be an implicit function $\Rightarrow u = f(x,y) = 0$ or const.

$$\Rightarrow du = 0. \qquad \text{....(i)}$$

Also when u be a function of x and y, i.e. u = f(x , y),

then the total differential of u is defined and written as $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy.$ ....(ii)

From (i) and (ii), we get $\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0 \Rightarrow \frac{dy}{dx} = -\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}} = -\frac{u_x}{u_y}$.

Total Differential Coefficient:

Let $u = f(x,y)$, where $x = \varphi(t)$, $y = \psi(t)$.

Then u is ultimately a function of t.

Then the total differential coefficient of u w.r.t. t is defined and written as

$$\frac{du}{dt} = \frac{\partial u}{\partial x}.\frac{dx}{dt} + \frac{\partial u}{\partial y}.\frac{dy}{dt}.$$

Similarly, if $u = f(x,y,z)$, where $x = \varphi(t)$, $y = \psi(t)$ and $z = \xi(t)$.

Then the total differential coefficient of u w.r.t. t is defined and written as

$$\frac{du}{dt} = \frac{\partial u}{\partial x}.\frac{dx}{dt} + \frac{\partial u}{\partial y}.\frac{dy}{dt} + \frac{\partial u}{\partial z}.\frac{dz}{dt}.$$

Remark: $u = f(x,y)$ and t = x, then from $\frac{du}{dt} = \frac{\partial u}{\partial x}.\frac{dx}{dt} + \frac{\partial u}{\partial y}.\frac{dy}{dt}$, we have

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx},$$

where $\frac{du}{dx}$ is the total differential coefficient of u w.r.t. x.

Now let us solve some problems related to Total Differentials and Total Differential Coefficient:

Q.No.1: Find the total differential of u in the following cases:-

(i) $u = \sqrt{x+y}$ and **(ii)** $u = \log(x^2 + y^2)$.

Sol.: (i) Since $u = \sqrt{x+y}$

$$\therefore \frac{\partial u}{\partial x} = \frac{1}{2\sqrt{x+y}} \text{ and } \frac{\partial u}{\partial y} = \frac{1}{2\sqrt{x+y}}.$$

Then $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = \frac{1}{2\sqrt{x+y}}(dx + dy)$.

(ii) Since $u = \log(x^2 + y^2) \therefore \frac{\partial u}{\partial x} = \frac{2x}{x^2 + y^2}$ and $\frac{\partial u}{\partial y} = \frac{2y}{x^2 + y^2}$.

Then $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = \frac{2xdx + 2ydy}{x^2 + y^2}$.

Q.No.2: If $x^3 + y^3 = 3axy$, find $\frac{dy}{dx}$.

Sol.: Given $x^3 + y^3 - 3axy = 0$. So let $u = x^3 + y^3 - 3axy = 0$.

$$\therefore \frac{\partial u}{\partial x} = 3x^2 - 3ay \text{ and } \frac{\partial u}{\partial y} = 3y^2 - 3ax.$$

Hence $\frac{dy}{dx} = -\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}} = -\frac{(3x^2 - 3ay)}{(3y^2 - 3ax)} = \frac{ay - x^2}{y^2 - ax}$.

Q.No.3: If $u = \sin^{-1}(x - y)$, where $x = 3t,\ y = 4t^3$. Prove that the total differential coefficient of u w. r. t. t is equal to $3(1 - t^2)^{-1/2}$.

Sol.: Given $u = \sin^{-1}(x - y)$, where $x = 3t, y = 4t^3$.

$\therefore \frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-(x-y)^2}}$, $\frac{\partial u}{\partial y} = \frac{-1}{\sqrt{1-(x-y)^2}}$, $\frac{du}{dx} = 3$ and $\frac{du}{dy} = 12t^2$.

Then total differential coefficient of u w. r. t. t is

$$\frac{du}{dt} = \frac{\partial u}{\partial x}.\frac{dx}{dt} + \frac{\partial u}{\partial y}.\frac{dy}{dt}$$

$$\Rightarrow \frac{du}{dt} = \frac{1}{\sqrt{1-(x-y)^2}}.3 + \frac{-1}{\sqrt{1-(x-y)^2}}.12t^2 = \frac{3(1-4t^2)}{\sqrt{1-(3t-4t^3)^2}} = \frac{3(1-4t^2)}{\sqrt{(1-4t^2)^2(1-t^2)}}$$

$$\Rightarrow \frac{du}{dt} = 3(1-t^2)^{-1/2}.$$

Q.No.4: If $u = \sin(x^2+y^2)$, where $a^2x^2 + b^2y^2 = c^2$, find the total differential coefficient of u w. r. t. x.

Sol.: Given $u = \sin(x^2+y^2)$, where $a^2x^2 + b^2y^2 = c^2$.

Let $f = a^2x^2 + b^2y^2 - c^2$.

$\therefore \frac{\partial f}{\partial x} = 2a^2x$, $\frac{\partial f}{\partial y} = 2b^2y$ and $\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{a^2x}{b^2y}$.

Since we know the total differential coefficient of u w. r. t. x. is

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}.\frac{dy}{dx}.$$

$$\Rightarrow \frac{du}{dx} = 2x\cos(x^2+y^2) + 2y.\cos(x^2+y^2)\left(-\frac{a^2x}{b^2y}\right) = 2\left(1-\frac{a^2}{b^2}\right)x\cos(x^2+y^2).$$

Q.No.5: Find the total differentials in the following cases:

(a) $u = (2x^2-4y^3)^3$, **(b)** $u = \tan\frac{x}{y}$.

Sol.: Since we know the total differential of u is $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$.

(a) Here $u = (2x^2-4y^3)^3$.

$$\therefore \frac{\partial u}{\partial x} = 3(2x^2-4y^3)^2.(4x) = 12x(2x^2-4y^3)^2,$$

and $\dfrac{\partial u}{\partial y} = 3\left(2x^2 - 4y^3\right)^2\left(-12y^2\right) = -36y^2\left(2x^2 - 4y^3\right)^2$.

Hence $du = \dfrac{\partial u}{\partial x}dx + \dfrac{\partial u}{\partial y}dy = 12x\left(2x^2 - 4y^3\right)^2 dx - 36y^2\left(2x^2 - 4y^3\right)^2 dy$

$$= 12\left(2x^2 - 4y^3\right)^2\left(xdx - 3y^2dy\right) \text{ . Ans.}$$

(b) Here $u = \tan\dfrac{x}{y}$. $\therefore \dfrac{\partial u}{\partial x} = \sec^2\left(\dfrac{x}{y}\right).\left(\dfrac{1}{y}\right)$ and $\dfrac{\partial u}{\partial y} = \sec^2\left(\dfrac{x}{y}\right).\left(-\dfrac{x}{y^2}\right)$.

Hence $du = \dfrac{\partial u}{\partial x}dx + \dfrac{\partial u}{\partial y}dy = \sec^2\left(\dfrac{x}{y}\right).\left(\dfrac{1}{y}\right)dx + \sec^2\left(\dfrac{x}{y}\right).\left(-\dfrac{x}{y^2}\right)dy$

$$= \sec^2\left(\frac{x}{y}\right).\left(\frac{ydx - xdy}{y^2}\right) \text{ . Ans.}$$

Q.No.6: Find the total differential coefficient of $u = \sin\left(\dfrac{x}{y}\right)$, where $x = e^t$,

$y = t^2$ w. r. t. t.

or

Given $u = \sin\left(\dfrac{x}{y}\right)$, where $x = e^t$, $y = t^2$, find $\dfrac{du}{dt}$ as a function of t.

Verify your result by direct substitution.

Sol.: We have $\dfrac{du}{dt} = \dfrac{\partial u}{\partial x}.\dfrac{dx}{dt} + \dfrac{\partial u}{\partial y}.\dfrac{dy}{dt} = \left(\cos\dfrac{x}{y}\right)\dfrac{1}{y}.e^t + \left(\cos\dfrac{x}{y}\right)\left(-\dfrac{x}{y^2}\right)2t$

$$= \cos\left(\frac{e^t}{t^2}\right).\frac{e^t}{t^2} - 2\cos\left(\frac{e^t}{t^2}\right).\frac{e^t}{t^3} = \left(\frac{t-2}{t^3}\right)e^t\cos\left(\frac{e^t}{t^2}\right).$$

Also $u = \sin\left(\dfrac{x}{y}\right) = \sin\left(\dfrac{e^t}{t^2}\right)$.

$\therefore \dfrac{du}{dt} = \cos\left(\dfrac{e^t}{t^2}\right).\dfrac{t^2e^t - e^t.2t}{t^4} = \left(\dfrac{t-2}{t^3}\right)e^t\cos\left(\dfrac{e^t}{t^2}\right)$ as before.

Q.No.7: If $u = x\log xy$, where $x^3 + y^3 + 3xy - 1 = 0$, find total differential coefficient of u w. r. t. x.

Sol.: Given $u = x\log xy$, where $x^3 + y^3 + 3xy - 1 = 0$.

Since we know that total differential coefficient of u w. r. t. x is

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}.\frac{dy}{dx}. \qquad .....(i)$$

Now $\frac{\partial u}{\partial x} = x.\left(\frac{1}{xy}.y\right) + 1.\log xy = 1 + \log xy$. Also $\frac{\partial u}{\partial y} = x.\left(\frac{1}{xy}.x\right) = \frac{x}{y}$.

Let $f = x^3 + y^3 + 3xy - 1$, then $\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = -\frac{(3x^2 + 3y)}{(3y^2 + 3x)} = -\frac{(x^2 + y)}{(y^2 + x)}$.

Substituting the values of $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$ and $\frac{dy}{dx}$ in (i), we get

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}.\frac{dy}{dx} = (1 + \log xy) + \left(\frac{x}{y}\right).\left(-\frac{x^2 + y}{y^2 + x}\right) = 1 + \log xy - \frac{x(x^2 + y)}{y(y^2 + x)}. \text{ Ans.}$$

Q.No.8: Find the total differential coefficient of $x^2 y$ w. r. t. x where x and y are connected by $x^2 + xy + y^2 = 1$.

Sol.: Let $u = x^2 y$, where x and y are connected by $x^2 + xy + y^2 = 1$.

Since we know that total differential coefficient of u w. r. t. x is

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}.\frac{dy}{dx}. \qquad .....(i)$$

Now $\frac{\partial u}{\partial x} = 2xy$. Also $\frac{\partial u}{\partial y} = x^2$.

Let $f = x^2 + xy + y^2 - 1$,

then $\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = -\frac{(2x + y)}{(2y + x)} = -\frac{(2x + y)}{(x + 2y)}$.

Substituting the values of $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$ and $\frac{dy}{dx}$ in (i), we get

$$\frac{du}{dx}=\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\cdot\frac{dy}{dx}=2xy+x^2\cdot\left(-\frac{2x+y}{x+2y}\right)=\frac{2xy(x+2y)-x^2(2x+y)}{(x+2y)}$$

$$\Rightarrow \frac{du}{dx}=\frac{x\left(xy+4y^2-2x^2\right)}{(x+2y)}.\ \text{Ans.}$$

Q.No.9: (i) If $f(x,y)=0$, $\varphi(y,z)=0$,

show that $\frac{\partial f}{\partial y}\cdot\frac{\partial \varphi}{\partial z}\cdot\frac{dz}{dx}=\frac{\partial f}{\partial x}\cdot\frac{\partial \varphi}{\partial y}$.

(ii) If the curves If $f(x,y)=0$ and $\varphi(x,y)=0$ touch,

show that at the point of contact $\frac{\partial f}{\partial x}\cdot\frac{\partial \varphi}{\partial y}=\frac{\partial f}{\partial y}\cdot\frac{\partial \varphi}{\partial x}$.

Sol.: (i) Given $f(x,y)=0$, $\varphi(y,z)=0 \Rightarrow df=0$ and $d\varphi=0$.

i.e. $\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy=0$ and $\frac{\partial \varphi}{\partial y}dy+\frac{\partial \varphi}{\partial z}dz=0$.

$$\Rightarrow \frac{dy}{dx}=-\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} \quad ....(i) \quad \text{and} \quad \frac{dz}{dy}=-\frac{\frac{\partial \varphi}{\partial y}}{\frac{\partial \varphi}{\partial z}}. \quad .....(ii)$$

$$(i) \Rightarrow \frac{\partial f}{\partial y}\cdot\frac{dy}{dx}=-\frac{\partial f}{\partial x} \Rightarrow \frac{\partial f}{\partial y}\cdot\frac{dy}{dz}\cdot\frac{dz}{dx}=-\frac{\partial f}{\partial x} \Rightarrow \frac{\partial f}{\partial y}\cdot\left[-\frac{\frac{\partial \varphi}{\partial z}}{\frac{\partial \varphi}{\partial y}}\right]\cdot\frac{dz}{dx}=-\frac{\partial f}{\partial x}$$

$$\Rightarrow \frac{\partial f}{\partial y}\cdot\frac{\partial \varphi}{\partial z}\cdot\frac{dz}{dx}=\frac{\partial f}{\partial x}\cdot\frac{\partial \varphi}{\partial y}.$$

(ii) Let the curves $f(x,y)=0$ and $\varphi(x,y)=0$ are touching at a point (a,b).

Now the slope of the tangent of the curve $f(x,y)=0$ at point of contact is

$$\frac{dy}{dx}=-\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}. \qquad ......(i)$$

and the slope of the tangent of the curve $\varphi(x,y)=0$ at point of contact is

$$\frac{dy}{dx} = -\frac{\frac{\partial \varphi}{\partial x}}{\frac{\partial \varphi}{\partial y}}. \qquad \text{.......(ii)}$$

Now since the curves $f(x,y)=0$ and $\varphi(x,y)=0$ are touching so that their slope of the tangents are same.

Hence from (i) and (ii), at the point of contact $\frac{\partial f}{\partial x}.\frac{\partial \varphi}{\partial y} = \frac{\partial f}{\partial y}.\frac{\partial \varphi}{\partial x}$.

Q.No.10: If $x^2 + y^2 + z^2 - 2xyz = 1$. Show that $\frac{dx}{\sqrt{1-x^2}} + \frac{dy}{\sqrt{1-y^2}} + \frac{dz}{\sqrt{1-z^2}} = 0$.

Sol.: Given $x^2 + y^2 + z^2 - 2xyz = 1$.

Let $u = x^2 + y^2 + z^2 - 2xyz - 1 = 0$.

Here u be an implicit function $\Rightarrow du = 0$. ...(i)

Here u be a function of x, y and z, i.e. $u = f(x, y, z)$.

Then the total differential of u is $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz$

$$\Rightarrow \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz = 0. \quad \text{[by (i)]} \qquad \text{....(ii)}$$

Evaluate: $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$ and $\frac{\partial u}{\partial z}$.

Since $u = x^2 + y^2 + z^2 - 2xyz - 1$.

$$\therefore \frac{\partial u}{\partial x} = 2(x - yz), \ \frac{\partial u}{\partial y} = 2(y - xz) \ \text{ and } \ \frac{\partial u}{\partial z} = 2(z - xy).$$

Hence (ii) becomes $(x - yz)dx + (y - xz)dy + (z - xy)dz = 0$. ....(iii)

Find: $(x - yz)$, $(y - xz)$ and $(z - xy)$.

Since we have given $x^2 + y^2 + z^2 - 2xyz = 1$

$$\Rightarrow x^2 - 2xyz = 1 - y^2 - z^2 \Rightarrow x^2 - 2xyz + y^2z^2 = 1 - y^2 - z^2 + y^2z^2$$

$$\Rightarrow (x - yz)^2 = (1 - y^2)(1 - z^2) \ \Rightarrow (x - yz) = \sqrt{(1 - y^2)(1 - z^2)}.$$

Similarly, $(y - xz) = \sqrt{(1 - x^2)(1 - z^2)}$

and $(z - xy) = \sqrt{(1 - x^2)(1 - y^2)}$.

Hence (iii) becomes $\sqrt{(1 - y^2)(1 - z^2)}dx + \sqrt{(1 - x^2)(1 - z^2)}dy + \sqrt{(1 - x^2)(1 - y^2)}dz = 0$.

Last step: Dividing by $\sqrt{(1 - x^2)(1 - y^2)(1 - z^2)}$, we get

$$\frac{dx}{\sqrt{1 - x^2}} + \frac{dy}{\sqrt{1 - y^2}} + \frac{dz}{\sqrt{1 - z^2}} = 0 .$$

Q.No.11: If $u = x^2 + y^2$, where $x = a\cos t$, $y = b\sin t$. Find $\frac{du}{dt}$ and verify the result.

Sol.: Given $u = x^2 + y^2$.

We have $\frac{du}{dt} = \frac{\partial u}{\partial x}.\frac{dx}{dt} + \frac{\partial u}{\partial y}.\frac{dy}{dt} = 2x.a(-\sin t) + 2y.b\cos t = 2(yb\cos t - xa\sin t)$

Also $u = x^2 + y^2 = a^2\cos^2 t + b^2\sin^2 t$.

$$\therefore \frac{du}{dt} = a^2 2\cos t(-\sin t) + b^2 2\sin t(\cos t) = 2[-(a\cos t)(a\sin t) + (b\sin t)(b\cos t)]$$

$$= 2(yb\cos t - xa\sin t) \text{ as before.}$$

Q.No.12: If $ax^2 + by^2 + cz^2 = 1$ and $lx + my + nz = 0$, then prove that

$$\frac{dx}{bny - cmz} = \frac{dy}{clz - anx} = \frac{dz}{amx - bly}.$$

Also find $\frac{dy}{dx}$ and $\frac{dz}{dx}$.

Sol.: Let $f = ax^2 + by^2 + cz^2 - 1$ and $\phi = lx + my + nz$.

$\therefore f = 0 \Rightarrow df = 0$ and $\phi = 0 \Rightarrow d\phi = 0$.

Now $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = 2axdx + 2bydy + 2czdz = axdx + bydy + czdz = 0$ ....(i)

Also $d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz = ldx + mdy + ndz = 0$ ...(ii)

Solving (i) and (ii), we get

$$\frac{dx}{bny - cmz} = \frac{dy}{clz - anx} = \frac{dz}{amx - bly}.$$

Now consider $\frac{dx}{bny - cmz} = \frac{dy}{clz - anx}$ and $\frac{dx}{bny - cmz} = \frac{dz}{amx - bly}$.

$\therefore \frac{dy}{dx} = \frac{clz - anx}{bny - cmz}$ and $\frac{dz}{dx} = \frac{amx - bly}{bny - cmz}$. Ans.

Q.No.13: If $x^2y - e^x + x\sin z = 0$ and $x^2 + y^2 + z^2 = a^2$. Find $\frac{dy}{dx}$ and $\frac{dz}{dx}$.

Sol.: Let $f = x^2y - e^x + x\sin z = 0$ and $\phi = x^2 + y^2 + z^2 - a^2 = 0$.

$\therefore f = 0 \Rightarrow df = 0$ and $\phi = 0 \Rightarrow d\phi = 0$.

Now $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = \left(2xy - e^x + \sin z\right)dx + x^2dy + x\cos z dz = 0$ ....(i)

Also $d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz = 2xdx + 2ydy + 2zdz = xdx + ydy + zdz = 0$ ...(ii)

Solving (i) and (ii), we get

$$\frac{dx}{x^2z - yx\cos z} = \frac{dy}{x^2\cos z - z\left(2xy - e^x + \sin z\right)} = \frac{dz}{y\left(2xy - e^x + \sin z\right) - x^3}.$$

Now consider $\frac{dx}{x^2z - yx\cos z} = \frac{dy}{x^2\cos z - z\left(2xy - e^x + \sin z\right)}$

and $\frac{dx}{x^2z - yx\cos z} = \frac{dz}{y\left(2xy - e^x + \sin z\right) - x^3}$.

We get $\frac{dy}{dx} = \frac{x^2\cos z - z\left(2xy - e^x + \sin z\right)}{x^2z - yx\cos z}$ and $\frac{dz}{dx} = \frac{y\left(2xy - e^x + \sin z\right) - x^3}{x^2z - yx\cos z}$.Ans.

Q.No.14: If $x^y = e^{x-y}$, then prove that $\frac{dy}{dx} = \frac{\log x}{(1 + \log x)^2}$.

Sol.: Given $x^y = e^{x-y}$.

Taking log on both sides, we get

$\log x^y = \log e^{x-y} \Rightarrow y\log x = x - y \Rightarrow y\log x - x + y = 0$.

Let $u = y\log x - x + y$. ....(i)

Differentiate (i) partially w. r. t. x and y separately, we get

$\therefore \frac{\partial u}{\partial x} = \frac{y}{x} - 1 = \frac{y - x}{x}$ and $\frac{\partial u}{\partial y} = (\log x + 1)$.

Hence $\frac{dy}{dx}=-\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}}=\frac{\frac{y-x}{x}}{(\log x+1)}=\frac{x-y}{x(\log x+1)}=\frac{y\log x}{x(1+\log x)}$. ...(i) $[\because x-y=y\log x]$

Now since $y\log x=x-y \Rightarrow \frac{y}{x}\log x=1-\frac{y}{x} \Rightarrow \frac{y}{x}(1+\log x)=1 \Rightarrow \frac{y}{x}=\frac{1}{(1+\log x)}$

Substituting the value of $\frac{y}{x}$ in (i), we get

$$\frac{dy}{dx}=\frac{\log x}{(1+\log x)^2}.$$

This completes the proof**.**

Q.No.15: Using partial differentiation, find $\frac{dy}{dx}$ when $x^y+y^x=C$.

Sol.: Given $x^y+y^x=C \Rightarrow x^y+y^x-C=0$

Let $f(x,y)=x^y+y^x-C$ .....(i)

Differentiate (i) partially w. r. t. x and y separately, we get

$$\frac{\partial f}{\partial x}=yx^{y-1}+y^x\log y \text{ and } \frac{\partial f}{\partial y}=x^y\log x+xy^{x-1}.$$

But $\frac{dy}{dx}=-\frac{f_x}{f_y}=-\frac{y\left(x^{y-1}+y^{x-1}\log y\right)}{x\left(x^{y-1}\log x+y^{x-1}\right)}$. Ans**.**

Q.No.16.: Find $\frac{du}{dt}$ as a total derivative and verify the result by direct substitution if

$$u=x^2+y^2+z^2 \text{ and } x=e^{2t},\ y=e^{2t}\cos 3t,\ z=e^{2t}\sin 3t.$$

Sol.: Here u is a function of x, y, z are and x, y, z are in turn functions of t. Thus u is a function 't' via the intermediate variables x, y, z. Then

$$\frac{du}{dt}=\frac{\partial u}{\partial x}\frac{dx}{dt}+\frac{\partial u}{\partial y}\frac{dy}{dt}+\frac{\partial u}{\partial z}\frac{dz}{dt}$$

$$=2x.2e^{2t}+2y.\left(2e^{2t}\cos 3t-3e^{2t}\sin 3t\right)+2z\left(2e^{2t}\sin 3t+3e^{2t}\cos 3t\right)$$

Rewriting in terms of x, y, z

$$=2x.2.x+2.y(2y-3.z)+2z(2z+3y)$$

$= 4\left(x^2 + y^2 + z^2\right)$

or in terms of t

$$\frac{du}{dt} = 4\left(e^{4t} + e^{4t}\left(\cos^2 3t + \sin^2 3t\right)\right) = 8e^{4t}$$

Verification by direct submission:

$$u = x^2 + y^2 + z^2 = e^{4t} + e^{4t}\cos^2 3t + e^{4t}\sin^2 3t = 2e^{4t}$$

$$\frac{du}{dt} = 8e^{4t}.$$

Q.No.17.: Find the total differential coefficient of x^2y w.r.t. x when x, y are connected by $x^2 + xy + y^2 = 1$.

Sol.: Let $u = x^2y$, then the total differential is

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

Thus the total differential coefficient of u w.r.t x is

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}$$

$$\frac{du}{dx} = 2xy + x^2\frac{dy}{dx}$$

From the Implicit relation $f = x^2xy + y = 1$, we calculate

$$\frac{dy}{dx} = \frac{f_x}{f_y} = -\frac{2x + y}{x + 2y}$$

so $\dfrac{du}{dx} = 2xy + x^2.\dfrac{dy}{dx} = 2xy + x^2\left(-\dfrac{(2x + y)}{(x + 2y)}\right)$

$$\frac{du}{dx} = 2xy - \frac{x^2(2x + y)}{(x + 2y)}.$$

Q.No.18.: The altitude of the right circular cone is 15 cm and is increasing at 0.2 cm/sec. The radius of the base is 10 cm and is decreasing at 0.3 cm/sec. How fast is the volume changing?

Sol.: Let x be the radius and y be the altitude of the cone. So volume V of the right circular cone is $V=\frac{1}{3}\pi x^2 y$.

Since x and y are changing w.r.t time t, differentiate V w.r.t. t.

$$\frac{dV}{dt}=\frac{\partial V}{\partial x}\frac{dx}{dt}+\frac{\partial V}{\partial y}\frac{dy}{dt}$$

$$=\frac{1}{3}\pi\left(2xy\frac{dx}{dt}+x^2\frac{dy}{dt}\right)$$

It is given that $x = 10$, $y = 15$, $\frac{dx}{dt}=-0.3$ and $\frac{dy}{dt}=0.2$, substituting these values

$$\frac{dV}{dt}=\frac{1}{3}\pi\left[2.10.15(-0.3)+10^2(0.2)\right]=\frac{-70}{3}\pi\ \text{cm}^3/\text{sec}$$

i.e, volume is decreasing at the rate of $\frac{70\pi}{3}$.

Home Assignments

Q.No.1.: Find $\frac{du}{dt}$ when $u=\sin\left(\frac{x}{y}\right)$ and $x=e^t$, $y=t^2$. Verify the result by direct substitution.

Ans.: $\frac{t-2}{t^3}e^t\cos\left(\frac{e^t}{t^2}\right)$.

Q.No.2.: Find $\frac{du}{dt}$ given $u=\sin^{-1}(x-y)$, $x = 3t$, $y=4t^3$. Verify the result by direct substitution.

Ans.: $3\left(1-t^2\right)^{-1/2}$

Q.No.3.: If $u = x^3ye^z$ where $x = t$, $y=t^2$ and $z=\text{In}\,t$, find $\frac{du}{dt}$ at $t = 2$.

Ans.: $6t^5$; 192.

Q.No.4.: Find $\frac{du}{dt}$, if $u = \tan^{-1}\left(\frac{y}{x}\right)$ and $x = e^t - e^{-t}$ and $y = e^t + e^{-t}$.

Ans.: $\frac{-2}{e^{2t} + e^{-2t}}$

Q.No.5.: If x, y are related by $x^2 - y^2 = 2$ and $u = \tan\left(x^2 + y^2\right)$, find $\frac{du}{dx}$.

Ans.: $4x\sec^2\left(2x^2 - 2\right)$.

Q.No.6.: If $u = \tan^{-1}\left(\frac{y}{x}\right)$ and $y = x^4$ find $\frac{du}{dx}$ at x = 1.

Ans.: $\frac{3x^2}{1+x^6}$; $\frac{3}{2}$ at x = 1.

Q.No.7.: In order that the function $u = 2xy - 3x^2y$ remains constant. What should be the rate of change of y (w.r.t. t) given that x increases at the rate of 2cm/sec at the instant when x = 3 cm and y = 1 cm.

Ans.: $\frac{dy}{dt} = -\frac{32}{21}\,\text{cm/sec}$; y must decrease at the rate of $\frac{32}{21}$ cm/sec.

Q.No.8.: Find the rate at which the area of a rectangle is increasing at a given instant when the sides of the rectangle are 4 ft and 3 ft and are increasing at the rate of 1.5 ft/sec. and 0.5 ft/sect respectively.

Ans.: 6.5 sq. ft/sec.

Q.No.9.: Find (**a**). $\frac{dz}{dx}$ and (**b**). $\frac{dz}{dy}$, given $z = xy^2 + x^2y$, $y = \text{In}\, x$.

Ans.: (a). Here x is the independent variable

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx} = y^2 + 2xy + 2y + x$$

(b). Here y is the independent variable

$$\frac{dz}{dy} = \frac{\partial z}{\partial y} + \frac{\partial z}{\partial x}\frac{dx}{dy} = xy^2 + 2x^2y + 2xy + x^2$$

Q.No.10.: Find the differential of the function $f(x, y) = x\cos y - y\cos x$.

Ans.: $df = (\cos y + y\sin x)dx - (x\sin y + \cos x)dy$

Q.No.11.: Find the differential of the function $u(x, y, z)) = e^{xyz}$.

Ans.: $du = e^{xyz}(yzdx + zxdy + xydz)$.

Q.No.12.: Find $\frac{du}{dt}$ for the functions $u = x^2 - y^2$, $x = e^t \cos t$, $y = e^t \sin t$ at $t = 0$.

Ans.: $2e^{2t}(\cos 2t - \sin 2t)$; At $t = 0$, $\frac{du}{dt} = 2$

Q.No.13.: Find $\frac{du}{dt}$ for the functions $u = \text{In}(x + y + z)$; $x = e^{-t}$, $y = \sin t$, $z = \cos t$.

Ans.: $\frac{\cos t - \sin t - e^{-t}}{\cos t + \sin t + e^{-t}}$

Q.No.14.: Find $\frac{du}{dt}$ for the functions $u = \sin(e^x + y)$, $x = f(t)$, $y = g(t)$.

Ans.: $\frac{du}{dt} = [\cos(e^x + y)]e^x f'(t) + [\cos(e^x + y)]g'(t)$.

Q.No.15.: Find $\frac{du}{dt}$ for the functions $u = x^y$ when $y = \tan^{-1} t$, $x = \sin t$.

Ans.: $y.x^{y-1} \cos t + x^y \text{In}\, x.\frac{1}{1+t^2}$.

Thank you

NEXT TOPIC

Transformation of independent variables (Composite Functions),

Jacobian, Properties of Jacobians

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Differential Calculus

Indeterminate Forms

$0^0, \quad \infty^0, \quad 1^\infty$

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Indeterminate forms-Problems of $0^0, \infty^0, 1^\infty$:

Q.No.1.: Evaluate $\underset{x\to 0}{\text{Lt}}\left(\dfrac{\sin x}{x}\right)^{\frac{1}{x}}$.

Sol.: Let $y = \underset{x\to 0}{\text{Lt}}\left(\dfrac{\sin x}{x}\right)^{\frac{1}{x}}$ $\qquad\qquad \left[1^\infty \text{ form}\right]$

Taking log of both sides, we get

$$\log y = \underset{x\to 0}{\text{Lt}}\log\left(\frac{\sin x}{x}\right)^{\frac{1}{x}} = \underset{x\to 0}{\text{Lt}}\frac{1}{x}\log\left(\frac{\sin x}{x}\right) = \underset{x\to 0}{\text{Lt}}\frac{\log\left(\frac{\sin x}{x}\right)}{x} \qquad \left[\frac{0}{0}\text{form}\right]$$

Now apply Cauchy's Rule, we get

$$\log y = \underset{x\to 0}{\text{Lt}}\left(\frac{x}{\sin x}\right)\left(\frac{x\cos x - \sin x}{x^2}\right) \qquad \left[\because \underset{x\to 0}{\text{Lt}}\frac{x}{\sin x} = 1\right]$$

$$= \underset{x\to 0}{\text{Lt}}\frac{x\cos x - \sin x}{x^2} \qquad \left[\frac{0}{0}\text{form}\right]$$

Applying Cauchy's Rule again, we get

$$\log y = \underset{x\to 0}{\text{Lt}}\frac{\cos x - x\sin x - \cos x}{2x} = \underset{x\to 0}{\text{Lt}}\frac{-x\sin x}{2x} = \underset{x\to 0}{\text{Lt}}\frac{-\sin x}{2} = 0$$

$\log y = 0$

$\therefore y = e^0 = 1$. Ans.

Q.No.2.: Evaluate $\underset{x\to 1}{\text{Lt}} \left(1-x^2\right)^{\frac{1}{\log(1-x)}}$.

Sol.: Let $y = \underset{x\to 1}{\text{Lt}} \left(1-x^2\right)^{\frac{1}{\log(1-x)}}$ $\qquad \left[0^0 \text{form}\right]$

Taking log of both sides, we get

$$\log y = \underset{x\to 1}{\text{Lt}} \log\left(1-x^2\right)^{\frac{1}{\log(1-x)}}$$

$$= \underset{x\to 1}{\text{Lt}} \frac{1}{\log(1-x)}.\log\left(1-x^2\right) \qquad \left[\frac{\infty}{\infty}\text{form}\right]$$

Applying Cauchy's Rule, we get

$$\log y = \underset{x\to 1}{\text{Lt}} \frac{\frac{1}{1-x^2}(-2x)}{\frac{1}{1-x}(-1)} = \underset{x\to 1}{\text{Lt}} \frac{2x(1-x)}{\left(1-x^2\right)} = \underset{x\to 1}{\text{Lt}} \frac{2x}{(1+x)} = 1$$

$$\log_e y = 1$$

$\therefore y = e^1 = e$. Ans.

Q.No.3.: Evaluate $\underset{x\to 0}{\text{Lt}} \left[\frac{a_1^x + a_2^x + \dots\dots\dots + a_n^x}{n}\right]^{\frac{1}{x}}$.

Sol.: Let $y = \underset{x\to 0}{\text{Lt}} \left[\frac{a_1^x + a_2^x + \dots\dots\dots + a_n^x}{n}\right]^{\frac{1}{x}}$ $\qquad \left[1^\infty \text{form}\right]$

Taking log of both sides, we get

$$\log y = \underset{x\to 0}{\text{Lt}} \log\left[\frac{a_1^x + a_2^x + \dots\dots\dots + a_n^x}{n}\right]^{\frac{1}{x}}$$

$$= \underset{x\to 0}{\text{Lt}} \frac{\log\left[\frac{a_1^x + a_2^x + \dots\dots\dots + a_n^x}{n}\right]}{x}$$

$$\left[\frac{0}{0}\text{form}\right]$$

Applying Cauchy's Rule, we get

$$\log y = \underset{x\to 0}{\text{Lt}} \frac{n}{a_1^x + a_2^x + \ldots\ldots + a_n^x} \times \frac{\left(a_1^x \log a_1 + a_2^x \log a_2 + \ldots\ldots\ldots + a_n^x \log a_n\right)}{n}$$

$$= \left(\frac{n}{n}\right) \cdot \frac{1}{n}\left(\log a_1 + \log a_2 + \ldots\ldots\ldots + \log a_n\right)$$

$$\log y = \log\left(a_1 a_2 \ldots\ldots\ldots\ldots + a_n\right)^{\frac{1}{n}}$$

$$\left[\frac{d}{dx}(\log u) = \frac{1}{u}\frac{du}{dx}\right]$$

$$y = \left(a_1 . a_2 \ldots\ldots\ldots . a_n\right)^{\frac{1}{n}}.$$ **Ans.**

Q.No.4.: Evaluate $\lim\limits_{x\to 0}(1+\tan x)^{\cot x}$.

Sol.: Let $y = \lim\limits_{x\to 0}(1+\tan x)^{\cot x}$. [$1^\infty$ form]

Taking log on both sides, we get

$$\log y = \log\left[\lim_{x\to 0}(1+\tan x)^{\cot x}\right] = \left[\lim_{x\to 0}\left\{\log(1+\tan x)^{\cot x}\right\}\right] = \left[\lim_{x\to 0}\{\cot x \log(1+\tan x)\}\right]$$

$$= \lim_{x\to 0}\left[\frac{\log(1+\tan x)}{\tan x}\right] = \lim_{x\to 0}\frac{1}{\tan x}\left[\tan x - \frac{\tan^2 x}{2} + \frac{\tan^3 x}{3} - \ldots\ldots\ldots\infty\right]$$

$$= \lim_{x\to 0}\left[1 - \frac{\tan x}{2} + \frac{\tan^2 x}{3} - \ldots\ldots\ldots\infty\right] = 1.$$

$$\therefore y = e^1 = e.$$

Hence $\lim\limits_{x\to 0}(1+\tan x)^{\cot x} = e$. **Ans.**

Q.No.5.: Evaluate $\lim\limits_{x\to 0}(\cos x)^{\frac{1}{x^2}}$.

Sol.: Let $y = \lim\limits_{x\to 0}(\cos x)^{\frac{1}{x^2}}$. [$1^\infty$ form]

Taking log on both sides, we get

$$\therefore \log y = \lim_{x\to 0}\log(\cos x)^{\frac{1}{x^2}} = \lim_{x\to 0}\frac{1}{x^2}\log(\cos x) = \lim_{x\to 0}\frac{\log(\cos x)}{x^2}.$$

$$\left[\frac{0}{0}\text{form}\right]$$

Apply Cauchy's rule, we get

$$\log y = \lim_{x\to 0}\frac{\left[\frac{1}{\cos x}\sin x\right]}{2x} = \lim_{x\to 0}\left(-\frac{\tan x}{2x}\right). \qquad \left[\frac{0}{0}\text{form}\right]$$

Apply Cauchy's rule, we get

$$\log y = \lim_{x\to 0}\left(-\frac{\sec^2 x}{2}\right) = -\frac{1}{2}.$$

$$\therefore y = e^{-1/2}.$$

Hence $\lim_{x\to 0}(\cos x)^{\frac{1}{x^2}} = e^{-1/2}$. Ans.

Q.No.6.: Evaluate $\lim_{x\to 0}\left(\frac{1}{x}\right)^{\tan x}$.

Sol.: Let $y = \lim_{x\to 0}\left(\frac{1}{x}\right)^{\tan x}$. $\qquad [\infty^0 \text{ form}]$

Taking log on both sides, we get

$$\log y = \lim_{x\to 0}\log\left(\frac{1}{x}\right)^{\tan x} = \lim_{x\to 0}\tan x\log\left(\frac{1}{x}\right) = \lim_{x\to 0}\left(\frac{\tan x}{x}\right)x\log\left(\frac{1}{x}\right)$$

$$= \lim_{x\to 0}x\log\left(\frac{1}{x}\right) \qquad \left[\therefore \lim_{x\to 0}\left(\frac{\tan x}{x}\right) = 1\right]$$

$$= \lim_{x\to 0}\frac{\log\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)}. \qquad \left[\frac{\infty}{\infty}\text{form}\right]$$

Apply Cauchy's rule, we get

$$\log y = \lim_{x\to 0}\frac{x\left(-\frac{1}{x^2}\right)}{\left(-\frac{1}{x^2}\right)} = \lim_{x\to 0}(x) = 0.$$

$\therefore y = e^0 = 1.$

Hence $\lim_{x\to 0}\left(\frac{1}{x}\right)^{\tan x} = 1$ .Ans.

Q.No.7.: Evaluate $\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}}$.

Sol.: Let $y = \lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}}$. [$1^\infty$ form]

Taking log on both sides, we get

$$\log y = \lim_{x\to 0}\log\left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}} = \lim_{x\to 0}\frac{1}{x^2}\log\left(\frac{\tan x}{x}\right) = \lim_{x\to 0}\frac{\log\left(\frac{\tan x}{x}\right)}{x^2}. \qquad \left[\frac{0}{0}\text{form}\right]$$

Apply Cauchy's rule, we get

$$\log y = \lim_{x\to 0}\frac{\left[\frac{x}{\tan x}\cdot\frac{x\sec^2 x - \tan x}{x^2}\right]}{2x} = \lim_{x\to 0}\left(\frac{x}{\tan x}\cdot\frac{x\sec^2 x - \tan x}{2x^3}\right)$$

$$= \lim_{x\to 0}\frac{x\sec^2 x - \tan x}{2x^3}. \qquad \left[\frac{0}{0}\text{form}\right] \qquad \left[\because \lim_{x\to 0}\left(\frac{x}{\tan x}\right) = 1\right]$$

Again, apply Cauchy's rule, we get

$$\log y = \lim_{x\to 0}\frac{\sec^2 x + x.2\sec x\sec x\tan x - \sec^2 x}{6x^2} = \lim_{x\to 0}\frac{2x\sec^2 x\tan x}{6x^2}.$$

$$= \lim_{x\to 0}\frac{\sec^2 x\tan x}{3x} \qquad = \lim_{x\to 0}\frac{\sec^2 x}{3}.\frac{\tan x}{x} = \frac{\sec^2 0}{3} = \frac{1}{3}.$$

$$\left[\because \lim_{x\to 0}\left(\frac{\tan x}{x}\right) = 1\right]$$

$\therefore y = e^{1/3}.$

Hence $\lim\limits_{x\to 0}\left(\dfrac{\tan x}{x}\right)^{\frac{1}{x^2}} = e^{1/3}$.Ans.

Q.No.8.: Evaluate $\lim\limits_{x\to 0}(\cot x)^{\sin x}$.

Sol.: Let $y = \lim\limits_{x\to 0}(\cot x)^{\sin x}$. $[\infty^0 \text{ form}]$

Taking log on both sides, we get

$$\log y = \lim_{x\to 0}\log(\cot x)^{\sin x} = \lim_{x\to 0}\sin x\log(\cot x) = \lim_{x\to 0}\frac{\log(\cot x)}{\operatorname{cosec} x}.$$

$$\left[\frac{\infty}{\infty}\text{form}\right]$$

Apply Cauchy's rule, we get

$$\log y = \lim_{x\to 0}\frac{\frac{1}{\cot x}\operatorname{cosec}^2 x}{\operatorname{cosec} x\cot x} = \lim_{x\to 0}\frac{\operatorname{cosec} x}{\cot^2 x} = \lim_{x\to 0}\frac{\sin^2 x}{\sin x\cos^2 x} = \lim_{x\to 0}\frac{\sin x}{\cos^2 x} = 0.$$

$\therefore y = e^0 = 1$.

Hence $\lim\limits_{x\to 0}(\cot x)^{\sin x} = 1$.Ans.

Q.No.9.: Evaluate $\lim\limits_{x\to 0}\left(\dfrac{a^x + b^x}{2}\right)^{\frac{1}{x}}$.

Sol.: Let $y = \lim\limits_{x\to 0}\left(\dfrac{a^x + b^x}{2}\right)^{\frac{1}{x}}$. $[1^\infty \text{ form}]$

Taking log on both sides, we get

$$\log y = \lim_{x\to 0}\log\left(\frac{a^x + b^x}{2}\right)^{\frac{1}{x}} = \lim_{x\to 0}\frac{1}{x}\log\left(\frac{a^x + b^x}{2}\right). \qquad \left[\frac{0}{0}\text{form}\right]$$

Apply Cauchy's rule, we get

$$\log y = \lim_{x\to 0}\frac{2}{a^x + b^x}\left(\frac{a^x\log x + b^x\log x}{2}\right) = \frac{\log a + \log b}{2} = \frac{1}{2}\log(ab) = \log(ab)^{1/2}.$$

$\therefore y = e^{\log(ab)^{1/2}} = \sqrt{ab}$.

Hence $\lim\limits_{x\to 0}\left(\dfrac{a^x+b^x}{2}\right)^{\frac{1}{x}} = \sqrt{ab}$. Ans.

Q.No.10.: Evaluate $\lim\limits_{x\to 0}\left(\dfrac{\tan x}{x}\right)^{\frac{1}{x}}$.

Sol.: Let $y = \lim\limits_{x\to 0}\left(\dfrac{\tan x}{x}\right)^{\frac{1}{x}}$. $[1^\infty$ form]

Taking log on both sides, we get

$$\log y = \lim_{x\to 0}\log\left(\frac{\tan x}{x}\right)^{\frac{1}{x}} = \lim_{x\to 0}\frac{1}{x}\log\left(\frac{\tan x}{x}\right) = \lim_{x\to 0}\frac{\log\left(\frac{\tan x}{x}\right)}{x}. \qquad \left[\frac{0}{0}\text{form}\right]$$

Apply Cauchy's rule, we get

$$\log y = \lim_{x\to 0}\frac{\left[\frac{x}{\tan x}.\frac{x\sec^2 x-\tan x}{x^2}\right]}{2} = \lim_{x\to 0}\left(\frac{x}{\tan x}.\frac{x\sec^2 x-\tan x}{2x^2}\right)$$

$$= \lim_{x\to 0}\frac{x\sec^2 x-\tan x}{2x^2}. \qquad \left[\frac{0}{0}\text{form}\right] \qquad \left[\because \lim_{x\to 0}\left(\frac{x}{\tan x}\right)=1\right]$$

Again, apply Cauchy's rule, we get

$$\log y = \lim_{x\to 0}\frac{\sec^2 x + x.2\sec x\sec x\tan x - \sec^2 x}{4x} = \lim_{x\to 0}\frac{2x\sec^2 x\tan x}{4x}.$$

$$= \lim_{x\to 0}\frac{x\sec^2 x}{2}.\frac{\tan x}{x} = \frac{0.\sec^2 0}{2} = 0. \qquad \left[\because \lim_{x\to 0}\left(\frac{\tan x}{x}\right)=1\right]$$

$\therefore y = e^0 = 1$.

Hence $\lim\limits_{x\to 0}\left(\dfrac{\tan x}{x}\right)^{\frac{1}{x}} = 1$ .Ans.

Q.No.11.: Evaluate $\lim\limits_{x\to 0}\left(\dfrac{\sinh x}{x}\right)^{\frac{1}{x^2}}$.

Sol.: Let $y = \lim\limits_{x\to 0}\left(\dfrac{\sinh x}{x}\right)^{\frac{1}{x^2}}$. [$1^\infty$ form]

Taking log on both sides, we get

$$\log y = \lim_{x\to 0}\log\left(\frac{\sinh x}{x}\right)^{\frac{1}{x^2}} = \lim_{x\to 0}\frac{1}{x^2}\log\left(\frac{\sinh x}{x}\right) = \lim_{x\to 0}\frac{\log\left(\frac{\sinh x}{x}\right)}{x^2}.$$

$\left[\dfrac{0}{0}\text{form}\right]$

Apply Cauchy's rule, we get

$$\log y = \lim_{x\to 0}\frac{\left[\frac{x}{\sinh x}.\frac{x\cosh x-\sinh x}{x^2}\right]}{2x} = \lim_{x\to 0}\left(\frac{x}{\sinh x}.\frac{x\cosh x-\sinh x}{2x^3}\right)$$

$$= \lim_{x\to 0}\frac{x\cosh x-\sinh x}{2x^3}. \qquad \left[\frac{0}{0}\text{form}\right] \qquad \left[\because \lim_{x\to 0}\left(\frac{x}{\sinh x}\right)=1\right]$$

Again, apply Cauchy's rule, we get

$$\log y = \lim_{x\to 0}\frac{x\sinh x+\cosh x-\cosh x}{6x^2} = \lim_{x\to 0}\frac{x\sinh x}{6x^2} = \lim_{x\to 0}\frac{1}{6}\frac{\sinh x}{x} = \frac{1}{6}.$$

$\therefore y = e^{1/6}$.

Hence $\lim\limits_{x\to 0}\left(\dfrac{\sinh x}{x}\right)^{\frac{1}{x^2}} = e^{1/6}$.Ans.

Q.No.12.: Evaluate $\lim\limits_{x\to 0}\dfrac{1-x^x}{x\log x}$.

Sol.: Let $y = \lim\limits_{x\to 0}\dfrac{1-x^x}{x\log x}$ $\left[\dfrac{0}{0}\text{ form}\right]$

Applying Cauchy's Rule, we get

$$y = \lim_{x\to 0}\frac{-x^x(1+\log x)}{(1+\log x)} = \lim_{x\to 0} -x^x$$

Taking log of both sides, we get

$$\log y = \lim_{x\to 0} -x\log x = \lim_{x\to 0} -\frac{\log x}{\frac{1}{x}} = \lim_{x\to 0}\frac{-\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0} x = 0$$

$\log y = 0$

$\therefore y = 1$. Ans.

Q.No.13.: Prove that $\lim_{x\to a}\left(2-\frac{x}{a}\right)^{\tan\frac{\pi x}{2a}} = e^{2/\pi}$.

Sol.: Let $y = \lim_{x\to a}\left(2-\frac{x}{a}\right)^{\tan\frac{\pi x}{2a}}$. $[1^\infty$ form$]$

Taking log of both sides, we get

$$\log y = \lim_{x\to a}\tan\left(\frac{\pi x}{2a}\right)\log\left(2-\frac{x}{a}\right) \qquad [\infty\times 0 \text{ form}]$$

$$= \lim_{x\to a}\frac{\log\left(2-\frac{x}{a}\right)}{\cot\left(\frac{\pi x}{2a}\right)} \qquad \left[\frac{0}{0}\text{ form}\right]$$

Apply Cauchy's rule, we get

$$\log y = \lim_{x\to a}\frac{\frac{1}{\left(2-\frac{x}{a}\right)}\cdot\left(-\frac{1}{a}\right)}{-\frac{\pi}{2a}\operatorname{cosec}^2\left(\frac{\pi x}{2a}\right)} = \frac{2}{\pi}.$$

$\therefore y = e^{2/\pi}$.

Hence $\lim_{x\to a}\left(2-\frac{x}{a}\right)^{\tan\frac{\pi x}{2a}} = e^{2/\pi}$.Ans

Q.No.14.: Prove that $\operatorname{Lt}_{x\to 2}\left(8-x^3\right)^{\frac{1}{\log(2-x)}} = e$.

Sol.: Let $y = \operatorname{Lt}_{x\to 2}\left(8-x^3\right)^{\frac{1}{\log(2-x)}}$

Taking log on both sides, we get

$$\log y = \underset{x\to 2}{\text{Lt}} \log\left(8-x^3\right)^{\frac{1}{\log(2-x)}} = \underset{x\to 2}{\text{Lim}} \frac{\log\left(\sqrt{8}-x^3\right)}{\log\left(\sqrt{2}-x\right)}$$

Apply Cauchy's rule, we get

$$\log y = \underset{x\to 2}{\text{Lim}} \frac{\dfrac{-3x^2}{8-x^3}}{\dfrac{-1}{2-x}} = \underset{x\to 2}{\text{Lim}} \frac{3x^2}{4+3x+x^2} = \frac{12}{12} = 1$$

$\log y = 1$

$\therefore y = e$. Ans.

Q.No.15.: Prove that $\underset{x\to 0}{\text{Lt}} \left[\dfrac{2(\cosh x - 1)}{x^2}\right]^{\frac{1}{x^2}} = e^{\frac{1}{12}}$.

Sol.: Let $y = \underset{x\to 0}{\text{Lt}} \left[\dfrac{2(\cosh x - 1)}{x^2}\right]^{\frac{1}{x^2}}$

$$\Rightarrow y = \underset{x\to 0}{\text{Lt}} \left[\frac{2}{x^2}\left(1+\frac{x^2}{2}+\frac{x^4}{4}+\text{..........}-1\right)\right]^{\frac{1}{x^2}}$$

Taking log on both sides, we get

$$\log y = \frac{\log\left(1+\dfrac{x^2}{12}+\text{...........}\infty\right)}{x^2} = \underset{x\to c}{\text{Lim}} \frac{\left(\dfrac{x}{6}+\dfrac{x^3}{3}+\text{........}\right)}{2x\left(1+\dfrac{x^2}{12}+\dfrac{x^4}{360}\text{.......}\right)}$$

$\log y = \dfrac{1}{12}$.

$y = e^{\frac{1}{12}}$. Ans.

Q.No.16.: Prove that $\underset{N\to\infty}{\text{Lt}} \left[\cos\dfrac{\beta}{N}\right]^{N^2} = e^{-\frac{1}{2}\beta^2}$.

Sol.: Let $y = \underset{N\to\infty}{\text{Lt}} \left[\cos\dfrac{\beta}{N}\right]^{N^2}$

$$y=\lim_{N\to\infty}\left[1-\frac{\beta^2}{2!N^2}+\frac{\beta^4}{4!N^4}+..........\right]^{N^2}$$

Taking log on both sides, we get

$$\log y=\lim_{N\to\infty}N^2\log\left[1-\frac{\beta^2}{2!N^2}+\frac{\beta^4}{4!N^4}+..........\right]$$

$$\Rightarrow \log y=\lim_{N\to\infty}N^2\log\left[1-\left(\frac{\beta^2}{2!N^2}-\frac{\beta^4}{4!N^4}+..........\right)\right]$$

$$\Rightarrow \log y=-\lim_{N\to\infty}N^2\left[\left(\frac{\beta^2}{2!N^2}-\frac{\beta^4}{4!N^4}+..........\right)+\frac{1}{2}\left(\frac{\beta^2}{2!N^2}-\frac{\beta^4}{4!N^4}+..........\right)^2+.........\right]$$

$$\Rightarrow \log y=-\lim_{N\to\infty}N^2\left[\left(\frac{\beta^2}{2!N^2}-\frac{\beta^4}{4!N^4}+..........\right)+\frac{\beta^4}{8N^4}\left(1-\frac{\beta^2}{12N^2}+..........\right)^2+.........\right]$$

$$\Rightarrow \log y=-\lim_{N\to\infty}N^2\left[\left(\frac{\beta^2}{2!N^2}-\frac{\beta^4}{4!N^4}+..........\right)+\frac{\beta^4}{8N^4}\left(1-\frac{2\beta^2}{12N^2}+..........\right)+.........\right]$$

$$\Rightarrow \log y=-\lim_{N\to\infty}N^2\left[\frac{\beta^2}{2!N^2}-\frac{1\beta^4}{12N^4}+..................\right]$$

$$\Rightarrow \log y=-\lim_{N\to\infty}\left[\frac{\beta^2}{2!}-\frac{1\beta^4}{12N^2}+..................\right]\Rightarrow \log y=-\frac{\beta^2}{2}$$

$$\therefore\ y=e^{-\frac{\beta^2}{2}}\ \textbf{. Ans.}$$

Q.No.17.: Prove that $\underset{x\to\infty}{\mathrm{Lt}}\left(\frac{ax+1}{ax-1}\right)^x=e^{\frac{2}{a}}$**.**

Sol.: Let $y=\underset{x\to\infty}{\mathrm{Lt}}\left(\frac{ax+1}{ax-1}\right)^x$ [1^∞ form]

Taking log on both sides, we get

$$\log y=\underset{x\to\infty}{\mathrm{Lt}}\ x\log\left(\frac{ax+1}{ax-1}\right)=\underset{x\to\infty}{\mathrm{Lt}}\frac{\log\left[1+\frac{1}{ax}\right]}{\frac{1}{x}\left[1-\frac{1}{ax}\right]}=\underset{x\to\infty}{\mathrm{Lt}}\frac{\log\left[1+\frac{1}{ax}\right]-\log\left[1-\frac{1}{ax}\right]}{\frac{1}{x}}$$

$$= \frac{\left[\frac{1}{ax} - \frac{1}{2a^2x^2} + \frac{1}{3a^3x^3} - \ldots\ldots\ldots\ldots\right] - \left[-\frac{1}{ax} - \frac{1}{2a^2x^2} - \frac{1}{3a^3x^3}\ldots\ldots\ldots\right]}{\frac{1}{x}}$$

$$\log y = \lim_{x\to\infty} \frac{2}{a}\left[1 + \frac{1}{3a^2x^2} + \ldots\ldots\right]$$

$$\log y = \frac{2}{a}.$$

$\therefore y = e^{\frac{2}{a}}$. Ans.

Q.No.18.: Evaluate $\operatorname*{Lt}_{x\to\infty}\left[\frac{1}{2}\left(\sqrt{\frac{a}{x}} + \sqrt{\frac{x}{a}}\right)\right]^{\frac{1}{x-a}}$.

Sol.: Let $y = \operatorname*{Lt}_{x\to\infty}\left[\frac{1}{2}\left(\sqrt{\frac{a}{x}} + \sqrt{\frac{x}{a}}\right)\right]^{\frac{1}{x-a}}$

Taking log on both sides, we get

$$\log y = \frac{\log\left[\frac{1}{2}\left(\sqrt{\frac{a}{x}} + \sqrt{\frac{x}{a}}\right)\right]}{x-a} = \frac{2\sqrt{xa}(x-a)}{a+x}\left[\sqrt{a}x^{-\frac{3}{2}} + \frac{x^{-\frac{1}{2}}}{\sqrt{a}}\right] = \frac{2a}{2a}\times 0$$

$\log y = 0$.

$\therefore y = 1$. Ans.

Q.No.19.: Prove that $\operatorname*{Lt}_{x\to 0}\frac{1-x^{\sin x}}{x\log x} = -1$.

Sol.: Let $y = \operatorname*{Lt}_{x\to 0}\frac{1-x^{\sin x}}{x\log x}$

$$y = \operatorname*{Lt}_{x\to 0}\frac{1-x^{\sin x}.x^x}{x\log x} = \lim_{x\to 0}\frac{1-x^x}{x\log x} = \operatorname*{Lt}_{x\to 0}\frac{-x^x(1+\log x)}{1+\log x}$$

$$y = \lim_{x\to 0} -x^x$$

Taking log on both sides, we get

$$\log y = \lim_{x\to 0} - x\log x \ = \lim_{x\to 0}\frac{-\log x}{\frac{1}{x}} \qquad \left[\frac{\infty}{\infty}\text{ form}\right]$$

Applying Cauchy's rule, we get

$$\log y = \frac{\frac{-1}{x}}{\frac{-1}{x^2}} = x = 0.$$

$\therefore\ y = 1$. Ans.

Q.No.20.: Evaluate $\underset{m\to\infty}{\mathrm{Lt}}\left(\cos\frac{x}{m}\right)^m$.

Sol.: Let $y = \underset{m\to\infty}{\mathrm{Lt}}\left(\cos\frac{x}{m}\right)^m$

Taking log on both sides, we get

$$\log y = \underset{m\to\infty}{\mathrm{Lt}}\ m\log\left(\cos\frac{x}{m}\right) = \underset{m\to\infty}{\mathrm{Lt}}\frac{\log\left(\cos\frac{x}{m}\right)}{1/m}$$

$\left[\frac{0}{0}\text{ form}\right]$

$$\log y = \underset{m\to\infty}{\mathrm{Lt}}\frac{\frac{1}{\cos\frac{x}{m}}\left(-\sin\frac{x}{m}\right)\left(-\frac{1}{m^2}\right)}{-\frac{1}{m^2}} = \underset{m\to\infty}{\mathrm{Lt}}\left(-\tan\frac{x}{m}\right) = 0$$

$\therefore\ y = e^0 = 1$. Ans.

Q.No.21.: Evaluate $\underset{x\to 0}{\mathrm{Lt}}\left(\frac{1}{x}\right)^{1-\cos x}$.

Sol.: Let $y = \underset{x\to 0}{\mathrm{Lt}}\left(\frac{1}{x}\right)^{1-\cos x}$

Taking log on both sides, we get

$$\log y = \lim_{x\to 0}(\cos x - 1)\log x = \lim_{x\to 0}\frac{(\cos x - 1)}{x}(x\log x) = \lim_{x\to 0}\frac{(\cos x - 1)}{x} \quad \left[\frac{0}{0}\text{ form}\right]$$

Apply Cauchy's rule, we get

$$\log y = \lim_{x\to 0}\frac{-\sin x}{1} = 0$$

$\log y = 0 \therefore y = e^0 = 1$. Ans.

Q.No.22.: Evaluate $\lim\limits_{x\to\frac{\pi}{2}}(\tan x)^{\cos x}$.

Sol.: Let $y = \lim\limits_{x\to\frac{\pi}{2}}(\tan x)^{\cos x}$ $\quad [\infty^0 \text{ form}]$

Taking log on both sides, we get

$$\log y = \lim_{x\to\frac{\pi}{2}}\cos x \log(\tan x) \qquad [0\times\infty \text{ form}]$$

$$= \lim_{x\to\frac{\pi}{2}}\frac{\log(\tan x)}{\sec x} \qquad \left[\frac{\infty}{\infty}\text{form}\right]$$

Apply Cauchy's rule, we get

$$= \lim_{x\to\frac{\pi}{2}}\frac{\frac{1}{\tan x}.\sec^2 x}{\sec x.\tan x} = \lim_{x\to\frac{\pi}{2}}\frac{\cos x}{\sin^2 x} = 0.$$

$\therefore y = e^0 = 1$. Ans.

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Differential Calculus

Partial Differentiation

[Transformation of independent variables (Composite Functions), Jacobian, Properties of Jacobians]

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Composite function:

If $u = f(x_1, x_2, x_3, \ldots\ldots\ldots\ldots)$ and the independent variables $x_1, x_2, x_3, \ldots\ldots\ldots$ are further functions of other variables $t_1, t_2, t_3, \ldots\ldots\ldots$.

by the relations, $x_1 = \phi(t_1, t_2, t_3, \ldots\ldots\ldots\ldots\ldots)$, $x_2 = \psi(t_1, t_2, t_3, \ldots\ldots\ldots\ldots)$ etc.

Then u is said to be a **composite function** of the variables $t_1, t_2, t_3, \ldots\ldots\ldots$,

For example if u to be function of x, y, i.e. $u = f(x, y)$ and further if x ,y are function of t_1, t_2, i.e. $x = \phi(t_1, t_2)$ and if $y = \psi(t_1, t_2,)$.

Then u is a **composite function** of variables t_1, t_2,

Transformation of independent variables:

Now the necessary formulae for changing of independent variables are obtained:

$$\frac{\partial u}{\partial t_1} = \frac{\partial u}{\partial x}.\frac{\partial x}{\partial t_1} + \frac{\partial u}{\partial y}.\frac{\partial y}{\partial t_1}, \quad \frac{\partial u}{\partial t_2} = \frac{\partial u}{\partial x}.\frac{\partial x}{\partial t_2} + \frac{\partial u}{\partial y}.\frac{\partial y}{\partial t_2}, \ldots\ldots\ldots\ldots$$

Further, if $u = f(x, y)$ and if $t_1 = f_1(x, y)$ and $t_2 = f_2(x, y)$.

Then the transformation equations are

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t_1}.\frac{\partial t_1}{\partial x} + \frac{\partial u}{\partial t_2}.\frac{\partial t_2}{\partial x},$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial t_1}.\frac{\partial t_1}{\partial y}+\frac{\partial u}{\partial t_2}.\frac{\partial t_2}{\partial y}.$$

Expansion:

Extending the above results, we may obtain.

In case $u=f(x,y,z)$ and $x=\varphi_1(t_1,t_2,t_3)$, $y=\varphi_2(t_1,t_2,t_3)$, $z=\varphi_3(t_1,t_2,t_3)$.

Then the transformation equations are

$$\frac{\partial u}{\partial t_1}=\frac{\partial u}{\partial x}.\frac{\partial x}{\partial t_1}+\frac{\partial u}{\partial y}.\frac{\partial y}{\partial t_1}+\frac{\partial u}{\partial z}.\frac{\partial z}{\partial t_1},$$

$$\frac{\partial u}{\partial t_2}=\frac{\partial u}{\partial x}.\frac{\partial x}{\partial t_2}+\frac{\partial u}{\partial y}.\frac{\partial y}{\partial t_2}+\frac{\partial u}{\partial z}.\frac{\partial z}{\partial t_2},$$

$$\frac{\partial u}{\partial t_3}=\frac{\partial u}{\partial x}.\frac{\partial x}{\partial t_3}+\frac{\partial u}{\partial y}.\frac{\partial y}{\partial t_3}+\frac{\partial u}{\partial z}.\frac{\partial z}{\partial t_3}.$$

Further, if $u=f(x,y,z)$, $t_1=f_1(x,y,z)$ $t_2=f_2(x,y,z)$ and $t_3=f_3(x,y,z)$. Then

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial t_1}.\frac{\partial t_1}{\partial x}+\frac{\partial u}{\partial t_2}.\frac{\partial t_2}{\partial x}+\frac{\partial u}{\partial t_3}.\frac{\partial t_3}{\partial x},$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial t_1}.\frac{\partial t_1}{\partial y}+\frac{\partial u}{\partial t_2}.\frac{\partial t_2}{\partial y}+\frac{\partial u}{\partial t_3}.\frac{\partial t_3}{\partial y},$$

$$\frac{\partial u}{\partial z}=\frac{\partial u}{\partial t_1}.\frac{\partial t_1}{\partial z}+\frac{\partial u}{\partial t_2}.\frac{\partial t_2}{\partial z}+\frac{\partial u}{\partial t_3}.\frac{\partial t_3}{\partial z}.$$

Jacobian:

Definition: If u and v are functions of two independent variables x and y, then the determinant $\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \frac{\partial u}{\partial x}.\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}.\frac{\partial v}{\partial x}$,

is called the functional determinant or **Jacobian** of u, v with respect to x, y, and is denoted by the symbol $J\left(\frac{u, v}{x, y}\right)$ or $\frac{\partial(u,\ v)}{\partial(x,\ y)}$.

Similarly, if u, v, w are functions of three independent variables x, y, z, then the Jacobian of u, v, w with respect to x, y, z is $J\left(\frac{u, v, w}{x, y, z}\right) = \frac{\partial(u,\ v, w)}{\partial(x,\ y,\ z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$.

Properties of Jacobians:

I. If u, v are functions of r, s where r, s are functions of x, y

then $\frac{\partial(u, v)}{\partial(x, y)} = \frac{\partial(u, v)}{\partial(r, s)} \times \frac{\partial(r, s)}{\partial(x, y)}$.

Proof: Since u, v are composite functions of x, y

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}.\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}.\frac{\partial s}{\partial x} = u_r r_x + u_s s_x,$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}.\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}.\frac{\partial s}{\partial y} = u_r r_y + u_s s_y,$$

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial r}.\frac{\partial r}{\partial x} + \frac{\partial v}{\partial s}.\frac{\partial s}{\partial x} = v_r r_x + v_s s_x,$$

$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial r}.\frac{\partial r}{\partial y} + \frac{\partial v}{\partial s}.\frac{\partial s}{\partial y} = v_r r_y + v_s s_y.$$

Now $\frac{\partial(u, v)}{\partial(r, s)} \times \frac{\partial(r, s)}{\partial(x, y)} = \begin{vmatrix} u_r & u_s \\ v_r & v_s \end{vmatrix} . \begin{vmatrix} r_x & r_y \\ s_x & s_y \end{vmatrix}$.

Interchanging rows and columns in the second determinant, we get

$$\begin{vmatrix} u_r & u_s \\ v_r & v_s \end{vmatrix} . \begin{vmatrix} r_x & s_x \\ r_y & s_y \end{vmatrix} = \begin{vmatrix} u_r r_x + u_s s_x & u_r r_y + u_s s_y \\ v_r r_x + v_s s_x & v_r r_y + v_s s_y \end{vmatrix} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \frac{\partial(u, v)}{\partial(x, y)}.$$

II. If J_1 is the Jacobian of u, v, with respect to x, y and J_2 is the Jacobian of x, y, with respect to u, v, then $J_1 J_2 = 1$ i. e. $\frac{\partial(u, v)}{\partial(x, y)} \times \frac{\partial(x, y)}{\partial(u, v)} = 1$.

Proof: Let u = u(x, y) and v = v(x, y), so that u and v are functions of x, y.

Suppose on solving for x and y, we get x = $\phi(u, v)$ and y = $\psi(u, v)$.

$$\frac{\partial u}{\partial u} = 1 = \frac{\partial u}{\partial x}.\frac{\partial x}{\partial u} + \frac{\partial u}{\partial y}.\frac{\partial y}{\partial u} = u_x x_u + u_y y_u$$

$$\frac{\partial u}{\partial v} = 0 = \frac{\partial u}{\partial x}.\frac{\partial x}{\partial v} + \frac{\partial u}{\partial y}.\frac{\partial y}{\partial v} = u_x x_v + u_y y_v$$

$$\frac{\partial v}{\partial u} = 0 = \frac{\partial v}{\partial x}.\frac{\partial x}{\partial v} + \frac{\partial v}{\partial y}.\frac{\partial y}{\partial u} = v_x x_u + v_y y_u$$

$$\frac{\partial v}{\partial v} = 1 = \frac{\partial v}{\partial x}.\frac{\partial x}{\partial v} + \frac{\partial v}{\partial y}.\frac{\partial y}{\partial v} = v_x x_v + v_y y_v$$

Now $\frac{\partial(u, v)}{\partial(x, y)} \times \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} . \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$.

Interchanging rows and columns in the second determinant, we get

$$\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} . \begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix} = \begin{vmatrix} u_x x_u + u_y y_u & u_x x_v + u_y y_v \\ v_x x_u + v_y y_u & v_x x_v + v_y y_v \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1.$$

Now let us solve some more problems:

Q.No.1.: If $r = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1}\frac{y}{x}$, evaluate $\frac{\partial(r, \theta)}{\partial(x, y)}$.

Sol.: Given $r = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1}\frac{y}{x}$.

Now $\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}}$, $\frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}}$

$$\frac{\partial\theta}{\partial x}=\frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right)=-\frac{y}{x^2+y^2},\quad \frac{\partial\theta}{\partial y}=\frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right)=\frac{x}{x^2+y^2}$$

$$\therefore \frac{\partial(r,\theta)}{\partial(x,y)}=\begin{vmatrix}\frac{\partial r}{\partial x} & \frac{\partial r}{\partial y}\\ \frac{\partial\theta}{\partial x} & \frac{\partial\theta}{\partial y}\end{vmatrix}=\begin{vmatrix}\frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}}\\ -\frac{y}{x^2+y^2} & \frac{x}{x^2+y^2}\end{vmatrix}$$

$$=\frac{x^2}{\left(x^2+y^2\right)^{3/2}}+\frac{y^2}{\left(x^2+y^2\right)^{3/2}}=\frac{x^2+y^2}{\left(x^2+y^2\right)^{3/2}}=\frac{1}{\sqrt{x^2+y^2}}.$$

Q.No.2.: If $x=r\sin\theta\cos\phi$, $y=r\sin\theta\sin\phi$, $z=r\cos\theta$,

show that $\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}=r^2\sin\theta$.

Sol.: $$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi}\\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial\theta} & \frac{\partial z}{\partial\phi}\end{vmatrix}=\begin{vmatrix}\sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi\\ \sin\theta\sin\phi & r\cos\theta\cos\phi & r\sin\theta\cos\phi\\ \cos\theta & -r\sin\theta & 0\end{vmatrix}$$

Taking out common factor (r from second column and $r\sin\theta$ from third column)

$$=r^2\sin\theta\begin{vmatrix}\sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi\\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi\\ \cos\theta & -\sin\theta & 0\end{vmatrix}$$

Expanding by third row

$$=r^2\sin\theta\left\{\cos\theta\begin{vmatrix}\cos\theta\cos\phi & -\sin\phi\\ \cos\theta\sin\phi & \cos\phi\end{vmatrix}+\sin\theta\begin{vmatrix}\sin\theta\cos\phi & -\sin\phi\\ \sin\theta\sin\phi & \cos\phi\end{vmatrix}\right\}$$

$$=r^2\sin\theta\left[\cos\theta\left(\cos\theta\cos^2\phi+\cos\theta\sin^2\phi\right)+\sin\theta\left(\sin\theta\cos^2\phi+\sin\theta\sin^2\phi\right)\right]$$

$$=r^2\sin\theta\left(\cos^2\theta+\sin^2\theta\right)=r^2\sin\theta$$

Q.No.3.: If $u=f(y-z,z-x,x-y)$, prove that $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0$.

Sol.: Suppose $u_1=y-z, u_2=z-x, u_3=x-y$. (i)

$\therefore u=f(y-z,z-x,x-y)$ becomes $u=f(u_1,u_2,u_3)$. (ii)

From (i) and (ii) we conclude that u is composite function of x, y, z.

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial u_1}.\frac{\partial u_1}{\partial x} + \frac{\partial u}{\partial u_2}.\frac{\partial u_2}{\partial x} + \frac{\partial u}{\partial u_3}.\frac{\partial u_3}{\partial x} \qquad \text{(iii)}$$

Now $\frac{\partial u_1}{\partial x} = 0$, $\frac{\partial u_2}{\partial x} = -1$, $\frac{\partial u_3}{\partial x} = 1$

$$\therefore \text{(iii) becomes } \frac{\partial u}{\partial x} = -\frac{\partial u}{\partial u_2} + \frac{\partial u}{\partial u_3}. \qquad \text{(iv)}$$

$$\text{Similarly } \frac{\partial u}{\partial y} = -\frac{\partial u}{\partial u_3} + \frac{\partial u}{\partial u_1}, \qquad \text{(v)}$$

$$\text{and } \frac{\partial u}{\partial z} = -\frac{\partial u}{\partial u_3} + \frac{\partial u}{\partial u_2}. \qquad \text{(vi)}$$

Adding (iv), (v) and (vi), we get

$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$, which is the required result**.**

Q.No.4.: If $w = f(x, y)$, $x = r\cos\theta$, $y = r\sin\theta$,

$$\text{show that } \left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2.$$

Sol.: The given equations define w as a composite function of r and θ.

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}.\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}.\frac{\partial y}{\partial r} = \frac{\partial w}{\partial x}.\cos\theta + \frac{\partial w}{\partial y}.\sin\theta$$

$$\Rightarrow \frac{\partial w}{\partial r} = \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta \qquad [\because w = f(x, y)] \qquad \text{(i)}$$

$$\text{Also } \frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}.\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}.\frac{\partial y}{\partial \theta} = \frac{\partial w}{\partial x}(-r\sin\theta) + \frac{\partial w}{\partial y}(r\cos\theta)$$

$$\Rightarrow \frac{1}{r}\frac{\partial w}{\partial \theta} = -\frac{\partial f}{\partial x}\sin\theta + \frac{\partial f}{\partial y}\cos\theta. \qquad \text{(ii)}$$

Squaring and adding (i) and (ii), we get

$$\left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2.$$

Q.No.5.: If $z = \sqrt{x^2 + y^2}$ and $x^3 + y^3 + 3axy = 5a^2$, find the value of $\frac{dz}{dx}$, when $x = y = a$.

Sol.: The given equation are of the form $z = f(x, y)$ and $\phi(x, y) = c$.

$\therefore$ z is the composite function of x.

$$\Rightarrow \frac{dz}{dx} = \frac{\partial z}{\partial x}.\frac{dx}{dx} + \frac{\partial z}{\partial y}.\frac{dy}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx}. \qquad \text{(i)}$$

Now $\frac{\partial z}{\partial x} = \frac{1}{2}\left(x^2 + y^2\right)^{-1/2}.2x = \frac{x}{\sqrt{x^2 + y^2}}$

Similarly, $\frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}}$

Also, differentiating $x^3 + y^3 + 3axy = 5a^2$, we get

$$3x^2 + 3y^2.\frac{dy}{dx} + 3ay + 3ax.\frac{dy}{dx} = 0 \Rightarrow \left(y^2 + ax\right)\frac{dy}{dx} = -\left(x^2 + ay\right)$$

$$\therefore \frac{dy}{dx} = -\frac{x^2 + ay}{y^2 + ax}$$

$\therefore$ From (i), we get $\frac{dz}{dx} = \frac{x}{\sqrt{x^2 + y^2}} + \frac{y}{\sqrt{x^2 + y^2}}\left(-\frac{x^2 + ay}{y^2 + ax}\right)$

$$\left[\frac{dz}{dx}\right]_{\substack{x=a\\y=a}} = \frac{a}{\sqrt{a^2 + a^2}} - \frac{a}{\sqrt{a^2 + a^2}}.\frac{a^2 + a^2}{a^2 + a^2} = 0.$$

Q.No.6.: If $u = xe^y z$, where $y = \sqrt{a^2 - x^2}$, $z = \sin^2 x$, find $\frac{du}{dx}$.

Sol.: Here u is a function of x, y and z while y and z are functions of x.

$$\therefore \frac{du}{dx} = \frac{\partial u}{\partial x}.\frac{dx}{dx} + \frac{\partial u}{\partial y}.\frac{dy}{dx} + \frac{\partial z}{\partial y}\frac{dz}{dx}$$

$$= e^y z.1 + xe^y z.\frac{1}{2}\left(a^2 - x^2\right)^{-1/2}(-2x) + xe^y.2\sin x\cos x$$

$$= e^y\left[z - \frac{x^2 z}{\sqrt{a^2 - x^2}} + x\sin 2x\right]. \text{ Ans.}$$

Q.No.7.: If $\varphi(x, y, z) = 0$, show that $\left(\frac{\partial y}{\partial z}\right)_x\left(\frac{\partial z}{\partial x}\right)_y\left(\frac{\partial x}{\partial y}\right)_z = -1$.

Sol.: The given relation defines y as a function of x and z. Treating x as constant

$$\left(\frac{\partial y}{\partial z}\right)_x = -\frac{\frac{\partial \varphi}{\partial z}}{\frac{\partial \varphi}{\partial y}}.$$

The given relation defines z as a function of x and y. Treating y as constant

$$\left(\frac{\partial z}{\partial x}\right)_y = -\frac{\frac{\partial \varphi}{\partial x}}{\frac{\partial \varphi}{\partial z}}.$$

Similarly, $\left(\frac{\partial x}{\partial y}\right)_y = -\frac{\frac{\partial \varphi}{\partial y}}{\frac{\partial \varphi}{\partial x}}.$

Multiplying, we get $\left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y \left(\frac{\partial x}{\partial y}\right)_z = -1$. Hence prove.

Q.No.8.: Prove that $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}$,

where $x = u\cos\alpha - v\sin\alpha$, $y = u\sin\alpha + v\cos\alpha$.

or

By changing the independent variables u and v to x by means of the relations $x = u\cos\alpha - v\sin\alpha$, $y = u\sin\alpha + v\cos\alpha$, show that $\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}$ transforms into $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}$.

Sol.: Here z is a composite function of u and v

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}.\frac{dx}{du} + \frac{\partial z}{\partial y}.\frac{dy}{du} = \cos\alpha\frac{\partial z}{\partial x} + \sin\alpha\frac{\partial z}{\partial y}$$

$$\Rightarrow \frac{\partial}{\partial u}(z) = \left(\cos\alpha\frac{\partial}{\partial x} + \sin\alpha\frac{\partial}{\partial y}\right)z \quad \Rightarrow \frac{\partial}{\partial u} = \cos\alpha\frac{\partial}{\partial x} + \sin\alpha\frac{\partial}{\partial y}. \qquad \text{(i)}$$

Also $\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}.\frac{dx}{dv} + \frac{\partial z}{\partial y}.\frac{dy}{dv} = -\sin\alpha\frac{\partial z}{\partial x} + \cos\alpha\frac{\partial z}{\partial y}$

$$\Rightarrow \frac{\partial}{\partial v}(z) = \left(-\sin\alpha\frac{\partial}{\partial x} + \cos\alpha\frac{\partial}{\partial y}\right)z \qquad \Rightarrow \frac{\partial}{\partial v} = -\sin\alpha\frac{\partial}{\partial x} + \sin\alpha\frac{\partial}{\partial y}. \qquad \text{(ii)}$$

Now we shall make use of the equivalence of operations as given by (i) and (ii)

$$\frac{\partial^2 z}{\partial u^2} = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right) = \left(\cos\alpha\frac{\partial}{\partial x} + \sin\alpha\frac{\partial}{\partial y}\right)\left(\cos\alpha\frac{\partial z}{\partial x} + \sin\alpha\frac{\partial z}{\partial y}\right)$$

$$= \cos^2\alpha\frac{\partial^2 z}{\partial x^2} + \cos\alpha\sin\alpha\frac{\partial^2 z}{\partial x\partial y} + \sin\alpha\cos\alpha\frac{\partial^2 z}{\partial y\partial x} + \sin^2\alpha\frac{\partial^2 z}{\partial y^2}$$

$$= \cos^2\alpha\frac{\partial^2 z}{\partial x^2} + 2\cos\alpha\sin\alpha\frac{\partial^2 z}{\partial x\partial y} + \sin^2\alpha\frac{\partial^2 z}{\partial y^2}\ . \qquad \text{(iii)}$$

$$\frac{\partial^2 z}{\partial v^2} = \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right) = \left(-\sin\alpha\frac{\partial}{\partial x} + \cos\alpha\frac{\partial}{\partial y}\right)\left(-\sin\alpha\frac{\partial z}{\partial x} + \cos\alpha\frac{\partial z}{\partial y}\right)$$

$$= \sin^2\alpha\frac{\partial^2 z}{\partial x^2} - \sin\alpha\cos\alpha\frac{\partial^2 z}{\partial x\partial y} - \cos\alpha\sin\alpha\frac{\partial^2 z}{\partial y\partial x} + \cos^2\alpha\frac{\partial^2 z}{\partial y^2}$$

$$= \sin^2\alpha\frac{\partial^2 z}{\partial x^2} - 2\cos\alpha\sin\alpha\frac{\partial^2 z}{\partial x\partial y} + \cos^2\alpha\frac{\partial^2 z}{\partial y^2}. \qquad \text{(iv)}$$

Adding (iii) and (iv), we get $\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} = \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}$. Hence prove.

Q.No.9: If $u = f(r,s)$, $r = x + y$, $s = x - y$, prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 2\frac{\partial u}{\partial r}$.

Sol.: Since $u = f(r,s)$ and r, s are the function of x and y.

$\therefore$ u is the composite function of x and y.

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}.\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}.\frac{\partial s}{\partial x} \Rightarrow \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} + \frac{\partial u}{\partial s} \qquad \left[\because \frac{\partial r}{\partial x} = 1 \text{ and } \frac{\partial s}{\partial x} = 1\right] \qquad \text{(i)}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}.\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}.\frac{\partial s}{\partial y} \Rightarrow \frac{\partial u}{\partial y} = \frac{\partial u}{\partial r} - \frac{\partial u}{\partial s} \qquad \left[\because \frac{\partial r}{\partial y} = 1 \text{ and } \frac{\partial s}{\partial y} = -1\right] \qquad \text{(ii)}$$

Now by adding (i) and (ii), we get

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = \frac{\partial u}{\partial r} + \frac{\partial u}{\partial s} + \frac{\partial u}{\partial r} - \frac{\partial u}{\partial s}$$

$$\Rightarrow \frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=2.\frac{\partial u}{\partial r}$$

Hence this proves the result**.**

Q.No.10: If $u=f(x,y)$, $x=r\cos\theta$, $y=r\sin\theta$, then

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2=\left(\frac{\partial u}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2.$$

Sol.: Here u is a composite function of r and θ

So we have

$$\frac{\partial u}{\partial r}=\frac{\partial u}{\partial x}.\frac{\partial x}{\partial r}+\frac{\partial u}{\partial y}.\frac{\partial y}{\partial r}\quad \text{since } \frac{\partial x}{\partial r}=\cos\theta,\ \frac{\partial y}{\partial r}=\sin\theta$$

$$=\frac{\partial u}{\partial x}\cos\theta+\frac{\partial u}{\partial y}\sin\theta$$

By squaring, we get

$$\left(\frac{\partial u}{\partial r}\right)^2=\left(\frac{\partial u}{\partial x}\right)^2\cos^2\theta+\left(\frac{\partial u}{\partial y}\right)^2\sin^2\theta+2\left(\frac{\partial u}{\partial x}\right).\left(\frac{\partial u}{\partial y}\right)\cos\theta.\sin\theta. \qquad \text{(i)}$$

Similarly we can get

$$\frac{\partial u}{\partial \theta}=\frac{\partial u}{\partial x}.\frac{\partial x}{\partial \theta}+\frac{\partial u}{\partial y}.\frac{\partial u}{\partial \theta}\quad \text{since } \frac{\partial x}{\partial \theta}=-r\sin\theta,\ \frac{\partial y}{\partial \theta}=r\cos\theta$$

$$=-r\frac{\partial u}{\partial x}.\sin\theta+r\frac{\partial u}{\partial y}.\cos\theta$$

By squaring, we get

$$\left(\frac{\partial u}{\partial \theta}\right)^2=\left[-r^2\left(\frac{\partial u}{\partial x}\right)^2.\sin^2\theta+\left(\frac{\partial u}{\partial y}\right)^2.\cos^2\theta-2\left(\frac{\partial u}{\partial x}\right).\left(\frac{\partial u}{\partial y}\right)\sin\theta\cos\theta\right]$$

$$\Rightarrow \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2=\left(\frac{\partial u}{\partial x}\right)^2\sin^2\theta+\left(\frac{\partial u}{\partial y}\right)^2\cos^2\theta-2\left(\frac{\partial u}{\partial x}\right).\left(\frac{\partial u}{\partial y}\right)\sin\theta\cos\theta. \qquad \text{(ii)}$$

Now by adding (i) and (ii), we get

$$\left(\frac{\partial u}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2=\left(\frac{\partial u}{\partial x}\right)^2\left(\cos^2\theta+\sin^2\theta\right)+\left(\frac{\partial u}{\partial y}\right)^2\left(\cos^2\theta+\sin^2\theta\right)$$

$$\Rightarrow\left(\frac{\partial u}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2=\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2.$$

Hence this proves the result.

Q.No.11: If z be a function of x and y, and u and v be two other variables, such that

$u=\ell x+my$, $v=\ell y-mx$. Show that

$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=\left(\ell^2+m^2\right)\left(\frac{\partial^2 z}{\partial u^2}+\frac{\partial^2 z}{\partial v^2}\right)$, assuming that z is a function of u and v.

Sol.: Let us assume that z is a function of u and v.

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}.\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}.\frac{\partial v}{\partial x}=\frac{\partial z}{\partial u}.\ell+\frac{\partial z}{\partial v}.(-m)=\ell\frac{\partial z}{\partial u}-m\frac{\partial z}{\partial v}$$

Let $\frac{\partial z}{\partial x}=f$. Since f is a composite function of x and y. Noting that f is also a function of u and v.

$$\Rightarrow\frac{\partial f}{\partial x}=\frac{\partial f}{\partial u}.\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}.\frac{\partial v}{\partial x}$$

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial u}.\left(\frac{\partial z}{\partial x}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial x}\right).\frac{\partial v}{\partial x}\qquad\left[\because\text{ By putting } f=\frac{\partial z}{\partial x}\right]$$

$$=\frac{\partial}{\partial u}.\left(\ell\frac{\partial z}{\partial u}-m\frac{\partial z}{\partial v}\right).\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\ell\frac{\partial z}{\partial u}-m\frac{\partial z}{\partial v}\right).\frac{\partial v}{\partial x}$$

$$\frac{\partial^2 z}{\partial x^2}=\left(\ell\frac{\partial^2 z}{\partial u^2}-m\frac{\partial^2 z}{\partial u\partial v}\right)\frac{\partial u}{\partial x}+\left(\ell\frac{\partial^2 z}{\partial v\partial u}-m\frac{\partial^2 z}{\partial v^2}\right).\frac{\partial v}{\partial x}\ .\qquad\text{(i)}$$

Similarly $\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}.\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}.\frac{\partial v}{\partial y}=m\frac{\partial z}{\partial u}+\ell\frac{\partial z}{\partial v}$

$$\frac{\partial^2 z}{\partial y^2}=\frac{\partial}{\partial u}.\left(\frac{\partial z}{\partial y}\right).\frac{\partial u}{\partial y}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial y}\right).\frac{\partial v}{\partial y}$$

$$\frac{\partial^2 z}{\partial y^2}=.\left(m\frac{\partial^2 z}{\partial u^2}+\ell\frac{\partial^2 z}{\partial u\partial v}\right)\frac{\partial u}{\partial y}+\left(m\frac{\partial^2 z}{\partial u\partial v}+\ell\frac{\partial^2 z}{\partial v^2}\right).\frac{\partial v}{\partial y}\ .\qquad\text{(ii)}$$

By adding (i) and (ii) we get,

$$\left(\frac{\partial^2 z}{\partial x^2}\right)+\left(\frac{\partial^2 z}{\partial y^2}\right)=\frac{\partial u}{\partial x}\left(\ell\frac{\partial^2 z}{\partial u^2}-m\frac{\partial^2 z}{\partial u\partial v}\right)+\left(\frac{\partial v}{\partial x}\right)\left(\ell\frac{\partial^2 z}{\partial u\partial v}-m\frac{\partial^2 z}{\partial v^2}\right)$$

$$+\left(\frac{\partial u}{\partial y}\right)\left(m\frac{\partial^2 z}{\partial v^2}+\ell\frac{\partial^2 z}{\partial u\partial v}\right)+\left(\frac{\partial v}{\partial y}\right)\left(m\frac{\partial^2 z}{\partial u\partial v}+\ell\frac{\partial^2 z}{\partial v^2}\right)$$

$$\Rightarrow \frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=\ell^2\frac{\partial^2 z}{\partial u^2}-\ell m\frac{\partial^2 z}{\partial u\partial v}-\ell m\frac{\partial^2 z}{\partial u\partial v}+m^2\frac{\partial^2 z}{\partial v^2}$$

$$+m^2\frac{\partial^2 z}{\partial v^2}+\ell m\frac{\partial^2 z}{\partial u\partial v}+\ell m\frac{\partial^2 z}{\partial u\partial v}+\ell^2\frac{\partial^2 z}{\partial v^2}$$

$$\Rightarrow \frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=\left(\ell^2+m^2\right)\left(\frac{\partial^2 z}{\partial u^2}+\frac{\partial^2 z}{\partial v^2}\right).$$

Hence this proves the result**.**

Q.No.12: If $z=f(u,v)$ and $u=x^2-2xy-y^2$ and $v=y$. Show that

$$(x+y)\frac{\partial z}{\partial x}+(x-y)\frac{\partial z}{\partial y}=(x-y)\frac{\partial z}{\partial v}.$$

Sol.: Clearly z is a composite function of x and y

$$\therefore \frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}.\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}.\frac{\partial v}{\partial x}\Rightarrow \frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}(2x-2y)+\frac{\partial z}{\partial v}(0)$$

$$\Rightarrow \frac{\partial z}{\partial x}=2(x-y)\frac{\partial z}{\partial u}. \qquad \text{(i)}$$

Also

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}.\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}.\frac{\partial v}{\partial y}$$

$$\Rightarrow \frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}(-2x-2y)+\frac{\partial z}{\partial v}(1)$$

$$\Rightarrow \frac{\partial z}{\partial y}=-2(x+y)\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}. \qquad \text{(ii)}$$

Taking L.H.S., we get

$$(x+y)\frac{\partial z}{\partial x}+(x-y)\frac{\partial z}{\partial y}=\left[(x+y)2(x-y)\frac{\partial z}{\partial u}\right]+\left[(x-y)\left\{(-2)(x+y)\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right\}\right]$$

$$=2(x+y)(x-y)\frac{\partial z}{\partial u}-2(x-y)(x+y)\frac{\partial z}{\partial u}+(x-y)\frac{\partial z}{\partial v}$$

$$=(x-y)\frac{\partial z}{\partial v}=\text{R.H.S.}$$

Hence this proves the result**.**

Q.No.13: Transform the equation $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ into polar co-ordinates**.**

Sol.: The relations connecting Cartesian co-ordinates (x, y) with polar co-ordinates (r,θ) are $x = r\cos\theta$, $y = r\sin\theta$.

Squaring and adding, we get $r^2 = x^2 + y^2$.

Dividing, we get $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$$\therefore\ r=\sqrt{x^2+y^2} \text{ and } \theta=\tan^{-1}\left(\frac{y}{x}\right)$$

$$\frac{\partial r}{\partial x}=\frac{x}{\sqrt{\left(x^2+y^2\right)}}=\frac{r\cos\theta}{r}=\cos\theta \quad \text{and}$$

$$\frac{\partial\theta}{\partial x}=\frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right)=-\frac{y}{\sqrt{\left(x^2+y^2\right)^2}}=-\frac{y}{x^2+y^2}=-\frac{r\sin\theta}{r^2}=-\frac{\sin\theta}{r}$$

$$\text{Now } \frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}.\frac{\partial r}{\partial x}+\frac{\partial u}{\partial\theta}.\frac{\partial\theta}{\partial x}=\frac{\partial u}{\partial r}.\cos\theta-\frac{\sin\theta}{r}.\frac{\partial u}{\partial\theta}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}.\left(\frac{\partial u}{\partial x}\right)=\frac{\partial f}{\partial x}, \text{ where } f=\frac{\partial u}{\partial x}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2}=\cos\theta\frac{\partial f}{\partial r}-\frac{\sin\theta}{r}.\frac{\partial f}{\partial\theta}=\cos\theta\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial x}\right)-\frac{\sin\theta}{r}.\frac{\partial}{\partial\theta}\left(\frac{\partial u}{\partial x}\right)$$

$$=\cos\theta\frac{\partial}{\partial r}\left(\cos\theta\frac{\partial u}{\partial r}-\frac{\sin\theta}{r}.\frac{\partial u}{\partial\theta}\right)-\frac{\sin\theta}{r}.\frac{\partial}{\partial\theta}\left(\cos\theta\frac{\partial u}{\partial r}-\frac{\sin\theta}{r}.\frac{\partial u}{\partial\theta}\right)$$

$$=\cos^2\theta\frac{\partial^2 u}{\partial r^2}-\frac{\cos\theta.\sin\theta}{r}.\frac{\partial^2 u}{\partial r\partial\theta}-\frac{\cos\theta.\sin\theta}{r}.\frac{\partial^2 u}{\partial\theta\partial r}+\frac{\sin^2\theta}{\partial^2 r}.\frac{\partial^2 u}{\partial\theta^2}$$

$$+\frac{\sin^2\theta}{r}\frac{\partial u}{\partial r}+\frac{\cos\theta.\sin\theta}{r^2}.\frac{\partial u}{\partial\theta}+\frac{\cos\theta.\sin\theta}{r^2}.\frac{\partial u}{\partial\theta}. \qquad \text{(i)}$$

Similarly, we get

$$\frac{\partial^2 u}{\partial y^2}=\frac{\partial}{\partial y}.\left(\frac{\partial u}{\partial y}\right)=\sin^2\theta\frac{\partial^2 u}{\partial r^2}+\frac{2\cos\theta.\sin\theta}{r}.\frac{\partial^2 u}{\partial r\partial\theta}+\frac{\cos^2\theta}{r^2}\frac{\partial^2 u}{\partial\theta^2}$$

$$+\frac{\cos^2\theta}{r}\frac{\partial u}{\partial r}-\frac{\cos\theta.\sin\theta}{r^2}.\frac{\partial u}{\partial\theta}-\frac{\cos\theta.\sin\theta}{r^2}.\frac{\partial u}{\partial\theta}. \qquad \text{(ii)}$$

Adding (i) and (ii) we get

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\left(\cos^2\theta+\sin^2\theta\right)\frac{\partial^2 u}{\partial r^2}+\frac{\cos^2\theta+\sin^2\theta}{r^2}\frac{\partial^2 u}{\partial\theta^2}+\frac{\cos^2\theta+\sin^2\theta}{r}\frac{\partial u}{\partial r}$$

$$=\frac{\partial^2 u}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}+\frac{1}{r}\frac{\partial u}{\partial r}$$

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 u}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}+\frac{1}{r}\frac{\partial u}{\partial r}=0.$$

Hence this proves the result**.**

Q.No.14: If $v=r^3$ and $r^2=x^2+y^2+z^2$ then show that

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}=\frac{\partial^2 v}{\partial r^2}+\frac{2}{r}\frac{\partial v}{\partial r}.$$

Sol.: Let $\frac{\partial v}{\partial x}=\frac{\partial v}{\partial r}.\frac{\partial r}{\partial x}=3r^2.\frac{x}{r}=3rx$

$$\Rightarrow\frac{\partial^2 v}{\partial x^2}=3r+3x.\frac{\partial r}{\partial x}=3r+3x.\frac{x}{r}=\frac{3\left(r^2+x^2\right)}{r}. \qquad \text{(i)}$$

Similarly we can find

$$\frac{\partial^2 v}{\partial y^2}=\frac{3\left(r^2+y^2\right)}{r} \qquad \text{(ii)}$$

$$\frac{\partial^2 v}{\partial z^2}=\frac{3\left(r^2+z^2\right)}{r}. \qquad \text{(iii)}$$

By adding (i), (ii) and (iii), we get

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}=\frac{3\left(3r^2+x^2+y^2+z^2\right)}{r}=\frac{3\left(3r^2+r^2\right)}{r}=\frac{3\times4r^2}{r}=12r.$$

(iv)

By differentiating $v=r^3$ w. r. t. r, we get

$$\frac{dv}{dr}=3r^2.$$

Again differentiating, we get $\frac{d^2v}{dr^2} = 6r$

$\therefore$ Let R. H. S. $\frac{d^2v}{dr^2} + \frac{2}{r}.\frac{dv}{dr} = 6r + \frac{2}{r}.3r^2 = 6r + 6r = 12r$. (v)

Hence from (iv) and (v), we get

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = \frac{\partial^2 v}{\partial r^2} + \frac{2}{r}\frac{\partial v}{\partial r}$$

Hence this proves the result**.**

Q.No.15: If $z = f(x, y)$, $x = u\cos\alpha - v\sin\alpha$, $y = u\sin\alpha + v\cos\alpha$, prove that

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}.$$

Sol.: Since z is a composite function of u and v

Thus $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial v} = \frac{\partial z}{\partial x}.\cos\alpha. + \frac{\partial z}{\partial y}.\sin\alpha = f$

Now, $\frac{\partial^2 z}{\partial u^2} = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right) = \frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}.\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}.\frac{\partial y}{\partial u}$

$$\Rightarrow \frac{\partial^2 z}{\partial u^2} = \cos\alpha\left(\cos\alpha\frac{\partial^2 z}{\partial x^2} + \sin\alpha\frac{\partial^2 z}{\partial x\partial y}\right) + \sin\alpha\left(\cos\alpha\frac{\partial^2 z}{\partial x\partial y} + \sin\alpha\frac{\partial^2 z}{\partial y^2}\right)$$

$$\Rightarrow \frac{\partial^2 z}{\partial u^2} = \cos^2\alpha\frac{\partial^2 z}{\partial x^2} + 2\cos\alpha\sin\alpha\frac{\partial^2 z}{\partial x\partial y} + \sin^2 z\frac{\partial^2 z}{\partial y^2}. \quad \text{(i)}$$

Similarly, $\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial v} = -\sin\alpha\frac{\partial z}{\partial x} + \cos\alpha\frac{\partial z}{\partial y} = g$

$$\Rightarrow \frac{\partial^2 z}{\partial v^2} = \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right) = \frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}.\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}.\frac{\partial y}{\partial v}$$

$$= -\sin\alpha\left(-\sin\alpha\frac{\partial^2 z}{\partial x^2} + \cos\alpha\frac{\partial^2 z}{\partial x\partial y}\right) + \cos\alpha\left(-\sin\alpha\frac{\partial^2 z}{\partial x\partial y} + \cos\alpha\frac{\partial^2 z}{\partial y^2}\right)$$

$$\Rightarrow \frac{\partial^2 z}{\partial v^2} = \sin^2\alpha\frac{\partial^2 z}{\partial x^2} - 2\cos\alpha\sin\alpha\frac{\partial^2 z}{\partial x\partial y} + \cos^2\alpha\frac{\partial^2 z}{\partial y^2}. \quad \text{(ii)}$$

Now by adding (i) and (ii), we get

$$\frac{\partial^2 z}{\partial u^2}+\frac{\partial^2 z}{\partial v^2}=\left(\cos^2\alpha+\sin^2\alpha\right)\frac{\partial^2 z}{\partial x^2}+\left(\cos^2\alpha+\sin^2\alpha\right)\frac{\partial^2 z}{\partial y^2}=\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}.$$

Hence this proves the result.

Q.No.16: If $f(p, t, v)=0$. Prove that $\left(\frac{dp}{dt}\right)_{v=c}\times\left(\frac{dt}{dv}\right)_{p=c}\times\left(\frac{dv}{dp}\right)_{t=c}=-1$.

Sol.: When $v=c$ then

$f_1=f(p, t, v)=f(p, t, c)=f(p, t)=0$

Now $$\left(\frac{dp}{dt}\right)_{v=c}=\frac{-\partial f_1/\partial t}{\partial f_1/\partial p}$$

(i)

Similarly $\left(\frac{dt}{dv}\right)_{p=c}=\frac{-\partial f_2/\partial v}{\partial f_2/\partial t}$ (ii)

and $\left(\frac{dv}{dp}\right)_{t=c}=\frac{-\partial f_3/\partial p}{\partial f_3/\partial v}$ (iii)

Multiplying (i), (ii) and (iii), we get

$$\left(\frac{dp}{dt}\right)_{v=c}\times\left(\frac{dt}{dv}\right)_{p=c}\times\left(\frac{dv}{dp}\right)_{p=c}=\frac{-\partial f_1/\partial t}{\partial f_1/\partial p}\times\frac{-\partial f_2/\partial v}{\partial f_2/\partial t}\times\frac{-\partial f_3/\partial p}{\partial f_3/\partial v}$$

$$\Rightarrow\left(\frac{\partial f}{\partial p}\right)_{v,t=0}=\left(\frac{\partial f_1}{\partial p}\right)_{v=c}=\left(\frac{\partial f_3}{\partial p}\right)_{t=c}$$

Similarly $\frac{\partial f_1}{\partial t}=\frac{\partial f_2}{\partial t}$ and $\frac{\partial f_2}{\partial v}=\frac{\partial f_3}{\partial v}$

Thus, we get

$$\left(\frac{dp}{dt}\right)_{v=c}\times\left(\frac{dt}{dv}\right)_{p=c}\times\left(\frac{dv}{dp}\right)_{t=c}=\frac{\partial f_1}{\partial t}\times\frac{1}{\partial f_1/\partial p}\times\frac{\partial f_2}{\partial v}=-1=\text{R. H. S.}.$$

Hence this proves the result.

Q.No.17: If $f(u, v)=0$, $u=\ell x+my+mz$ and $v=x^2+y^2+z^2$. Hence show that

$$(\ell y-mx)+(ny-mz)\frac{\partial z}{\partial x}+(\ell z-nx)\frac{\partial z}{\partial y}=0.$$

Sol.: Since f is a composite function of x, y, and z. Then we have

$$\frac{\partial f}{\partial x}=\frac{\partial f}{\partial u}.\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}.\frac{\partial v}{\partial x}\Rightarrow\frac{\partial f}{\partial x}=\ell\frac{\partial f}{\partial u}+2x\frac{\partial f}{\partial v} \qquad \text{(i)}$$

and $$\frac{\partial f}{\partial y}=\frac{\partial f}{\partial u}.\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}.\frac{\partial v}{\partial y}\Rightarrow\frac{\partial f}{\partial y}=m\frac{\partial f}{\partial u}+2y\frac{\partial f}{\partial v} \qquad \text{(ii)}$$

and $$\frac{\partial f}{\partial z}=n\frac{\partial f}{\partial u}+2z\frac{\partial f}{\partial v} \qquad \text{(iii)}$$

Solving (i) and (ii), we get

$$\frac{\partial f}{\partial u}=\frac{y\frac{\partial f}{\partial x}-x\frac{\partial f}{\partial y}}{\ell y-mx} \text{ and } \frac{\partial f}{\partial v}=\frac{m\frac{\partial f}{\partial x}-\ell\frac{\partial f}{\partial y}}{2xm-2\ell y}$$

$$\therefore \frac{\partial f}{\partial z}=n\frac{\partial f}{\partial u}+2z\frac{\partial f}{\partial v}=n\left(\frac{y\frac{\partial f}{\partial x}-x\frac{\partial f}{\partial y}}{\ell y-mx}\right)+2z\left(\frac{m\frac{\partial f}{\partial x}-\ell\frac{\partial f}{\partial y}}{2(mx-\ell y)}\right)$$

$$\Rightarrow(\ell y-mx)\frac{\partial f}{\partial z}=ny\frac{\partial f}{\partial x}-nx\frac{\partial f}{\partial y}-mz\frac{\partial f}{\partial x}+z\ell\frac{\partial f}{\partial y}$$

$$\Rightarrow(\ell y-mx)\frac{\partial f}{\partial z}=(ny-mz)\frac{\partial f}{\partial x}+(\ell z-nx)\frac{\partial f}{\partial y}$$

$$\Rightarrow(\ell y-mx)-(ny-mz)\frac{\partial f/\partial x}{\partial f/\partial z}-(\ell z-nx)\frac{\partial f/\partial y}{\partial f/\partial z}=0$$

$$\Rightarrow(\ell y-mx)+(ny-mz)\frac{\partial z}{\partial x}+(\ell z-nx)\frac{\partial z}{\partial y}=0. \qquad \left[\begin{array}{l}\therefore \frac{\partial z}{\partial x}=-\frac{\partial f/\partial x}{\partial f/\partial z}\\ \frac{\partial z}{\partial y}=-\frac{\partial f/\partial y}{\partial f/\partial z}\end{array}\right]$$

Hence this proves the result.

Q.No.18.: If $z=f(x,y)$, $x=u+v$, $y=uv$, prove that

(i) $(u-v)\frac{\partial z}{\partial x}=u\frac{\partial z}{\partial u}-v\frac{\partial z}{\partial v}$.

(ii) $(u-v)\frac{\partial z}{\partial y}=\frac{\partial z}{\partial v}-\frac{\partial z}{\partial u}$.

Sol.: Here z is a composite function of u and v

Hence $$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}.\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}.\frac{\partial y}{\partial u}=(1)\frac{\partial z}{\partial x}+v\frac{\partial z}{\partial y}=\frac{\partial z}{\partial x}+v\frac{\partial z}{\partial y} \qquad \text{(i)}$$

Similarly we get

$$\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}.\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}.\frac{\partial y}{\partial v}=(1)\frac{\partial z}{\partial x}+u\frac{\partial z}{\partial y}\;=\frac{\partial z}{\partial x}+u\frac{\partial z}{\partial y} \qquad \text{(ii)}$$

Let $u\frac{\partial z}{\partial u}-v\frac{\partial z}{\partial v}=u\frac{\partial z}{\partial x}+uv\frac{\partial z}{\partial y}-v\frac{\partial z}{\partial x}-uv\frac{\partial z}{\partial y}=u\frac{\partial z}{\partial x}-v\frac{\partial z}{\partial x}\Rightarrow(u-v)\frac{\partial z}{\partial x}$.

Hence this prove the (i) relation**.**

Let us subtract (ii) from (i), we get

$$\frac{\partial z}{\partial v}-\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}+u\frac{\partial z}{\partial y}-\frac{\partial z}{\partial x}-v\frac{\partial z}{\partial y}=(u-v)\frac{\partial z}{\partial y}.$$

Hence this proves the (ii) relation**.**

Q.No.19.: If $z=f(r,s,t)$ and $r=\frac{x}{y}$, $s=\frac{y}{z}$ and $t=\frac{z}{x}$, prove that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=0.$$

Sol.: Here $\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}.\frac{\partial r}{\partial x}+\frac{\partial u}{\partial s}.\frac{\partial s}{\partial x}+\frac{\partial u}{\partial t}.\frac{\partial t}{\partial x}=\frac{\partial u}{\partial r}.\frac{1}{y}+\frac{\partial u}{\partial s}.(0)+\left(-\frac{z}{x^2}\right).\frac{\partial u}{\partial t}$

$$=\frac{1}{y}\frac{\partial u}{\partial r}-\frac{z}{x^2}\frac{\partial u}{\partial t}.$$

Similarly, we get

$$\frac{\partial u}{\partial y}=\frac{1}{z}\frac{\partial u}{\partial s}-\frac{x}{y^2}\frac{\partial u}{\partial r}\text{ and }\frac{\partial u}{\partial z}=\frac{1}{x}\frac{\partial u}{\partial s}-\frac{y}{z^2}\frac{\partial u}{\partial s}$$

$$\Rightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=\frac{x}{y}\frac{\partial u}{\partial r}-\frac{z}{x}\frac{\partial u}{\partial t}+\frac{y}{z}\frac{\partial u}{\partial s}-\frac{x}{y}\frac{\partial u}{\partial r}+\frac{z}{x}\frac{\partial u}{\partial t}-\frac{y}{z}\frac{\partial u}{\partial s}$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=0.$$

Hence this proves the result**.**

Q.No.20: If $z=f(x,y)$ and $x=r\cos\theta$, $y=r\sin\theta$ express the equation $\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=0$ in terms of $r\,\theta$. Is the equation in terms of r and θ valid at $r=0$.

Sol.: Let $x=r\cos\theta$ and $y=r\sin\theta$ $\Rightarrow r=\sqrt{x^2+y^2}$

And $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$$\Rightarrow \frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}} = \cos\theta$$

$$\Rightarrow \frac{\partial \theta}{\partial x} = -\frac{y}{\left(\sqrt{x^2+y^2}\right)^2} = -\frac{y}{x^2+y^2} = -\frac{\sin\theta}{r}$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}.\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}.\frac{\partial \theta}{\partial x} = \frac{\partial u}{\partial r}.\cos\theta - \frac{\sin\theta}{r}.\frac{\partial u}{\partial \theta}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial f}{\partial x} \quad \text{where } f = \frac{\partial u}{\partial x}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = \cos\theta.\frac{\partial f}{\partial x} - \frac{\sin\theta}{r}\frac{\partial f}{\partial \theta} = \cos\theta\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial x}\right) - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial x}\right)$$

$$= \cos\theta.\frac{\partial}{\partial r}\left(\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}.\frac{\partial u}{\partial \theta}\right) - \frac{\sin\theta}{r}.\frac{\partial}{\partial \theta}\left(\cos\theta.\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}.\frac{\partial u}{\partial \theta}\right)$$

$$= \cos^2\theta\frac{\partial^2 u}{\partial r^2} - \frac{2\sin\theta.\cos\theta}{r}.\frac{\partial^2 u}{\partial r.\partial \theta} + \frac{\sin^2\theta}{r^2}.\frac{\partial^2 u}{\partial^2 \theta} + \frac{\sin^2\theta}{r}.\frac{\partial u}{\partial r}$$

$$+ \frac{\sin\theta\cos\theta}{r^2}.\frac{\partial u}{\partial \theta} - \frac{\sin\theta.\cos\theta}{r^2}.\frac{\partial u}{\partial \theta}. \qquad \text{(i)}$$

Similarly, we get

$$\frac{\partial^2 u}{\partial y^2} = \sin^2\theta\frac{\partial^2 u}{\partial r^2} + \frac{2\sin\theta.\cos\theta}{r}.\frac{\partial^2 u}{\partial r.\partial \theta} + \frac{\cos^2\theta}{r^2}.\frac{\partial^2 u}{\partial \theta^2} + \frac{\cos^2\theta}{r}.\frac{\partial u}{\partial r}$$

$$- \frac{\sin\theta\cos\theta}{r^2}.\frac{\partial u}{\partial r} + \frac{\sin\theta.\cos\theta}{r^2}.\frac{\partial u}{\partial r}. \qquad \text{(ii)}$$

By adding (i) and (ii), we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \left(\sin^2\theta + \cos^2\theta\right)\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}.\left(\sin^2\theta + \cos^2\theta\right)\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r}.\left(\sin^2\theta + \cos^2\theta\right)\frac{\partial u}{\partial r}$$

$$= \frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}.\frac{\partial^2 \theta}{\partial \theta^2} + \frac{1}{r}.\frac{\partial u}{\partial r}$$

From this equation, we get

$$r^2\frac{\partial^2 z}{\partial r^2}+r\frac{\partial z}{\partial r}+\frac{\partial^2 z}{\partial \theta^2}=0.$$

When r = 0 then we have

$\frac{\partial^2 z}{\partial \theta^2}=0$. Thus the equation is valid.

Hence this proves the result.

Q.No.21: If $x=u+v+w$, $y=u^2+v^2+w^2$, $z=u^3+v^3+w^3$ then prove that

$$\frac{\partial u}{\partial x}=\frac{vw}{(u-v)(u-w)}.$$

Sol.: Let $x=u+v+w$

By differentiating w. r. t. x, we get

$$\frac{\partial x}{\partial x}=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial x}+\frac{\partial w}{\partial x}\Rightarrow\frac{\partial u}{\partial x}+\frac{\partial v}{\partial x}+\frac{\partial w}{\partial x}=1 \qquad \text{(i)}$$

Also $y=u^2+v^2+w^2$

Again by differentiating partially w. r. t. x, we get

$$0=2u\frac{\partial u}{\partial x}+2v\frac{\partial v}{\partial x}+2w\frac{\partial w}{\partial x}\Rightarrow u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial x}+w\frac{\partial w}{\partial x}=0 \qquad \text{(ii)}$$

and $z=u^3+v^3+w^3$

Again by differentiating partially w. r. t. x, we get

$$0=3u^2\frac{\partial u}{\partial x}+3v^2\frac{\partial v}{\partial x}+3w^2\frac{\partial w}{\partial x}\Rightarrow u^2\frac{\partial u}{\partial x}+v^2\frac{\partial v}{\partial x}+w^2\frac{\partial w}{\partial x}=0 \qquad \text{(iii)}$$

Let $\frac{\partial u}{\partial x}=a$, $\frac{\partial v}{\partial x}=b$ and $\frac{\partial w}{\partial x}=c$

Putting these values in (i), (ii) and (iii), we get

$$a+b+c=1 \qquad \text{(iv)}$$

$$ua+vb+wc=0 \qquad \text{(v)}$$

$$u^2a+v^2b+w^2c=0 \qquad \text{(vi)}$$

$$a+b+c=1 \quad \Rightarrow wa+wb+wc=w \qquad \text{(vii)}$$

$$ua+vb+wc=0 \quad \Rightarrow wua+wvb+w^2c=0 \qquad \text{(viii)}$$

Now subtracting (v) from (vii), we get

$(w-u)a+(w-v)b=w$

Now subtracting (vi) from (viii), we get

$\left(wu-u^2\right)a+\left(wv-v^2\right)b=0$ i. e.

$(w-u)a+(w-v)b=w \Rightarrow v(w-u)a+v(w-v)b=wv$ (ix)

and $u(w-v)a-v(w-v)b=0$ (x)

By solving, we get

$$v(w-u)a-u(w-v)a=vw \Rightarrow (v-u)(w-v)a=vw \Rightarrow a=\frac{vw}{(v-u)(w-v)}$$

Hence $\dfrac{\partial u}{\partial x}=\dfrac{uw}{(u-v)(u-w)}$.

Hence this proves the result.

Q.No.22: If $x=\cosh\theta.\cos\phi$, $y=\sinh\theta.\sin\phi$ then show that

$$J\left(\frac{x,y}{\theta,\phi}\right)=\frac{1}{2}(\cosh 2\theta-\cos 2\phi).$$

Sol.: Let $J\left(\dfrac{x,y}{\theta,\phi}\right)=\begin{vmatrix}\dfrac{\partial x}{\partial\theta} & \dfrac{\partial x}{\partial\phi}\\ \dfrac{\partial y}{\partial\theta} & \dfrac{\partial y}{\partial\theta}\end{vmatrix}=\dfrac{\partial x}{\partial\theta}.\dfrac{\partial y}{\partial\phi}-\dfrac{\partial x}{\partial\phi}.\dfrac{\partial y}{\partial\theta}$

$$\Rightarrow \frac{\partial x}{\partial\theta}=\sinh\theta\cos\phi\,;\quad \frac{\partial x}{\partial\phi}=-\cosh\theta.\sin\phi$$

and $\dfrac{\partial y}{\partial\theta}=\cosh\theta\sin\phi\,;\quad \dfrac{\partial x}{\partial\phi}=\sinh\theta.\cos\phi$

$$\therefore J\left(\frac{x,y}{\theta,\phi}\right)=(\sinh\theta.\cos\phi).(\sinh\theta.\cos\phi)+(\cosh\theta.\sin\phi).(\cosh\theta.\sin\phi)$$

$$=\cos^2\phi.\sinh^2\theta+\cosh^2\theta\sin^2\phi$$

Now here $\sinh\theta=\dfrac{e^x-e^{-x}}{2}$ and $\cosh\theta=\dfrac{e^x+e^{-x}}{2}$

$$\Rightarrow \cosh^2\theta=\frac{e^{x^2}+e^{-x^2}+2e^{x-x}}{4}=\frac{e^{x^2}+e^{-x^2}}{4}+\frac{1}{2}$$

and $\sinh^2\theta = \dfrac{e^{x^2} + e^{-x^2}}{4} - \dfrac{1}{2}$

$$J\left(\frac{x, y}{\theta, \phi}\right) = \cos^2\phi\left(\frac{e^{x^2} + e^{-x^2}}{4} - \frac{1}{2}\right) + \sin^2\phi\left(\frac{e^{x^2} + e^{-x^2}}{4} + \frac{1}{2}\right)$$

$$= \frac{e^{x^2} + e^{-x^2}}{4}\left(\cos^2\phi + \sin^2\phi\right) - \frac{1}{2}\left(\cos^2\phi - \sin^2\phi\right)$$

$$= \frac{e^{x^2} + e^{-x^2}}{4} - \frac{1}{2}\cos 2\phi = \frac{1}{2}.\frac{e^{x^2} + e^{-x^2}}{2} - \frac{1}{2}\cos 2\phi$$

$$= \frac{1}{2}.\left(\frac{e^{x^2} + e^{-x^2}}{2} - \cos 2\phi\right) = \frac{1}{2}.(\cos 2h\theta - \cos 2\phi) \qquad \left[\because \frac{e^x + e^{-x}}{2} = \cosh\theta\right]$$

Hence this proves the result**.**

Q.No.23.: If $u = \dfrac{yz}{x}$, $v = \dfrac{zx}{y}$, $w = \dfrac{xy}{z}$, show that $\dfrac{\partial(u, v, w)}{\partial(x, y, z)} = 4$.

Sol.: Here $\dfrac{\partial u}{\partial x} = -\dfrac{yz}{x^2}$, $\dfrac{\partial v}{\partial y} = -\dfrac{zx}{y^2}$ and $\dfrac{\partial w}{\partial z} = -\dfrac{xy}{z^2}$

$\dfrac{\partial u}{\partial y} = \dfrac{z}{x}$, $\dfrac{\partial v}{\partial z} = \dfrac{x}{y}$ and $\dfrac{\partial w}{\partial x} = \dfrac{y}{z}$

and $\dfrac{\partial u}{\partial z} = \dfrac{y}{x}$, $\dfrac{\partial v}{\partial x} = \dfrac{z}{y}$ and $\dfrac{\partial w}{\partial y} = \dfrac{x}{z}$

$\therefore$ Taking L. H. S., we get

$$J\left(\frac{u, v, w}{x, y, z}\right) = \frac{\partial(u, v, w)}{\partial(x, y, z)}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} -\frac{yz}{x^2} & \frac{z}{x} & \frac{y}{z} \\ \frac{z}{y} & -\frac{zx}{y^2} & \frac{x}{y} \\ \frac{y}{z} & \frac{x}{z} & -\frac{xy}{z^2} \end{vmatrix} = \frac{1}{x\,y\,z}\begin{vmatrix} -\frac{yz}{x} & z & y \\ z & -\frac{zx}{y} & x \\ y & x & -\frac{xy}{z} \end{vmatrix}$$

$$= \frac{1}{x\,y\,z}\left[-\frac{yz}{x}\left(\frac{(zx)(xy)}{zy} - x^2\right) - z(-xy - yx) + y(zx + zx)\right]$$

$$= \frac{1}{x\,y\,z}\left[(-yzx + yzx) - z(-2xy) + y(2zx)\right]$$

$$= \frac{1}{x\,y\,z}\left((0) + 2xyz + 2xyz\right) = \frac{1}{x\,y\,z}(4xyz) = 4 = \text{R. H. S.}.$$

Hence this proves the result.

Q.No.24.: If $x = r\cos\theta$, $y = r\sin\theta$, prove that $J\left(\frac{r,\theta}{x,y}\right) = \frac{1}{r}$.

Sol.: Given that $x = r\cos\theta$ (i)

And $y = r\sin\theta$ (ii)

From (i) and (ii), we get

$$r = \sqrt{x^2 + y^2} \quad \text{and} \quad \theta = \tan^{-1}\frac{y}{x}$$

So we get $\frac{\partial r}{\partial x} = \frac{2x}{2\sqrt{x^2+y^2}} = \frac{x}{\sqrt{x^2+y^2}} = \cos\theta$

And $\frac{\partial r}{\partial y} = \frac{2y}{2\sqrt{x^2+y^2}} = \frac{y}{\sqrt{x^2+y^2}} = \sin\theta$

Similarly $\frac{\partial \theta}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2}$

And $\frac{\partial \theta}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{1}{x} = \frac{x}{x^2+y^2}$

Let L. H. S.

$$J\left(\frac{r,\theta}{x,y}\right) = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix} = \frac{\partial r}{\partial x}\cdot\frac{\partial \theta}{\partial y} - \frac{\partial \theta}{\partial x}\cdot\frac{\partial r}{\partial y} = \frac{x}{\sqrt{x^2+y^2}}\cdot\frac{x}{x^2+y^2} + \frac{y}{\sqrt{x^2+y^2}}\cdot\frac{y}{x^2+y^2}$$

$$= \frac{x^2}{x^2+y^2\sqrt{x^2+y^2}} + \frac{y^2}{x^2+y^2\sqrt{x^2+y^2}} = \frac{x^2+y^2}{x^2+y^2\sqrt{x^2+y^2}} = \frac{1}{\sqrt{x^2+y^2}} = \frac{1}{r}$$

$= \text{R. H. S.}.$

Hence this proves the result.

Q.No.25.: If $x = \rho\cos\theta$, $y = \rho\sin\theta$, $z = z$, show that $\dfrac{\partial(x, y, z)}{\partial(\rho, \theta, z)} = \rho$.

Sol.: Let $x = \rho\cos\theta$, $y = \rho\sin\theta$ and $z = z$

$$\Rightarrow \frac{\partial x}{\partial \rho} = \cos\theta, \frac{\partial y}{\partial \rho} = \sin\theta \text{ and } \frac{\partial z}{\partial \rho} = 0$$

$$\frac{\partial x}{\partial \theta} = -\rho\sin\theta, \quad \frac{\partial y}{\partial \theta} = \rho\cos\theta \text{ and } \frac{\partial z}{\partial \theta} = 0$$

$$\text{and } \frac{\partial x}{\partial z} = 0, \quad \frac{\partial y}{\partial z} = 0 \text{ and } \frac{\partial z}{\partial z} = 1$$

Taking L. H. S., we get

$$\frac{\partial(x, y, z)}{\partial(\rho, \theta, z)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\theta & -\rho\sin\theta & 0 \\ \sin\theta & \rho\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1\left(\rho\cos^2\theta + \rho\sin^2\theta\right)$$

$$= \rho\left(\cos^2\theta + \sin^2\theta\right) = \rho = \text{R. H. S.}.$$

Hence this proves the result**.**

Q.No.26.: If $x = f(u, v)$, $y = \phi(u, v)$ are two functions which satisfy the equations

$\dfrac{\partial f}{\partial u} = \dfrac{\partial \phi}{\partial v}$, $\dfrac{\partial f}{\partial v} = -\dfrac{\partial \phi}{\partial u}$ and z is a function of x and y, then prove that

$$\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} = \left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right)\left[\left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2\right].$$

Sol.: Given that $\dfrac{\partial z}{\partial u} = \dfrac{\partial z}{\partial x}.\dfrac{\partial x}{\partial u} + \dfrac{\partial z}{\partial y}.\dfrac{\partial y}{\partial u}$

$$\Rightarrow g = \frac{\partial z}{\partial x}.\frac{\partial f}{\partial u} + \frac{\partial z}{\partial y}.\frac{\partial \phi}{\partial u} \Rightarrow \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right) = \frac{\partial}{\partial u} g$$

$$= \frac{\partial g}{\partial x}.\frac{\partial x}{\partial u} + \frac{\partial g}{\partial y}.\frac{\partial y}{\partial u}$$

$$= \frac{\partial f}{\partial u}\left(\frac{\partial f}{\partial u}.\frac{\partial^2 z}{\partial x^2} + \frac{\partial z}{\partial x}.\frac{\partial^2 f}{\partial x \partial u} + \frac{\partial^2 z}{\partial x \partial y}.\frac{\partial \phi}{\partial u} + \frac{\partial^2 \theta}{\partial x \partial u}.\frac{\partial z}{\partial y}\right)$$

$$+\frac{\partial\phi}{\partial u}\left(\frac{\partial f}{\partial u}.\frac{\partial^2 z}{\partial x\partial y}+\frac{\partial z}{\partial x}.\frac{\partial^2 f}{\partial y\partial u}+\frac{\partial^2 z}{\partial y^2}.\frac{\partial\phi}{\partial u}+\frac{\partial^2\theta}{\partial y\partial u}.\frac{\partial z}{\partial y}\right).$$

Now we have $\frac{\partial^2 f}{\partial x\partial u}=\frac{\partial^2 f}{\partial u\partial x}-\frac{\partial}{\partial u}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial}{\partial u}(1)=0-\frac{\partial^2 f}{\partial y\partial v}$.

Similarly, we can have $\frac{\partial^2\phi}{\partial y\partial u}=0=\frac{\partial^2\phi}{\partial y\partial v}$.

So that

$$\frac{\partial^2 z}{\partial u^2}=\frac{\partial f}{\partial u}\left(\frac{\partial f}{\partial u}.\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial x\partial y}.\frac{\partial\phi}{\partial v}+\frac{\partial^2\phi}{\partial x\partial u}.\frac{\partial z}{\partial y}\right)+\frac{\partial\phi}{\partial u}\left(\frac{\partial f}{\partial u}.\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial z}{\partial x}.\frac{\partial f}{\partial y\partial u}+\frac{\partial\phi}{\partial u}.\frac{\partial^2 z}{\partial y^2}\right) \qquad \text{(i)}$$

Similarly, we can find

$$\frac{\partial^2 z}{\partial v^2}=\frac{\partial f}{\partial v}\left(\frac{\partial f}{\partial v}.\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial x\partial y}.\frac{\partial\phi}{\partial v}+\frac{\partial^2\phi}{\partial x\partial u}.\frac{\partial z}{\partial y}\right)+\frac{\partial\phi}{\partial v}\left(\frac{\partial f}{\partial v}.\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial z}{\partial x}.\frac{\partial^2 f}{\partial y\partial v}+\frac{\partial\phi}{\partial v}.\frac{\partial^2 z}{\partial y^2}\right) \qquad \text{(ii)}$$

Since $\frac{\partial f}{\partial u}=\frac{\partial\phi}{\partial v}$ and $\frac{\partial f}{\partial v}=\frac{\partial\phi}{\partial u}$

Taking L. H. S., we get

$$\frac{\partial^2 z}{\partial u^2}+\frac{\partial^2 z}{\partial v^2}=\frac{\partial f}{\partial u}\left(\frac{\partial f}{\partial u}.\frac{\partial^2 z}{\partial x^2}-\frac{\partial^2 z}{\partial x\partial y}.\frac{\partial f}{\partial v}-\frac{\partial^2 f}{\partial x\partial v}.\frac{\partial z}{\partial y}\right)-\frac{\partial f}{\partial v}\left(\frac{\partial f}{\partial u}.\frac{\partial^2 z}{\partial x\partial y}+\frac{\partial^2 f}{\partial y\partial u}.\frac{\partial z}{\partial x}-\frac{\partial f}{\partial v}.\frac{\partial^2 z}{\partial y^2}\right)$$

$$+\frac{\partial f}{\partial v}\left(\frac{\partial f}{\partial v}.\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial x\partial y}.\frac{\partial f}{\partial u}+\frac{\partial^2 f}{\partial x\partial u}.\frac{\partial z}{\partial y}\right)+\frac{\partial f}{\partial u}\left(\frac{\partial f}{\partial v}.\frac{\partial^2 z}{\partial x\partial y}+\frac{\partial^2 f}{\partial y\partial v}.\frac{\partial z}{\partial x}+\frac{\partial f}{\partial u}.\frac{\partial^2 z}{\partial y^2}\right)$$

$$=\left(\frac{\partial f}{\partial u}\right)^2.\frac{\partial^2 z}{\partial x^2}+\left(\frac{\partial f}{\partial u}\right)^2.\frac{\partial^2 z}{\partial y^2}+\left(\frac{\partial f}{\partial v}\right)^2.\frac{\partial^2 z}{\partial x^2}+\left(\frac{\partial f}{\partial v}\right)^2.\frac{\partial^2 z}{\partial y^2}$$

$$+\frac{\partial f}{\partial v}\left(-\frac{\partial z}{\partial x}.\frac{\partial^2 f}{\partial u\partial y}+\frac{\partial z}{\partial y}.\frac{\partial^2 f}{\partial u\partial x}\right)+\frac{\partial f}{\partial u}\left(-\frac{\partial z}{\partial x}.\frac{\partial^2 f}{\partial v\partial y}-\frac{\partial z}{\partial y}.\frac{\partial^2 f}{\partial v\partial x}\right)$$

$$\left[\because \frac{\partial f}{\partial u\partial x}=\frac{\partial f}{\partial v\partial x}=0\right]$$

$$=\left(\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}\right)\left[\left(\frac{\partial f}{\partial u}\right)^2+\left(\frac{\partial f}{\partial v}\right)^2\right]+\frac{\partial f}{\partial v}\left(-\frac{\partial z}{\partial x}-\frac{\partial^2\phi}{\partial v\partial y}\right)+\frac{\partial f}{\partial u}\left(-\frac{\partial z}{\partial x}-\frac{\partial^2\phi}{\partial u\partial y}\right)$$

$$\left[\because \frac{\partial^2 u}{\partial v \partial y} = \frac{\partial^2 \phi}{\partial u \partial y} = 0\right]$$

$$= \left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right)\left[\left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2\right] = \text{R. H. S.}$$

Hence this proves the result.

Q.No.27: If $z = u^2 + v^2$, $x = u^2 - v^2$ and $y = uv$. Find the value of $\frac{\partial z}{\partial x}$.

Sol.: Here $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}.\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}.\frac{\partial v}{\partial x} \Rightarrow \frac{\partial z}{\partial x} = \frac{\partial u}{\partial x}2u + 2v\frac{\partial v}{\partial x}$

Now $u^2 - v^2 = x$.

Differentiating w.r.t. to x, we get

$$2u\frac{\partial u}{\partial x} - 2v\frac{\partial v}{\partial x} = \frac{\partial x}{\partial x} = 2u\frac{\partial u}{\partial x} - 2v\frac{\partial v}{\partial x} = 1, \qquad \text{(i)}$$

and $vu = y$.

Differentiating w. r. t. to x, we get

$$u.\frac{\partial v}{\partial x} + v.\frac{\partial u}{\partial x} = 0. \qquad \text{(ii)}$$

Solving (i) and (ii), we get

$$\frac{\frac{\partial u}{\partial x}}{0+u} = \frac{\frac{\partial v}{\partial x}}{-v-0} = \frac{1}{2u^2 + 2v^2}$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{u}{2\left(u^2 + v^2\right)} = \frac{u}{2z} \text{ and } \frac{\partial v}{\partial x} = \frac{-v}{2\left(u^2 + v^2\right)} = \frac{-v}{2z}$$

$$\Rightarrow \frac{\partial z}{\partial x} = 2u.\frac{u}{2z} - 2v.\frac{v}{2z} = \frac{u^2 - v^2}{z} = \frac{x}{z}$$

Hence $\frac{\partial z}{\partial x} = \frac{x}{z}$. Ans.

Thank you

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Differential Calculus

Errors and Approximations

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Definition: Suppose u is a function of x and y i.e. $u = f(x,\ y)$.

If there is a change in the value of x from x to $(x + \delta x)$ and a change in the value of y from y to $(y + \delta y)$ (where δx and δy are small and may be positive or negative), then there will be a change in the value of u (say) from u to $u + \delta u$.

We may call this change in the value of x i.e. δx as 'increment in x' or 'error in x'.

Similarly, δy may be called as 'increment in y' or 'error in y' and so δu is the 'increment in the value of u' or 'error in the value of u'.

Now we have $u = f(x,\ y)$. (i)

$\therefore u + \delta u = f(x + \delta x, y + \delta y)$

$\Rightarrow \delta u = f(x + \delta x,\ y + \delta y) - f(x,\ y)$. (ii)

Expanding $f(x + \delta x,\ y + \delta y)$ by Taylor's theorem on two variables

$$f(x + \delta x, y + \delta y) = f(x, y) + \left(\frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y\right) + \frac{1}{2!}\left(\frac{\partial^2 f}{\partial x^2}\delta x^2 + 2\delta x \delta y \frac{\partial^2 f}{\partial x \partial y} + \delta y^2 \frac{\partial^2 f}{\partial y^2}\right).....$$

We know that, Taylor's series for a function of one variable is

$$f(x+h) = f(h) + xf'(h) + \frac{x^2}{2!} f''(h) + \ldots\ldots\ldots\ldots$$

$$= f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \ldots\ldots\ldots\ldots$$

Also, Taylor's series for a function of two variables is

$$f(x+h, y+k) = f(x, y) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(x, y) + \frac{1}{2!}\left(h^2\frac{\partial^2}{\partial x^2} + 2hk\frac{\partial^2}{\partial x \partial y} + k^2\frac{\partial^2}{\partial y^2}\right) f(x, y) + \ldots\ldots$$

Substituting the expansion of $f(x+\delta x,\ y+\delta y)$ in (ii), we obtain

$$\delta u = \left[f(x, y) + \left(\frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y\right) + \frac{1}{2!}\left(\frac{\partial^2 f}{\partial x^2}\delta x^2 + 2\delta x \delta y \frac{\partial^2 f}{\partial x \partial y} + \delta y^2 \frac{\partial^2 f}{\partial y^2}\right) \ldots\ldots \right] - f(x,\ y).$$

As δx and δy are supposed to be very-very small, therefore their squares and higher powers can be neglected.

Thus $\delta u = \dfrac{\partial f}{\partial x}\delta x + \dfrac{\partial f}{\partial y}\delta y$.

Replacing $\delta x,\ \delta y,\ \delta u$ by $dx,\ dy,\ dz$ respectively, we have

$$du = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \quad \Rightarrow \delta u = \frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y \quad [\because u = f(x,\ y)].$$

This formula is used in calculating the effect of small errors or increments in measured quantities and is useful in correcting the effect of small errors.

Remarks: If $u = f(x,\ y,\ z, \ldots\ldots)$ then, $\delta u = \dfrac{\partial u}{\partial x}\delta x + \dfrac{\partial u}{\partial y}\delta y + \dfrac{\partial u}{\partial z}\delta z + \ldots\ldots\ldots\ldots$

Percentage error:

Definition: $\dfrac{\delta x}{x} \times 100$ is called percentage error in the value of x, where δx is the change or actual error in the value of x.

Similarly, $\dfrac{\delta y}{y} \times 100$ is called the percentage error in y, and

$\dfrac{\delta u}{u} \times 100$ is called the percentage error in u.

where δy and δu are actual errors in y and u respectively.

Relative error: If δx is the error in x, then relative error $= \dfrac{\delta x}{x}$.

Now let us solve some problems related to errors and approximations:

Q.No.1.: Find the percentage error in the area of an ellipse, when an error $+1$ percent is made in semi-major axis and -1 is made in measuring the semi-minor axis.

Sol.: Since, the area A of an ellipse is given by the relation $A = \pi ab$, where a, b are its semi-major and semi-minor axis.

Here error in a and b are given, therefore we will treat a and b variables. Since, when a and b are treated as variables $\Rightarrow$ A is also a variable.

Taking differentials, we get

$$d(A) = d(\pi ab) = \pi d(ab) = \pi[d(a).b + a.d(b)]$$

$$\Rightarrow dA = \pi b\,.\,da + \pi a\,.\,db$$

$$\Rightarrow \frac{dA}{A} = \frac{\pi b}{A}.da + \frac{\pi a}{A}.db = \frac{da}{a} + \frac{db}{b} \qquad \left[\because a = \frac{A}{\pi b},\ \ b = \frac{A}{\pi a}\right]$$

$$\therefore \frac{dA}{A}\times 100 = \frac{da}{a}\times 100 + \frac{db}{b}\times 100 = \text{percentage error in a} + \text{percentage error in b}$$

$$= (+1) + (-1) = 0.$$

Hence percentage error in the area of an ellipse is zero.

2nd method:

Since $A = \pi ab$

Taking logarithms on both sides, we get

$$\log A = \log(\pi ab) = \log \pi + \log a + \log b.$$

Now taking differentials, we get

$$\frac{1}{A}.dA = 0 + \frac{1}{a}.da + \frac{1}{b}.db$$

$$\Rightarrow \frac{dA}{A}\times 100 = \frac{da}{a}\times 100 + \frac{db}{b}\times 100 = (+1) + (-1) = 0\ \text{.Ans.}$$

Q.No.2.: If an error committed in measuring the side of a square be 2%. Find the error in calculating the area.

Sol.: Since the area A of a square is $A = x^2$, where x is the side of a square.

Taking log on both sides, we get

$$\log A = \log x^2 = 2\log x .$$

Taking differentials on both sides, we get

$$\frac{1}{A}dA = 2.\frac{1}{x}dx$$

$$\therefore \frac{dA}{A}\times 100 = 2\left[\frac{dx}{x}\times 100\right]$$

$\Rightarrow$ %age error in $A = 2(\%\text{ age error in } x) = 2\times 2 = 4\%$ .Ans.

Q.No.3.: Find the % error in the area of an ellipse, when an error of +1% is made in measuring the semi-major and semi-minor axis.

Sol.: Since $A = \pi ab$, where a, b are its semi-major and semi-minor axis.

Taking logarithms on both sides, we get

$$\log A = \log(\pi ab) = \log\pi + \log a + \log b .$$

Now taking differentials, we get

$$\frac{1}{A}.dA = 0 + \frac{1}{a}.da + \frac{1}{b}.db$$

$$\Rightarrow \frac{dA}{A}\times 100 = \frac{da}{a}\times 100 + \frac{db}{b}\times 100 = (+1) + (+1) = 2\%$$ .Ans.

Q.No.4.: The time of swing t, of a pendulum, of length ℓ, under certain conditions is given by $t = 2\pi\sqrt{\frac{\ell}{g'}}$, where $g' = g\left(\frac{r}{r+h}\right)^2$. Find the %age error in t due to the errors of p% in h and q% in ℓ.

Sol.: Given $t = 2\pi\sqrt{\frac{\ell}{g'}}$

Taking log on both sides, we get $\log t = \log 2\pi + \frac{1}{2}\log\ell - \frac{1}{2}\log g'$

$$\Rightarrow \log t = \log 2\pi + \frac{1}{2}\log\ell - \frac{1}{2}\log g\left(\frac{r}{r+h}\right)^2$$

$$\Rightarrow \log t = \log 2\pi + \frac{1}{2}\log \ell - \frac{1}{2}\log g - \log r + \log(r+h)$$

Taking differentials, we get

$$\Rightarrow \frac{dt}{t} = 0 + \frac{1}{2}\frac{d\ell}{\ell} - 0 - 0 + \frac{1}{r+h}dh$$

$$\Rightarrow \frac{dt}{t}\times 100 = \frac{1}{2}\left(\frac{d\ell}{\ell}\times 100\right) + \left(\frac{dh}{r+h}\times 100\right)$$

$$\Rightarrow \%\text{age error in } t = \frac{1}{2}q + \frac{1}{r+h}\left(\frac{dh}{h}\times 100\right)h = \left(\frac{1}{2}q + \frac{ph}{r+h}\right)\%\text{ .Ans.}$$

Q.No.5.: Using the concept of small errors, find an approximate value of $f(10.02,\ 40.05,\ 29.97)$ where $f(x,\ y,\ z) = x\,y\,z$.

or

Let $f(10.02,\ 40.05,\ 29.97)$ where $f(x,\ y,\ z) = x\,y\,z$.

Using the concept of small errors, find relative error, actual error and approximate value of f.

Sol.: Let $x = 10,\ \delta x = 0.02,\ y = 40,\ \delta y = 0.05,\ z = 30,\ \delta z = -0.03$.

Now $f(x, y, z) = xyz$.

Taking log on both sides, we get

$\log f = \log x + \log y + \log z$.

Taking differentials, we get

$$\frac{\delta f}{f} = \frac{\delta x}{x} + \frac{\delta y}{y} + \frac{\delta z}{z} = \frac{0.02}{10} + \frac{0.05}{40} + \frac{-0.03}{30} = 0.002 + .00125 + (-0.001) = 0.00225$$

which is the relative error in f.

$\delta f = 0.00225\,f$, but $f = 10\times 40\times 30 = 12000$. $\delta f = 12000\times 0.00225 = 27$.

$\therefore$ Approximate value of $f = f + \delta f = 12000 + 27 = 12027$.**Ans.**

Actual value = 12026.991

Q.No.6.: If $f(x, y, z) = x^{\ell}y^{m}z^{n}$ and errors of p %, q % and r % are made in measuring x, y, z respectively. Find the error in $f(x, y, z)$.

Sol.: Given $f(x, y, z) = x^{\ell}y^{m}z^{n}$.

Taking log on both sides, we get

$\log f = \ell \log x + m \log y + n \log z$.

Taking differentials, we get

$$\frac{\delta f}{f} = \ell \frac{\delta x}{x} + m \frac{\delta y}{y} + n \frac{\delta z}{z}$$

$$\Rightarrow \frac{\delta f}{f} \times 100 = \ell\left(\frac{\delta x}{x} \times 100\right) + m\left(\frac{\delta y}{y} \times 100\right) + n\left(\frac{\delta z}{z} \times 100\right)$$

Hence %age error in $f(x, y, z) = (\ell p + qm + r\,n)\%$. Ans.

Q.No.7.: The area S of a triangle is calculated from the length of sides a, b, and c. If a be diminished and b be increased by small amounts x, prove that the consequent change in area is given by $\frac{\delta S}{S} = \frac{2(a-b)x}{c^2 - (a-b)^2}$.

Sol.: Hero's Formula: A formula connecting the area of a Δ with its sides

$\Delta = \sqrt{s(s-a)(s-b)(s-c)}$, $s = \frac{a+b+c}{2}$ is semi-parameter.

$$\therefore \text{ Area } S = \sqrt{\left(\frac{a+b+c}{2}\right)\left(\frac{a+b+c}{2} - a\right)\left(\frac{a+b+c}{2} - b\right)\left(\frac{a+b+c}{2} - c\right)}$$

$$= \sqrt{\left(\frac{a+b+c}{2}\right).\left(\frac{b+c-a}{2}\right).\left(\frac{c+a-b}{2}\right).\left(\frac{a+b-c}{2}\right)}.$$

Taking log on both sides, we get

$$\log S = \frac{1}{2}\left[(\log(a+b+c) + \log(b+c-a) + \log(c+a-b) + \log(a+b-c)) - 4\log 2\right]$$

Taking differentials, we get

$$\frac{\delta S}{S} = \frac{1}{2}\left[\frac{\delta a + \delta b}{(a+b+c)} + \frac{\delta b - \delta a}{(b+c-a)} + \frac{\delta a - \delta b}{(c+a-b)} + \frac{\delta a + \delta b}{a+b-c}\right]$$

$$= \frac{1}{2}\left[0 + \frac{2x}{(b+c-a)} + \frac{(-2x)}{(c+a-b)} + 0\right]$$

$$= \frac{x}{(b+c-a)} - \frac{x}{(a+c-b)} = x\left[\frac{1}{c-(a-b)} - \frac{1}{c+(a-b)}\right]$$

$$= x\left[\frac{c+a-b-c+a-b}{c^2 - (a-b)^2}\right] = \frac{2(a-b)x}{c^2 - (a-b)^2}.\ \text{Ans.}$$

Q.No.8.: The edge of a cube is measured with a positive error of 0.05 cm. Find the relative error in the computed volume, when the edge is found to be 7.5 cm. Also find percentage error in the computed volume.

Sol.: Let x be the edge of the cube.

$\therefore$ volume V of the cube $= x^3$. (i)

Taking log on both sides, we get

$\log V = \log x^3 = 3\log x$. (ii)

Taking differentials, we get

$\frac{1}{V}.dV = 3.\frac{1}{x}dx$. (iii)

$\therefore$ Error in the computed volume $= dV = \frac{3V}{x}dx = \frac{3x^3}{x}dx = 3x^2dx$

$\Rightarrow dV = 3\times(7.5)^2\times(0.05) = 8.44$ cubic cm**.**

Thus, relative error in the computed volume $= \frac{dV}{V} = \frac{3dx}{x} = \frac{3\times 0.05}{7.5} = 0.02$. Ans.

Now, percentage error in the computed volume $= \frac{dV}{V}\times 100 = 0.02\times 100 = 2\%$ **.** Ans**.**

Q.No.9.; The diameter and altitude of a can in the shape of right circular cylinder are measured as 4 cm and 6 cm respectively. The possible error in each measurement is 0.1 cm. Find approximately the maximum possible error in the value computed for the volume and lateral surface.

Sol.: Let the diameter and altitude of the can be denoted by D and H respectively.

Then radius $= \frac{D}{2}$.

(i) The volume V of the can is given by $V = \pi r^2 h = \frac{\pi}{4}D^2H$ $[= f(D, H)]$

$\therefore dV = \frac{\partial V}{\partial D}dD + \frac{\partial V}{\partial H}dH = \frac{\pi}{4}\left[2DHdD + D^2dH\right]$

$= \frac{\pi}{4}\left[2\times 4\times 6\times 0.1 + 4^2\times 0.1\right] = \frac{\pi}{4}(6.4) = 1.6\pi$ cubic cm. Ans.

(ii) The lateral surface S of the can is given by $S = 2\pi rh = \pi DH$ $[= f(D, H)]$

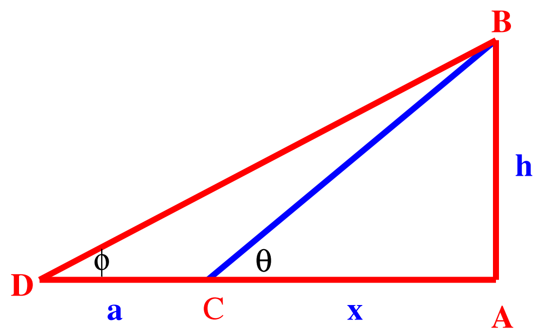
$\therefore dS = \frac{\partial S}{\partial D}dD + \frac{\partial S}{\partial H}dH = \pi[HdD + DdH] = \pi[6\times 0.1 + 4\times 0.1] = \pi$ sq. cm. Ans.

Q.No.10.: The height of a tower is determined by observing the elevation θ and ϕ of its summit from two points in a direct line with the foot of the tower and at a distance 'a' apart. Show that the error in the calculated height due to small errors $d\theta$ and $d\phi$ is approximately $a\left(\sin^2\theta d\phi - \sin^2\phi d\theta\right)\text{cosec}^2(\theta-\phi)$.

Sol.: Let h be the height of the tower AB and C and D, the two points of observation so that

$CD = a$, $\angle ACB = \theta$,

$\angle ADB = \phi$. Let $AC = x$



From right angle ΔBAC, $x = h\cot\theta$ (i)

From right angle ΔBAD, $x + a = h\cot\phi$ (ii)

Subtracting (i) from (ii), we get

$$a = h(\cot\phi - \cot\theta) = h\left(\frac{\cos\phi}{\sin\phi} - \frac{\cos\theta}{\sin\theta}\right) = h\left[\frac{\sin\theta\cos\phi - \cos\theta\sin\phi}{\sin\theta\sin\phi}\right] = \frac{h\sin(\theta-\phi)}{\sin\theta\sin\phi}$$

$$\Rightarrow h = \frac{a\sin\theta\sin\phi}{\sin(\theta-\phi)}[= f(\theta,\phi)] \qquad \text{(iii)}$$

Taking log on both sides, we get $\log h = \log a + \log\sin\theta + \log\sin\phi - \log\sin(\theta-\phi)$

Taking differentials, we get $\frac{dh}{h} = 0 + \frac{\cos\theta}{\sin\theta}d\theta + \frac{\cos\phi}{\sin\phi}d\phi - \frac{\cos(\theta-\phi)}{\sin(\theta-\phi)}(d\theta - d\phi)$

$$\Rightarrow \frac{dh}{h} = \left[\frac{\cos\theta}{\sin\theta} - \frac{\cos(\theta-\phi)}{\sin(\theta-\phi)}\right]d\theta + \left[\frac{\cos\phi}{\sin\phi} + \frac{\cos(\theta-\phi)}{\sin(\theta-\phi)}\right]d\phi$$

$$= \frac{\sin(\theta-\phi)\cos\theta - \cos(\theta-\phi)\sin\theta}{\sin\theta\sin(\theta-\phi)}d\theta + \frac{\sin(\theta-\phi)\cos\phi + \cos(\theta-\phi)\sin\phi}{\sin\phi\sin(\theta-\phi)}d\phi$$

$$= \frac{\sin[(\theta-\phi)-\theta]}{\sin\theta\sin(\theta-\phi)}d\theta + \frac{\sin[(\theta-\phi)+\phi]}{\sin\phi\sin(\theta-\phi)}d\phi$$

$$= \frac{\sin(-\phi)}{\sin\theta\sin(\theta-\phi)}d\theta + \frac{\sin\theta}{\sin\phi\sin(\theta-\phi)}d\phi = \frac{\sin^2\theta d\phi - \sin^2\phi d\theta}{\sin\theta\sin\phi\sin(\theta-\phi)} \quad [\because \sin(-\phi) = -\sin\phi]$$

$$\therefore dh = h.\frac{\sin^2\theta d\phi - \sin^2\phi d\theta}{\sin\theta\sin\phi\sin(\theta-\phi)} = \frac{a\sin\theta\sin\phi}{\sin(\theta-\phi)}.\frac{\sin^2\theta d\phi - \sin^2\phi d\theta}{\sin\theta\sin\phi\sin(\theta-\phi)} \qquad \text{[using (iii)]}$$

$$= a\left(\sin^2\theta d\phi - \sin^2\phi d\theta\right)\text{cosec}^2(\theta-\phi).$$

Hence prove.

Q.No.11.: In some torsion experiment an error of 0.5%, was made in measuring the diameter x. Calculate the corresponding %age error in the stress f, where

$$T + \frac{\pi}{16} f\, x^3 = 0.$$

Sol.: Here $T + \frac{\pi}{16} f\, x^3 = 0 \Rightarrow f = \frac{-16}{11}\frac{T}{x^3}$ $\left[\begin{array}{l}\text{Here only f and x will be treated as} \\ \text{variables as error occurs in these.}\end{array}\right]$

Taking log on both sides, we get

$$\log f = \log\left(\frac{-16}{11}\right) + \log T - 3\log x$$

Taking differentials on both sides, we get

$$\frac{df}{f} = 0 + 0 - 3\frac{dx}{x} \Rightarrow \frac{df}{f}\times 100 = -3\frac{dx}{x}\times 100 = -3\times(0.5) = -1.5\%$$ **.Ans.**

Q.No.12.: In an experiment carried out to find the value of g error of 0.5% and 1% are possible in the value of t and ℓ respectively. Show that the maximum error in the calculated value of g could not be more than 2%.

Sol.: Time period of pendulum is given by $T = 2\pi\sqrt{\frac{\ell}{g}}$ (i)

Where T = time period, ℓ = length of pendulum, g = acceleration due to gravity.

On squaring (i), we get

$$T^2 = 4\pi^2\frac{\ell}{g} \Rightarrow g = 4\pi^2\frac{\ell}{T^2}. \qquad \text{(ii)}$$

On differentiating, we get

$$dg = 4\pi^2\left[\frac{1}{T^2}d\ell - \frac{2}{T^2}\ell dT\right] = \frac{4\pi^2}{T^2}\ell\left[\frac{d\ell}{\ell} - 2\frac{dT}{T}\right]$$

$$dg = g\left[\frac{d\ell}{\ell} - 2\frac{dT}{T}\right]$$

$$\Rightarrow \frac{dg}{g} = \left[\frac{d\ell}{\ell} - 2\frac{dT}{T}\right] \qquad \text{(iii)}$$

Given, %age error in length, $\frac{d\ell}{\ell}\times 100 = 1\%$ (iv)

%age error in Time period, $\frac{dT}{T}\times 100 = .5\%$ (v)

Putting the values from (iv) and (v) in (iii), we get

$$\frac{dg}{g}\times 100 = 1 - 2(-.5) = (1+1)\% = 2\% .$$

Therefore maximum error in the value of g = 2%. Ans.

Q.No.13.: In measuring the value of angle θ, an error of 0.1^0 was made. Find the corresponding error in the value of the sine of the angle**.**

Sol.: Given $d\theta = 0.1^0 = \frac{0.1\times\pi}{180}\text{Radian} = 0.00175\text{Radian}$

$dy = ?$ where $y = \sin\theta$

$\therefore dy = \cos\theta d\theta = 0.00175\cos\theta$ **.**Ans**.**

Q.No.14.: If H. .P. required to propel a steamer is proportional to the cube of its velocity and square of its length, prove that a 2% increase in velocity and 3% increase in length will require approximately a 12% increase in H. P**.**

Sol.: Given $P \propto v^3\ell^2 \Rightarrow P = kv^3\ell^2$, where k is the constant of proportionality.

Taking log on both sides, we get

$\log P = \log k + 3\log v + 2\log\ell$.

Taking differentials, we get

$$\frac{dP}{P} = 0 + 3\frac{dv}{v} + 2\frac{d\ell}{\ell}$$

$$\Rightarrow \frac{dP}{P}\times 100 = 3\left(\frac{dv}{v}\times 100\right) + 2\left(\frac{d\ell}{\ell}\times 100\right) = (3\times 2) + (2\times 3) = 6 + 6 = 12\ \%$$ **.**Ans**.**

Q.No.15.: The indicated horse power I of an engine is calculated from the formula $I = \frac{PLAN}{33000}$, where $A = \frac{\pi}{4}d^2$. Assuming that error of r % may have been made in measuring P, L, N and d. Find the greatest possible error in I**.**

Sol.: Given $I = \frac{PLAN}{33000}$.

Taking log on both sides, we get

$\log I = \log P + \log L + \log A + \log N - \log 33000$

Taking differentials, we get

$$\Rightarrow \frac{dI}{I} = \frac{dP}{P} + \frac{dL}{L} + 2.\frac{d(d)}{dN} + \frac{dN}{N} - 0 \qquad \left[\begin{aligned} \because \log A &= \log\left(\frac{\pi}{4}d^2\right) = \log\frac{\pi}{4} + \log d^2 \\ &= \log\frac{\pi}{4} + 2\log d \end{aligned}\right]$$

$$\Rightarrow \frac{dI}{I}\times 100 = \frac{dP}{P}\times 100 + \frac{dL}{L}\times 100 + 2\frac{d(d)}{d}\times 100 + \frac{dN}{N}\times 100$$

$$= r + r + 2r + r = 5r\% \text{. Ans.}$$

Q.No.16.: The time period of a simple pendulum is given by $t = 2\pi.\sqrt{\frac{\ell}{g}}$.

Find the error in t due to error $\delta\ell$ and δg in ℓ and g. What is the max. %age error in t if there is an error of 1% in ℓ and g.

Sol.: Given $t = 2\pi.\sqrt{\frac{\ell}{g}}$.

Taking log on both sides, we get

$$\log t = \log 2\pi + \frac{1}{2}\log\ell - \frac{1}{2}\log g .$$

Taking differentials, we get

$$\frac{dt}{t} = 0 + \frac{1}{2}\frac{d\ell}{\ell} - \frac{1}{2}\frac{dg}{g}$$

$$\Rightarrow \left(\frac{dt}{t}\times 100\right) = \frac{1}{2}\left(\frac{d\ell}{\ell}\times 100\right) - \frac{1}{2}\left(\frac{dg}{g}\times 100\right)$$

(i) $= \frac{1}{2}(+1) - \frac{1}{2}(+1) = 0\%$ (Not max). Ans.

(ii) $= \frac{1}{2}(+1) - \frac{1}{2}(-1) = +1\%$ (Max)

(iii) $= \frac{1}{2}(-1) - \frac{1}{2}(+1) = -1\%$ (Max). Ans.

(iv) $= \frac{1}{2}(-1) - \frac{1}{2}(-1) = 0\%$ (Not Max)

$\therefore$ Max. %age error in t $= \pm 1$ %. Ans.

Q.No.17.: The slope of a hanging rod of uniform strength is given by $y = A\exp.\left(\frac{w}{f}x\right)$, where y is the radius at any height x above a fixed point at A is constant. Find the change in y produced by small changes δw in w and δf in f. Show that the %age error in y is $\frac{wx}{f}$ times the difference in the %age errors in w and f.

Sol.: Given $y = A e^{wx/f}$.

Taking log on both sides, we get $\log y = \log A + \frac{wx}{f}\log e$.

Taking differentials on both sides, we get

$$\frac{\delta y}{y} = 0 + x\left(\frac{f\delta w - w\delta f}{f^2}\right)$$

$$\Rightarrow \delta y = \frac{xy}{f}\left(\frac{f\delta w - w\delta f}{f}\right) = \frac{wxy}{f}\left(\frac{f\delta w - w\delta f}{wf}\right).\ \text{Ans.}$$

Also $\frac{\delta y}{y}\times 100 = x\left[\frac{f\delta y - w\delta f}{f^2}\right]\times 100 = \frac{wx}{f}\left[\frac{f\delta w - w\delta f}{wf}\right]\times 100$

$$= \frac{wx}{f}\left[\frac{\delta w}{w}\times 100 - \frac{\delta f}{f}\times 100\right].\ \text{Ans.}$$

Q.No.18.: If $R = \frac{E}{C}$, find the max. error and the %age error in R if $C = 20$ with a possible error of ± 0.1 and $E = 120$ with a possible error of ± 0.05.

Sol.: Given $R = \frac{E}{C}$.

Taking log on both sides, we get

$\log R = \log E - \log C$.

Taking differentials on both sides, we get

$$\frac{\delta R}{R} = \frac{\delta E}{E} - \frac{\delta C}{C} \qquad \Rightarrow \delta R = R\left(\frac{\delta E}{E} - \frac{\delta C}{C}\right) = 6\left[\frac{0.05}{120} - \left(\frac{-0.1}{20}\right)\right] = 0.0324.(\text{max})$$

which is the required max. error in R.

Now $\frac{\delta R}{R}\times 100 = \frac{\delta E}{E}\times 100 - \frac{\delta C}{C}\times 100$

$$\Rightarrow \frac{\delta R}{R}\times 100 = \frac{+0.05}{120}\times 100 - \frac{-0.1}{20}\times 100 = \frac{5}{120}+\frac{1}{2} = 0.54\% \text{ (max)}. \text{ Ans.}$$

which is the required max. % error in R.

Q.No.19.: In calculating the volume of a right circular cone, errors of +2% and minus one percent are made in the height and radius of the base respectively. Find the %age error in the volume. What is the percentage error in calculating value of the surface area of the cone ?

Sol.: Given %age error in height = 2% and %age error in radius $= -1\%$.

Since we know that the volume of a right circular cone is $V = \frac{1}{3}\pi r^2 h$.

$$\log V = \log\frac{1}{3} + \log\pi + 2\log r + \log h$$

Taking differentials, we get

$$\frac{dV}{V} = \frac{2dr}{r} + \frac{dh}{h} \quad \Rightarrow \frac{dV}{V}\times 100 = \frac{2dr}{r}\times 100 + \frac{dh}{h}\times 100.$$

$\therefore$ %age error in volume $= 2(-1) + 2 = 0\%$. Ans.

Q.No.20.: In estimating the cost of a pile of bricks measured as 6 by 50 by 4 feet, the tape is stretched 1% beyond the standard length. If the count is 12 bricks to 1foot^3 and bricks cost Rs. 100 per 1000, find the approximate error in the cost**.**

Sol.: Let ℓ, b and h feet be the length, breadth and height of the pile so that its volume $V = \ell\times b\times h$.

Taking log on both sides, we get

$$\log V = \log\ell + \log b + \log h.$$

Taking differentials, we get

$$\frac{\delta V}{V} = \frac{\delta\ell}{\ell} + \frac{\delta b}{b} + \frac{\delta h}{h}.$$

Since $V = 6\times 50\times 4 = 1200\,\text{ft}^3$ and $\frac{\delta\ell}{\ell}\times 100 = \frac{\delta b}{b}\times 100 = \frac{\delta h}{h}\times 100 = 1\%$.

$$\therefore \delta V = 1200\left(\frac{3}{100}\right) = 36\ \text{ft}^3.$$

Number of bricks in $\delta V = 36\times 12 = 432$.

Thus error in the cost $= 432 \times \dfrac{100}{1000} = \text{Rs. } 43.20$,

which is less to the brick seller.

Q.No.21.: Two quantities x_1 and x_2 are related to each other by the formula,

$x_2 = a(x_1)^n$, where a and n are constant quantities. Small errors of p % and q% are made in measuring a and n, show that the calculated value of x_2 for a given value of X' of x_1 will have a percentage error of $p + nq\log_e X'$.

Sol.: Given that %age error in a = p%. %age error in n = q%

Since given $x_2 = a(x_1)^n$.

Taking log on both sides, we get $\log x_2 = \log a + n\log x_1$.

Differentiating on both sides, we get $\dfrac{dx_2}{x_2} = \dfrac{da}{a} + dn\log x_1 + \dfrac{ndx_1}{x_1}$

$$\Rightarrow \frac{dx_2}{x_2} \times 100 = \frac{da}{a} \times 100 + \left(\frac{dn}{n} \times 100\right) n\log x_1 + \frac{ndx_1}{x_1} \times 100$$

$$\Rightarrow \frac{dx_2}{x_2} \times 100 = p + qn\log_e X' + 0 = p + nq\log_e X'.$$

Thus %age error in $x_2 = (p + nq\log_e X')\%$.

Q.No.22.: The acceleration of a piston is equal to $rw^2\cos\theta + \dfrac{r^2w^2}{\ell}\cos 2\theta$. In measuring $\theta\ (= 30°)$ and w small error minus 1 percent each was detected. Prove that calculated value of acceleration is minus 1.5%. Take $4r = \ell$.

Sol.: Given acceleration of a piston $a = rw^2\cos 2\theta + \dfrac{r^2w^2}{\ell}\cos 2\theta$. (i)

Putting $4r = \ell$ in (i), we get

$$a = rw^2\cos\theta + \frac{rw^2}{4}\cos 2\theta$$

$$\delta a = 2rw\cos\theta\delta w + rw^2(-\sin\theta)\delta\theta + \frac{r\cos 2\theta}{4}2w\delta w + \frac{rw^2}{4}2(-\sin 2\theta)\delta\theta$$

$$\delta a = rw^2\left[2\cos\theta\left(\frac{\delta w}{w}\right) + (-\sin\theta)\left(\frac{\delta\theta}{\theta}\right)\theta + \frac{\cos 2\theta}{2}\left(\frac{\delta w}{w}\right) - \frac{\sin 2\theta}{2}\left(\frac{\delta\theta}{\theta}\right)\theta\right]$$

$$\frac{\delta a}{a}=\frac{rw^2\left[2\cos\theta\left(\frac{\delta w}{w}\right)-\sin\theta\left(\frac{\delta\theta}{\theta}\right)\theta+\frac{\cos 2\theta}{2}\left(\frac{\delta w}{w}\right)-\frac{\sin 2\theta}{2}\left(\frac{\delta\theta}{\theta}\right)\theta\right]}{rw^2\left[\cos\theta+\frac{\cos 2\theta}{4}\right]}$$

Dividing and multiplying by 100 and putting $\frac{\partial w}{w}=-1\%$, $\frac{\partial\theta}{\theta}=-1\%$, $\theta=30°$, we get

$$\frac{\delta a}{a}=-\left[\frac{\left[2(\cos 30°)1\%-\sin 30°\times 1\%\times 30°+\frac{\cos(2\times 30°)}{2}1\%-\frac{(\sin 60°)}{2}\times 1\%\times 30°\right]}{\cos 30°+\frac{\cos 60°}{4}}\right]$$

$$\therefore\frac{\delta a}{a}=-\left[\frac{\left(2\times\frac{\sqrt{3}}{2}\right)-\left(\frac{1}{2}\times\frac{\pi}{6}\right)+\left(\frac{1}{2}\times\frac{1}{2}\right)-\left(\frac{\sqrt{3}}{2}\times\frac{1}{2}\times\frac{\pi}{6}\right)}{\left(\frac{\sqrt{3}}{2}+\frac{1}{8}\right)}\right]\%=-1.5\%\text{ . Hence prove.}$$

Q.No.23.: In a plane triangle ABC, if the sides and angles receive small variations, prove that $\delta a\cos C+\delta c\cos A=0$; b, B being constant.

Sol.: To prove: $\delta a\cos C+\delta c\cos A=0$**,** b and B as constants

Here using projection formula: $b=a\cos C+c\cos A$.

Differentiating, we get

$$db=da\cos C+a(-\sin C)dC+dc\cos A+c(-\sin A)dA$$

$$0=da\cos C+a(-\sin C)dC+dc\cos A-c\sin A dA$$

Now $A+B=\pi-C$

$$dA+dB=-dC\Rightarrow dA=-dC$$

$$0=da\cos C+a\sin C dA+dc\cos A-c\sin A dA$$

Now using sin formula: $\frac{\sin A}{a}=\frac{\sin C}{c}\Rightarrow c\sin A=a\sin C$

$$\therefore 0=da\cos C+dc\cos A+c\sin A dA-c\sin A dA$$

$$\Rightarrow da\cos C+dc\cos A=0.$$

Hence this proves the result.

Q.No.24.: The side a of a triangle ABC is calculated from b, c, A. Small errors db, dc, dA occur in the measured values of b,c, and A respectively. Prove that the error in a is given by $da = \cos B dc + \cos C db + b\sin C dA$.

Sol.: To prove: $da = \cos B dc + \cos C db + b\sin C dA$.

Here using projection formula: $a = b\cos C + c\cos B$.

Differentiating, we get

$da = db\cos C + dc\cos B - b\sin C dC - c\sin B dB$.

Using sine formula: $c\sin B = b\sin C$

$\therefore da = db\cos C + dc\cos B - b\sin C(dB + dC)$

But $B + C = \pi - A \Rightarrow dB + dC = -dA$.

$\therefore da = db\cos C + dc\cos B + b\sin C dA$.

Hence this proves the result.

Q.No.25.: Given the formula $\frac{1}{z} = \frac{1}{x} + \frac{1}{y}$. If x and y are both in the error by r %, prove that z is also in the error of r %**.**

Sol.: Since $\frac{1}{z} = \frac{1}{x} + \frac{1}{y}$.

Taking differentials, we get $-z^{-2}dz = -x^{-2}dx - y^{-2}dy$

$$\Rightarrow -\frac{1}{z}\left(\frac{dz}{z}\times 100\right) = -\frac{1}{x}\left(\frac{dx}{x}\times 100\right) - \frac{1}{y}\left(\frac{dy}{y}\times 100\right) = -\frac{r}{x} - \frac{r}{y} = -r\left(\frac{1}{x} + \frac{1}{y}\right) = -\frac{r}{z}$$

$\Rightarrow \left(\frac{dz}{z}\times 100\right) = r\ \%$**.** Ans**.**

Hence z is also in the error of r %.

Q.No.26.: The quantity Q of water flowing over a notch is given by $Q = \frac{8}{15}\times 0.64\times\sqrt{2g}\times(H)^{5/2}$, where H is the head at the notch. What is the % age error in Q caused by measuring H as 0.198 instead of 0.2?

Sol.: Since $Q = \frac{8}{15}\times 0.64\times\sqrt{2g}\times(H)^{5/2}$.

Taking log on both sides, we get

$\log Q = \log\frac{8}{15} + \log 0.64 + \log\sqrt{2g} + \frac{5}{2}\log H$.

Taking differentials, we get $\frac{\delta Q}{Q}=0+0+0+\frac{5}{2}.\frac{\delta H}{H}$

$$\Rightarrow \frac{\delta Q}{Q}\times 100=\frac{5}{2}.\left(\frac{\delta H}{H}\times 100\right)=\frac{5}{2}.\left(\frac{0.002}{0.2}\times 100\right)=\frac{5}{2} \qquad [\because \delta H=0.2-0.198=0.002]$$

Hence %age error in Q = 2.5 % **.** Ans**.**

Q.No.27.: A closed rectangular box with unequal sides a, b, c has its edges slightly altered in length by amount δa, δb and δc respectively. Show that its volume and surface area remain unchanged then $\frac{\delta a}{a^2(b-c)}=\frac{\delta b}{b^2(c-a)}=\frac{\delta c}{c^2(a-b)}$**.**

Sol.: Given volume of a rectangular box is $V=abc$.

Taking log on both sides, we get

$\log V=\log a+\log b+\log c$

Taking differentials, we get

$\frac{\delta V}{V}=\frac{\delta a}{a}+\frac{\delta b}{b}+\frac{\delta c}{c}$. Now since $\delta V=0\Rightarrow \frac{\delta a}{a}+\frac{\delta b}{b}+\frac{\delta c}{c}=0$

$$\Rightarrow \frac{\delta a}{a}=-\frac{\delta b}{b}-\frac{\delta c}{c}\Rightarrow \delta a=-\left(\frac{a}{b}\delta b+\frac{a}{c}\delta c\right)$$

Also $S=2(ab+bc+ca)$

Taking differentials, we get

$0=2(a\delta b+b\delta a+b\delta c+c\delta b+c\delta a+a\delta c)$ (since $\delta S=0$)

$$\Rightarrow (a+c)\delta b+(a+b)\delta c+(b+c)\delta a=0$$

$$\Rightarrow \delta a=-\frac{(a+c)\delta b+(a+b)\delta c}{(b+c)}\Rightarrow \frac{a}{b}\delta b+\frac{a}{c}\delta c=\frac{(a+c)\delta b+(a+b)\delta c}{(b+c)}$$

$$\Rightarrow \left[\frac{a}{b}-\frac{a+c}{b+c}\right]\delta b=\left[\frac{a+b}{b+c}-\frac{a}{c}\right]\delta c$$

$$\Rightarrow \frac{\delta b}{b^2(c-a)}=\frac{\delta c}{c^2(a-b)}.$$

Similarly $\frac{\delta a}{a^2(b-c)}=\frac{\delta c}{c^2(a-b)}$.

Hence $\frac{\delta a}{a^2(b-c)}=\frac{\delta b}{b^2(c-a)}=\frac{\delta c}{c^2(a-b)}$**.** Hence prove.

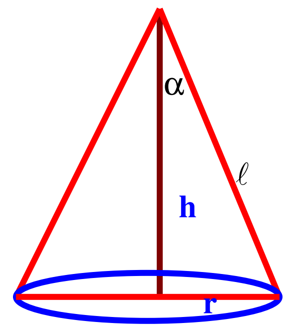
Q.No.28.: The height h and semi-vertical angle a of a cone are measured and from then A the total area of the surface of the cone including the base is calculated. If h and a are in error by small quantity δh and δa respectively. Find the corresponding error in the area. Show further that, if $\alpha = \frac{\pi}{6}$, an error of +1% in h will be approximately compensated by an error of -0.33^o in α.

Sol.: Base radius $r = h\tan\alpha$.

Slant height $\ell = h\sec\alpha$.

Area of base $= \pi r^2$.

Area of curved surface $= \pi r\ell$.



Total surface area $A = \pi r^2 + \pi r\ell = \pi r(r+\ell) = \pi r\left(r + \sqrt{h^2+r^2}\right)$

$$= \pi.h\tan\alpha\left(h\tan\alpha + \sqrt{h^2 + h^2\tan^2\alpha}\right) = \pi h\tan\alpha(h\tan\alpha + h\sec\alpha)$$

$$= \pi h^2\tan\alpha(\tan\alpha + \sec\alpha). \qquad \Big| = f(h, \alpha)$$

$$\therefore \ \delta A = \frac{\partial A}{\partial h}\delta h + \frac{\partial A}{\partial \alpha}\delta\alpha$$

$$= 2\pi h\left(\tan^2\alpha + \tan\alpha\sec\alpha\right)\delta h + \pi h^2\left(2\tan\alpha\sec^2\alpha + \sec^3\alpha + \tan\alpha\sec\alpha\tan\alpha\right)\delta\alpha$$

$$= 2\pi h\tan\alpha(\tan\alpha + \sec\alpha)\delta h + \pi h^2\sec\alpha\left(2\tan\alpha\sec\alpha + \sec^2\alpha + \tan^2\alpha\right)\delta\alpha, \qquad \text{(i)}$$

which gives the error in A.

Putting $\alpha = \frac{\pi}{6}$ and $\delta h = 1\%$ of $h = \frac{h}{100}$ in (i), we have

$$\delta A = 2\pi h.\frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{3}} + \frac{2}{\sqrt{3}}\right).\frac{h}{100} + \pi h^2.\frac{2}{\sqrt{3}}\left(\frac{2}{\sqrt{3}}.\frac{2}{\sqrt{3}} + \frac{4}{4} + \frac{1}{3}\right)\delta\alpha = \frac{2\pi h^2}{100} + 2\sqrt{3}\pi h^2\delta\alpha.$$

Since the error in h is to be compensated by the error in α $\Rightarrow \delta A = 0$

$$\Rightarrow \frac{1}{100} + \sqrt{3}\,\delta\alpha = 0 \Rightarrow \delta\alpha = -\frac{1}{100\sqrt{3}} \text{ radians}$$

$$\Rightarrow \delta\alpha = -\frac{0.01}{1.732}\times 57.3 \text{ degree} \qquad \left[\because 1 \text{ radian} = 57.3^o \text{ nearly}\right]$$

$$= -0.33 \text{ degree.}$$

Q.No.29.: At a distance of 30 meter from the foot of the tower the elevation of its top is

$30°$. If the possible error in measuring the distance and elevation are 2cm. and 0.05degrees. Find the approximate error in calculating the height.

Sol.: $h = x\tan\alpha$.

Taking log on both sides, we get

$\log h = \log x + \log\tan x$

Differentiating, we get

$$\frac{\delta h}{h} = \frac{\delta x}{x} + \frac{\sec^2\alpha}{\tan\alpha}.\delta x \Rightarrow \delta h = \frac{h}{x}\delta x + h\frac{\sec^2\alpha}{\tan\alpha}.\delta x = \tan\alpha.\delta x + x\sec^2\alpha.\delta\alpha$$

Given $\delta x = 0.02$, $\delta\alpha = 0.05° = 0.05.\frac{\pi}{180}$ rad.

$$\delta h = \tan 30°(0.02) + 30.\sec^2 30°\left(0.05.\frac{\pi}{180}\right) = 0.0464\,\text{m} = 4.64\text{ cm. Ans.}$$

Q.No.30.: Find the %age error in the area of an ellipse if one % error is made in measuring the major and minor axes.

Sol.: Area of an ellipse $A = \pi ab$.

Taking log on both sides, we get

$\log A = \log\pi + \log a + \log b$

Differentiating, we get

$$\frac{\delta A}{A} = \frac{\delta a}{a} + \frac{\delta b}{b} \Rightarrow \frac{\delta A}{A}\times 100 = \frac{\delta a}{a}\times 100 + \frac{\delta b}{b}\times 100 = 1\% + 1\% = 2\%.$$

$\therefore$ %age error in area of an ellipse = 2%. Ans.

Q.No.31.: Two sides a, b of a triangle and included angle C are measured. Show that the error δc in the computed length of third side c due to a small error in the angle C is given by $\delta c = a\sin B\delta C$.

Sol.: Given $c^2 = a^2 + b^2 - 2ab\cos C$

Differentiating, we get

$$2c\delta c = 2a\delta a + 2b\delta b - 2[\delta a.b\cos C + a\delta b\cos C + ab(-\sin C)\delta C]$$

As $\delta a = \delta b = 0$

$\therefore\ 2c\delta c = -2[ab(-\sin C)\delta C] \Rightarrow c\delta c = ab\sin C.\delta C$

By sin law in ΔABC, $\quad b\sin C = c\sin B$

$\therefore\ c\delta c = ac\sin B.\delta C \Rightarrow \delta c = a\delta C.\sin B$.

Hence this proves the result.

Q.No.32.: Let $T = 2\pi\sqrt{\frac{\ell}{g}}$. Find the maximum %age error in T due to possible error of 1% in ℓ and g respectively.

Sol.: Given $T = 2\pi\sqrt{\frac{\ell}{g}}$.

Taking log on both sides, we get

$$\log T = \log 2\pi + \frac{1}{2}\log \ell - \frac{1}{2}\log g.$$

Differentiating, we get

$$\frac{\delta T}{T} = 0 + \frac{1}{2}\frac{\delta \ell}{\ell} - \frac{1}{2}\frac{\delta g}{g} = \frac{1}{2}\left(\frac{\delta \ell}{\ell} - \frac{\delta g}{g}\right)$$

$$\frac{\delta T}{T} \times 100 = \%\text{age error in } T = \frac{1}{2}\left(\frac{\delta \ell}{\ell} - \frac{\delta g}{g}\right) \times 100.$$

Maximum %age error in $T = \frac{1}{2}\left(\pm\frac{\delta \ell}{\ell} \pm \frac{\delta g}{g}\right).100 = \frac{1}{2}(\pm 1 \pm 2) = \pm 1.5\%$. Ans.

Q.No.33.: Let $R = \frac{V^2 \sin 2\theta}{g}$, find the %age error in R due to an error of 1% in v and $\frac{1}{2}\%$ in θ.

Sol.: Given $R = \frac{V^2 \sin 2\theta}{g}$. (i)

Given $\frac{dv}{v} \times 100 = 1\%$, $\frac{d\theta}{\theta} \times 100 = \frac{1}{2}\%$

Taking log on both sides of (i), we get

$$\log R = 2\log V + \log \sin 2\theta - \log g$$

Differentiate on both sides, we get

$$\frac{dR}{R} = 2\frac{dV}{V} + \frac{\cos 2\theta}{\sin 2\theta}.2d\theta.$$

Multiplying by 100, we get

$$\frac{dR}{R}\times 100 = 2\left(\frac{dV}{V}\times 100\right) + (\theta\cot 2\theta)\left(\frac{d\theta}{\theta}\times 100\right) = 2.1 + \theta\cot 2\theta.\frac{1}{2} = 2 + \frac{\theta}{2}\cot 2\theta.\ \text{Ans.}$$

Q.No.34.: If $S = \frac{A}{A-W}$, find the maximum relative error in S and maximum error in S.

If the values of A and W are 1.1 and 0.6 respectively with possible error in 0.01 and 0.02 in A and W respectively.

Sol.: Given $S = \frac{A}{A-W}$. (i)

Now differentiating (i) w. r. t. to A, we get

$$\frac{\partial S}{\partial A} = \frac{(A-W).1 - A.1}{(A-W)^2} = \frac{A-W-A}{(A-W)^2} = \frac{-W}{(A-W)^2} \qquad \text{(ii)}$$

Differentiating (i) w. r. t. W, we get

$$\frac{\partial S}{\partial W} = \frac{(A-W).0 - A.(-1)}{(A-W)^2} = \frac{A}{(A-W)^2} \qquad \text{(iii)}$$

We know that $dS = \frac{\partial S}{\partial A}dA + \frac{\partial S}{\partial W}dW$ (iv)

Now putting the values of (ii) and (iii) in (iv), we get

$$dS = \frac{-W}{(A-W)^2}dA + \frac{A}{(A-W)^2}dW$$

Now given $A = 1.1$, $W = 0.6$, $dA = 0.01$, $dW = 0.02$.

Maximum error in S=

$$dS = \frac{.6}{(1.1-0.6)^2}\times 0.01 + \frac{1.1}{(1.1-0.6)^2}\times 0.02 = \frac{0.6}{0.25}\times 0.01 + \frac{1.1}{0.25}\times 0.02$$

$$= 0.024 + 0.088 = 0.112.$$

$\therefore$ Maximum error in S = 0.112. Ans.

Maximum relative error in S is given by $\frac{dS}{S}$.

Now $S = \frac{A}{A-W} = \frac{1.1}{1.1-0.5} = \frac{1.1}{0.5} = 2.2$.

$$\frac{dS}{S} = \frac{0.112}{2.2} = 0.0509 = 0.51$$

$\therefore$ Maximum relative error in S = 0.51. Ans.

Q.No.35.: Use differentials to compute $f(0.9,\ -1.2)$ approximately where

$$f(x, y) = \tan^{-1}(xy).$$

Sol.: Given $f(0.9,\ -1.2)$.

Let $x = 1,\quad \delta x = -0.1,$

$y = -1,\ \delta y = -0.2.$

$\therefore\ f(x, y) = \tan^{-1}(-1)$

Let $f(x, y) = \theta,\quad \therefore \theta = \dfrac{3\pi}{4}.$

Now $\tan^{-1}(xy) = \theta = \dfrac{3\pi}{4}.$

Taking log on both sides, we get $\quad \log\theta = \log\tan^{-1}(xy).$

Differentiating, we get

$$\frac{\delta\theta}{\theta} = \frac{1}{\tan^{-1}(xy)}\frac{x\delta y + y\delta x}{1 + x^2y^2} = \frac{1}{\tan^{-1}(xy)}\frac{-0.2 + 0.1}{1+1} = \frac{1}{\theta}\frac{-0.2+0.1}{1+1} \Rightarrow \delta\theta = -\frac{0.1}{2} = -0.05.$$

$\therefore f(0.9, -1.2) = f(x + \delta x, y + \delta y) = \theta + \delta\theta = \dfrac{3\pi}{4} - 0.05 = 2.307$. Ans.

Q.No.36.: If the sides and angles of a triangle ABC vary in such a manner that its circum-radius remains constant, prove that $\dfrac{da}{\cos A} + \dfrac{db}{\cos B} + \dfrac{dc}{\cos C} = 0.$

Sol.: To prove: $\dfrac{da}{\cos A} + \dfrac{db}{\cos B} + \dfrac{dc}{\cos C} = 0.$

We know that, the circum-radius R of a ΔABC is given by

$$R = \frac{a}{2\sin A} = \frac{b}{2\sin B} = \frac{c}{2\sin C}.$$

Now $a = 2R\sin A$. [R is constant]

Differentiating, we get $\quad da = 2R\cos A dA \Rightarrow \dfrac{da}{\cos A} = 2RdA.$

Similarly $\quad db = 2R\cos B dB \Rightarrow \dfrac{db}{\cos B} = 2RdB.$

$\quad dc = 2R\cos C dC \Rightarrow \dfrac{dc}{\cos C} = 2RdC.$

Adding, we get $\frac{da}{\cos A}+\frac{db}{\cos B}+\frac{dc}{\cos C}=2R(dA+dB+dC)=2Rd(A+B+C)$ (i)

Also $A+B+C=\pi$. $\Rightarrow d(A+B+C)=0$.

$\therefore$ From (i), we get $\frac{da}{\cos A}+\frac{db}{\cos B}+\frac{dc}{\cos C}=0$.

This completes the proof.

Q.No.37.: The area of a rectangle is found from measurements of side a and angle B and C. Prove that error in the calculated value of area due to small error $\delta a,\ \delta B,\ \delta C$ is given by $\left(\frac{2}{a}\delta a+\frac{c}{a}\frac{\delta B}{\sin B}+\frac{b}{a}\frac{\delta C}{\sin C}\right)\Delta$.

Sol.: We know that area $\Delta=\frac{1}{2}ab\sin C$ (i)

Now $\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}\Rightarrow b=\frac{a\sin B}{\sin A}$

Putting in (i), we get

$$\Delta=\frac{1}{2}(a)\left(\frac{a\sin B}{\sin A}\right)\sin C=\frac{1}{2}a^2\frac{\sin B\sin C}{\sin[\pi-(B+C)]} \qquad \text{(ii)} \quad [\because A+B+C=\pi]$$

Taking log of (ii) on both sides, we get

$$\log\Delta=\log\frac{1}{2}+2\log a+\log\sin B+\log\sin C-\log\sin[(B+C)] \qquad \text{(iii)}$$

$$[\because \sin[\pi-(B+C)]=\sin(B+C)]$$

Differentiating (iii), we get

$$\frac{\delta\Delta}{\Delta}=\frac{2\delta a}{a}+\frac{\cos B}{\sin B}\delta B+\frac{\cos C}{\sin C}\delta C-\frac{\cos(B+C)}{\sin(B+C)}(\delta B+\delta C)$$

$$=\frac{2\delta a}{a}+\delta B\left[\frac{\cos B}{\sin B}-\frac{\cos(B+C)}{\sin(B+C)}\right]+\delta C\left[\frac{\cos C}{\sin C}-\frac{\cos(B+C)}{\sin(B+C)}\right]$$

$$=\frac{2\delta a}{a}+\delta B\left[\frac{\cos B\sin(B+C)-\sin B\cos(B+C)}{\sin(B+C)\sin B}\right]$$

$$+\delta C\left[\frac{\cos C\sin(B+C)-\sin C\cos(B+C)}{\sin(B+C)\sin C}\right]$$

$$= \frac{2\delta a}{a} + \delta B \frac{\sin C}{\sin B \sin(B+C)} + \frac{\delta C}{\sin C} \cdot \frac{\sin B}{\sin(B+C)} \qquad \text{(iv)}$$

$A + B + C = \pi \Rightarrow B + C = \pi - A \Rightarrow \sin(B+C) = \sin(\pi - A)$

Putting this value in (iv), we get

$$\frac{\delta\Delta}{\Delta} = \frac{2\delta a}{a} + \frac{\delta B}{\sin B} \frac{\sin C}{\sin A} + \frac{\delta C}{\sin C} \cdot \frac{\sin B}{\sin A}. \qquad \text{(v)}$$

According to sine formula: $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$.

$\Rightarrow \frac{c}{a} = \frac{\sin C}{\sin A}, \quad \frac{b}{a} = \frac{\sin B}{\sin A}$

Putting these values in (v), we get

$$\frac{\delta\Delta}{\Delta} = \frac{2\delta a}{a} + \frac{c}{a} \frac{\delta B}{\sin B} + \frac{b}{a} \frac{\delta C}{\sin C}$$

$\delta\Delta = \left(\frac{2}{a}\delta a + \frac{c}{a}\frac{\delta B}{\sin B} + \frac{b}{a}\frac{\delta C}{\sin C}\right)\Delta$, which is the required proof.

Q.No.38.: In a plane triangle, if the sides a, b be constant, prove that the variations of its angles are given by the relations

$$\frac{dA}{\sqrt{a^2 - b^2 \sin^2 A}} = \frac{dB}{\sqrt{b^2 - a^2 \sin^2 B}} = -\frac{dC}{c},$$

the letters have their usual significance.

Sol.: By sine formula, $\frac{a}{\sin A} = \frac{b}{\sin B} \Rightarrow a \sin B = b \sin A \qquad \text{(i)}$

Taking differentials, we get $a \cos B . dB = b \cos A . dA$

$\Rightarrow \frac{dA}{a\cos B} = \frac{dB}{b \cos A} = \frac{dA + dB}{a\cos B + b\cos A}. \qquad \text{(ii)} \quad \left[\because \text{If } \frac{a}{b} = \frac{c}{d} \text{ then each} = \frac{a+c}{b+d}\right]$

Now $a\cos B = a\sqrt{1 - \sin^2 B} = \sqrt{a^2 - a^2 \sin^2 B} = \sqrt{a^2 - b^2 \sin^2 A}$ [using (i)]

$b\cos A = b\sqrt{1 - \sin^2 A} = \sqrt{b^2 - b^2 \sin^2 A} = \sqrt{b^2 - a^2 \sin^2 B}$ [using (i)]

$a\cos B + b\cos A = c$ [By projection formula]

Also $A + B + C = \pi \Rightarrow A + B = \pi - C$ so that $dA + dB = -dC$

$\therefore$ From (ii), $\frac{dA}{\sqrt{a^2 - b^2 \sin^2 A}} = \frac{dB}{\sqrt{b^2 - a^2 \sin^2 B}} = -\frac{dC}{c}$.

Q.No.39.: If there is a small error δc in measuring the side c in a triangle, show that relative error in the area of the triangle is equal to

$$\left[\frac{1}{s}+\frac{1}{s-a}+\frac{1}{s-b}-\frac{1}{s-c}\right]\frac{\delta c}{4}.$$

or

If A be the area of a triangle, prove that the error in A resulting from a small error in 'c' is given by

$$\delta A=\frac{A}{4}\left[s^{-1}+(s-a)^{-1}+(s-b)^{-1}-(s-c)^{-1}\right]\delta c.$$

Sol.: Let δA is the error in A, then relative error in A $=\dfrac{\delta A}{A}$.

Now to prove: $\dfrac{\delta A}{A}=\left[\dfrac{1}{s}+\dfrac{1}{s-a}+\dfrac{1}{s-b}-\dfrac{1}{s-c}\right]\dfrac{\delta c}{4}$.

or $\qquad \delta A=\dfrac{A}{4}\left[s^{-1}+(s-a)^{-1}+(s-b)^{-1}-(s-c)^{-1}\right]\delta c$

Since, we know that $A=\sqrt{s(s-a)(s-b)(s-c)}$.

Taking log on both sides, we get

$$\log A=\frac{1}{2}\log s+\frac{1}{2}\log(s-a)+\frac{1}{2}\log(s-b)+\frac{1}{2}\log(s-c).$$

Differentiating on both sides, we get

$$\frac{\delta A}{A}=\frac{1}{2}\left[\frac{\delta s}{s}+\frac{\delta s}{s-a}+\frac{\delta s}{s-b}+\frac{\delta s-\delta c}{s-c}\right]. \qquad \text{(i)}$$

Since $s=\dfrac{a+b+c}{2}\Rightarrow \delta s=\dfrac{\delta c}{2}$.

Putting in (i), we get

$$\frac{\delta A}{A}=\frac{1}{4}\left[\frac{\delta c}{s}+\frac{\delta c}{s-a}+\frac{\delta c}{s-b}-\frac{\delta c}{s-c}\right]=\frac{\delta c}{4}\left[\frac{1}{s}+\frac{1}{s-a}+\frac{1}{s-b}-\frac{1}{s-c}\right]$$

$$\Rightarrow \delta A=\frac{A}{4}\left[s^{-1}+(s-a)^{-1}+(s-b)^{-1}-(s-c)^{-1}\right]\delta c.$$

Hence this proves the result.

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Differential Calculus

Taylor's and Maclaurin's Infinite Series

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Maclaurin's Theorem:

Statement: If f(x) can be expanded in ascending powers of x, then

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots\ldots + \frac{x^n}{n!}f^n(0) + \ldots\ldots..$$

Proof: Suppose $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots\ldots\ldots + a_nx^n + \ldots\ldots\ldots$ (i)

where a_0, a_1, a_2, a_3, ... are constants to be evaluated.

Differentiating (i) w. r. t. x, we get

$$f'(x) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \ldots\ldots + na_nx^{n-1} + \ldots\ldots \quad \text{(ii)}$$

Differentiating (ii) w. r. t. x, we get

$$f''(x) = 2a_2 + 3.2a_3x + 4.3a_4x^2 + \ldots\ldots + n(n-1)a_nx^{n-2} + \ldots\ldots \quad \text{(iii)}$$

Differentiating (iii) w. r. t. x, we get

$$f'''(x) = 3.2.1a_3 + 4.3.2a_4x + \ldots\ldots + n(n-1)(n-2)a_nx^{n-3} + \ldots\ldots \quad \text{(iv)}$$

Similarly, if we go on differentiating, we get

$$f^n(x) = n(n-1)(n-2)\ldots\ldots 3.2.1.a_n + \text{ terms containing x} \quad \text{(v)}$$

Putting x = 0 in (i) to (v) , we get

$$a_0 = f(0),\ a_1 = f'(0),\ a_2 = \frac{f''(0)}{2!},\ a_3 = \frac{f'''(0)}{3!}, \ldots\ldots..,\ a_n = \frac{f^n(0)}{n!}.$$

Putting these values of constants in (i), we get

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots\ldots + \frac{x^n}{n!}f^n(0) + \ldots\ldots\ldots\ldots .$$

This completes the proof**.**

Taylor's Theorem:

Statement: If f (x + h) can be expanded in ascending powers of x, then

$$f(x+h) = f(h) + xf'(h) + \frac{x^2}{2!}f''(h) + \frac{x^3}{3!}f'''(h) + \ldots\ldots + \frac{x^n}{n!}f^n(h) + \ldots\ldots\ldots\ldots .$$

Proof: Suppose $f(x+h) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots\ldots + a_nx^n + \ldots\ldots$ (i)

where a_0, a_1, a_2, a_3, ... are constants to be evaluated.

Differentiating (i) w.r.t. x, we get

$$f'(x+h) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \ldots\ldots + na_nx^{n-1} + \ldots\ldots \quad \text{(ii)}$$

Differentiating (ii) w.r.t. x, we get

$$f''(x+h) = 2a_2 + 3.2a_3x + 4.3a_4x^2 + \ldots\ldots + n(n-1)a_nx^{n-2} + \ldots\ldots \quad \text{(iii)}$$

Differentiating (iii) w.r.t. x, we get

$$f'''(x+h) = 3.2.1a_3 + 4.3.2a_4x + \ldots\ldots + n(n-1)(n-2)a_nx^{n-3} + \ldots\ldots \quad \text{(iv)}$$

Similarly, if we go on differentiating, we get

$$f^n(x+h) = n(n-1)(n-2)\ldots\ldots 3.2.1.a_n + \text{ terms containing x} \quad \text{(v)}$$

Putting x = 0 in (i) to (v), we get

$$a_0 = f(h),\ a_1 = f'(h),\ a_2 = \frac{f''(h)}{2!},\ a_3 = \frac{f'''(h)}{3!},\ldots\ldots,\ a_n = \frac{f^n(h)}{n!}.$$

Putting these values of constants in (i), we get

$$f(x+h) = f(h) + xf'(h) + \frac{x^2}{2!}f''(h) + \frac{x^3}{3!}f'''(h) + \ldots\ldots + \frac{x^n}{n!}f^n(h) + \ldots\ldots\ldots\ldots .$$

This completes the proof.

Remarks:

(i) Put h = 0, we get

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots\ldots + \frac{x^n}{n!}f^n(0) + \ldots\ldots\ldots\ldots .$$

$$y = y(0) + xy_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + \ldots\ldots + \frac{x^n}{n!}y_n(0) + \ldots\ldots\ldots\ .$$

which is a Maclaurin's expansion.

(ii) If we interchange x and h, we get

$$f(h+x) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \ldots\ldots + \frac{h^n}{n!}f^n(x) + \ldots\ldots\ldots\ .$$

(iii) If we replace x by a and h by (x - a), we get

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \ldots\ldots + \frac{(x-a)^n}{n!}f^n(a) + \ldots\ldots$$

Taylor's Theorem for functions of two variables:

Statement: Prove that

$$f(x+h, y+k) = f(x,y) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)f + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f + \ldots\ldots\ .$$

Proof: Considering $f(x+h, y+k)$ as a function of a single variable x, we have

$$f(x+h, y+k) = f(x, y+k) + h\frac{\partial f(x, y+k)}{\partial x} + \frac{h^2}{2!}\frac{\partial^2 f(x, y+k)}{\partial x^2} + \ldots\ldots \qquad \ldots(i)$$

Now expanding $f(x, y+k)$ as a function of y only, we obtain

$$f(x, y+k) = f(x,y) + k\frac{\partial f(x,y)}{\partial y} + \frac{k^2}{2!}\frac{\partial^2 f(x,y)}{\partial y^2} + \ldots\ldots \qquad \ldots(ii)$$

$\therefore$(i) takes the form

$$f(x+h, y+k) = f(x,y) + k\frac{\partial f(x,y)}{\partial y} + \frac{k^2}{2!}\frac{\partial^2 f(x,y)}{\partial y^2} + \ldots\ldots$$

$$+ h\frac{\partial}{\partial x}\left\{f(x,y) + k\frac{\partial f(x,y)}{\partial y} + \frac{k^2}{2!}\frac{\partial^2 f(x,y)}{\partial y^2} + \ldots\ldots\right\}$$

$$+ \frac{h^2}{2!}\frac{\partial^2}{\partial x^2}\left\{f(x,y) + k\frac{\partial f(x,y)}{\partial y} + \frac{k^2}{2!}\frac{\partial^2 f(x,y)}{\partial y^2} + \ldots\ldots\right\}$$

Hence $f(x+h, y+k) = f(x,y) + h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} + \frac{1}{2!}\left(h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right) + \ldots\ldots\ .$

In symbols we write it as

$$f(x+h, y+k) = f(x,y) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)f + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f + \ldots\ldots .$$

Now let us expand some well known functions by using these theorems:

Q.No.1.: Prove that $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots + \frac{x^n}{n!} + \ldots\ldots.$

Sol.: Let $f(x) = e^x$, $\therefore f'(x) = e^x$, $f''(x) = e^x$, ..., $f^n(x) = e^x$

$\therefore f(0) = 1$, $\therefore f'(0) = 1$, $f''(0) = 1$, ..., $f^n(0) = 1$

Maclaurin's expansion is $f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots\ldots + \frac{x^n}{n!}f^n(0)\ldots\ldots$

Substituting all the above values in this equation, we get

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots + \frac{x^n}{n!} + \ldots\ldots$$

Note: If we replace x by –x, we get

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots\ldots\ldots\ldots\ldots.$$

Q.No.2.: Prove that $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots\ldots.$

Sol.: Since $\cosh x = \frac{1}{2}\left(e^x + e^{-x}\right)$

$$= \frac{1}{2}\left[\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots\right) + \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots\ldots\right)\right]$$

Hence, $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots\ldots.$

Similarly, $\sinh x = \frac{1}{2}\left(e^x - e^{-x}\right) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots\ldots.$

Q.No.3.: Prove that $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\ldots.$

Sol.: Let $y = \sin x \Rightarrow y_n = \sin\left(x + \frac{n\pi}{2}\right)$

$\therefore y(0) = 0$, $y_1(0) = 1$, $y_2(0) = 0$, $y_3(0) = -1$, $y_n(0) = 0$ and so on

Now Maclaurin's series is

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \dots\dots..$$

$$= y(0) + xy_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + \dots\dots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots\dots..$$

Similarly, $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\dots..$

Q.No.4.: Prove that $\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \dots\dots..$

Sol.: Let $y = f(x) = \tan x$.

$\therefore y_1 = f'(x) = \sec^2 x = 1 + \tan^2 x$.

$y_2 = f''(x) = 2\tan x.\sec^2 x = 2\tan x\left(1 + \tan^2 x\right) = 2\tan x + 2\tan^3 x$

$y_3 = f'''(x) = 2\sec^2 x + 6\tan^2 x \sec^2 x = 2\sec^2 x\left(1 + 3\tan^2 x\right)$

$= 2\left(1 + \tan^2 x\right)\left(1 + 3\tan^2 x\right) = 2\left(1 + 4\tan^2 x + 3\tan^4 x\right).$

$y_4 = f^{iv}(x) = 16\tan x + 40\tan^3 x + 24\tan^5 x$.

$y_5 = f^{v}(x) = 16\left(1 + \tan^2 x\right) + 120\tan^2 x\left(1 + \tan^2 x\right) + 120\tan^4 x\left(1 + \tan^2 x\right).$

Putting $x = 0$, we get

$y(0) = 0$, $y_1(0) = 1$, $y_2(0) = 0$, $y_3(0) = 2$, $y_4(0) = 0$, $y_5(0) = 16$ and so on.

Now, Maclaurin's series is

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{iv}(0) + \frac{x^5}{5!}f^{v}(0)\dots\dots\dots..$$

$$\Rightarrow \tan x = x + \frac{x^3}{3!}.2 + \frac{x^5}{5!}.16 + \dots\dots\dots.$$

$$\Rightarrow \tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \dots\dots..$$

Q.No.5.: Prove that $\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \dots\dots\dots..$

Sol.: $y = \log(1+x)$

$$\therefore y_1 = \frac{1}{(1+x)} = (1+x)^{-1}, \quad y_2 = (-1)(1+x)^{-2}, \quad y_3 = (-1)(-2)(1+x)^{-3}$$

$$y_4 = (-1)(-2)(-3)(1+x)^{-4}, \quad y_5 = (-1)(-2)(-3)(-4)(1+x)^{-5}$$

Putting $x = 0$, we get

$$y(0) = 0, \quad y_1(0) = 1, \quad y_2(0) = -1, \quad y_3(0) = 2, \quad y_4(0) = -6, \quad y_5(0) = 24, \ldots\ldots\ldots$$

Now, Maclaurin's series is

$$f(x) = y(0) + xy_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + \frac{x^4}{4!}y_4(0) + \frac{x^5}{5!}y_5(0)\ldots\ldots\ldots$$

$$= x - \frac{x^2}{2!} + \frac{x^3}{3!}.2 - \frac{x^4}{4!}.6 + \frac{x^5}{5!}.24\ldots\ldots\ldots$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}\ldots\ldots\ldots\ldots.$$

Q.No.6.: Expand $\tan^{-1} x$ in powers of x by Maclaurin's theorem**.**

Sol.: Here $f(x) = \tan^{-1} x$

$$\therefore f'(x) = \frac{1}{1+x^2} = \left(1+x^2\right)^{-1} = 1 - x^2 + x^4 - x^6 + x^8 \ldots\ldots\ldots \text{[by Binomial theorem]}$$

$$f''(x) = -2x + 4x^3 - 6x^5 + 8x^7\ldots\ldots\ldots\ldots$$

$$f'''(x) = -2 + 12x^2 - 30x^4 + 56x^6\ldots\ldots\ldots\ldots$$

$$f^{iv}(x) = 24x - 120x^3 + \ldots\ldots\ldots\ldots$$

$$f^{v}(x) = 24 - 360x^2 + \ldots\ldots\ldots\ldots$$

Putting $x = 0$, we get

$$f(0) = 0, \quad f'(0) = 1, \quad f''(0) = 0, \quad f'''(0) = -2, \quad f^{iv}(0) = 0, \quad f^{v}(0) = 24.$$

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{iv}(0) + \frac{x^5}{5!}f^{v}(0)\ldots\ldots\ldots$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5}\ldots\ldots\ldots.$$

Q.No.7.: Prove that $e^{x \sin x} = 1 + x^2 + \frac{x^4}{3} + \frac{x^6}{120} + \ldots\ldots\ldots.$

or

Find the expansion of $e^{x\sin x}$ as far as the terms in x^6.

Sol.: Here if we use Maclaurin's theorem, we need successive derivatives of $e^{x\sin x}$, which is inconvenient to obtain. We, therefore, make use of the two standard series.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} ...., \quad \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...$$

$$e^{x\sin x} = 1 + (x\sin x) + \frac{(x\sin x)^2}{2!} + \frac{(x\sin x)^3}{3!} + \frac{(x\sin x)^4}{4!} + ...$$

$$= 1 + x\sin x + \frac{x^2}{2!}(\sin x)^2 + \frac{x^3}{3!}(\sin x)^3 + \frac{x^4}{4!}(\sin x)^4 + ...$$

$$= 1 + x\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...\right) + \frac{x^2}{2!}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...\right)^2$$

$$+ \frac{x^3}{3!}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...\right)^3 + \frac{x^4}{4!}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...\right)^4 + ...$$

$$= 1 + x^2\left(1 - \frac{x^2}{6} + \frac{x^4}{120}.....\right) + \frac{x^4}{2}\left(1 - \frac{x^2}{6} + \frac{x^4}{120}.....\right)^2 + \frac{x^6}{6}\left(1 - \frac{x^2}{6} + \frac{x^4}{120}.....\right)^3$$

$$+ \frac{x^8}{24}\left(1 - \frac{x^2}{6} + \frac{x^4}{120}.....\right)^4 + ...$$

Now expanding by binomial theorem, we get

$$= 1 + x^2\left(1 - \frac{x^2}{6} + \frac{x^4}{120}.....\right) + \frac{x^4}{2}\left[1 - 2\left(\frac{x^2}{6} - \frac{x^4}{120}.....\right)......\right] + \frac{x^6}{6}\left[1 - 3\left(\frac{x^2}{6} - \frac{x^4}{120}.....\right) + ...\right] + ...$$

Collecting terms of the same nature, we get

$$e^{x\sin x} = 1 + x^2 + \left(-\frac{1}{6} + \frac{1}{2}\right)x^4 + \left(\frac{1}{120} - \frac{1}{6} + \frac{1}{6}\right)x^6 + ...$$

$$e^{x\sin x} = 1 + x^2 + \frac{x^4}{3} + \frac{x^6}{120} + ...$$

Q.No.8.: Apply Maclaurin's theorem to find the expansion of $\frac{e^x}{e^x + 1}$ as far as the term in x^3.

Sol.: Let $f(x)=\dfrac{e^x}{e^x+1}$, $f(0)=\dfrac{1}{2}$

$$f'(x)=\frac{\left(e^x+1\right)e^x-e^x.e^x}{\left(e^x+1\right)}=\frac{e^x}{\left(e^x+1\right)^2},\quad \therefore f'(0)=\frac{1}{4}$$

$$f''(x)=\frac{\left(e^x+1\right)e^x-e^x.2\left(e^x+1\right)e^x}{\left[\left(e^x+1^2\right)\right]^2}=\frac{\left(e^x+1\right)e^x\left[\left(1-e^x\right)\right]e^x-2e^{2x}}{\left(e^x+1\right)^4}=\frac{e^x+e^{2x}-2e^{2x}}{\left(e^x+1\right)^3}$$

$$=\frac{\left(e^x\,''\,e^{2x}\right)}{\left(e^x+1\right)^3},\quad \therefore f''(0)=0$$

$$f'''(x)=\frac{\left(e^x+1\right)^3\left(e^x-2e^2x\right)-\left(e^x-e^{2x}\right)3\left(e^x+1\right)^3.e^x}{\left[\left(e^x+1\right)^3\right]^2},\quad \therefore f'''(0)=-\frac{1}{8}$$

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\ldots\ldots\ldots\ldots\ldots$$

$$\therefore \frac{e^x}{e^x+1}=\frac{1}{2}+\frac{1}{4}x-\frac{x^3}{48}+\ldots\ldots\ldots$$

Q.No.9.: Prove that $\log(\sec x)=\dfrac{x^2}{2}+\dfrac{1}{3}.\dfrac{x^4}{4}+\dfrac{2}{15}.\dfrac{x^6}{16}+\ldots\ldots\ldots$.

Sol.: Let $y=\log\sec x$, then

$$\frac{dy}{dx}=\frac{1}{\sec x}.\frac{\sec x\tan x}{1}=\tan x$$

But $\tan x=x+\dfrac{x^3}{3}+\dfrac{2}{15}x^5+\ldots\ldots\ldots$

$$\therefore \frac{dy}{dx}=x+\frac{x^3}{3}+\frac{2}{15}x^5+\ldots\ldots\ldots$$

Integrating w. r. t. x , we get

$$y=\frac{x^2}{2}+\frac{x^4}{12}+\frac{2}{15}.\frac{x^6}{6}+\ldots\ldots\ldots+a_0$$

where a_0 is the constant of integration

To evaluate a_0, But $x=0$, $y=0$, and $\therefore a_0=0$

Hence $\log(\sec x) = \frac{x^2}{2} + \frac{1}{3}.\frac{x^4}{4} + \frac{2}{15}.\frac{x^6}{16} + ...$

Remarks: $\log\cos x = -\log\sec x = -\left[\frac{x^2}{2} + \frac{1}{3}.\frac{x^4}{4} + \frac{2}{15}.\frac{x^6}{16} + ...\right]$

Q.No.10.: If $x^3 + 2xy^2 - y^3 + x = 1$, expand y in the ascending powers of x.

Sol.: Here y is implicit function of x and so to get an expansion for y, we shall obtain first $(y)_0$, $(y')_0$, $(y'')_0$, etc. by a little different method and use Maclaurin's theorem.

Here $x^3 + 2xy^2 - y^3 + x = 0$ (i)

Put x = 0, we get $-(y)^3{}_0 1 = 0 \quad \therefore (y)_0 = -1$ (ii)

Differentiating (i) w. r. t. x , we get

$3x^2 + 2y^2 4xy\,y' - 3y^2 y' + 1 = 0$ (iii)

Put x = 0 and $(y)_0 = -1$ in (iii), we get

$2 - 3(y')_0 + 1 = 0 \quad \therefore (y')_0 = 1$ (iv)

Differentiating (iii) w. r. t. x, we get

$6x + 8yy' 4x\,y'^2 - 4xyy'' - 6yy'^2 - 3y^2 y^{2^2} = 0$ (v)

Putting x = 0, $\therefore (y)_0 = -1, (y')_0 = 1$ in (v), we get

$-8 + 6 - 3(y'')_0 = 0 \quad \therefore (y'')_0 = -\frac{2}{3}$ (vi)

Now by Maclaurin's theorem

$$y = (y)_0 + x(y')_0 + \frac{x^2}{2!}(y'')_0 + \frac{x^3}{3!}(y''')_0 + ...$$

$$y = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + ...$$

Substituting the values of

$(y)_0 = -1$, $(y')_0 = 1$, $(y'')_0 = -\frac{2}{3}$ from (ii), (iv) and (vi) etc.

Then the required expansion of y is $y = 1 + x - \frac{x^2}{2!}.\frac{2}{3}.........$

Hence $y = 1 + x - \frac{1}{3}x^2.........$

Q.No.11.: Expand $\log\cos(x+h)$ in powers of h by Taylor's theorem**.**

Sol.: Let $f(x+h) = \log\cos(x+h)$

$\therefore\ f(x) = \log\cos x$

$f'(x) = \frac{1}{\cos x}(-\sin x) = -\tan x$

$f''(x) = -\sec^2 x$

$f'''(x) = -2\sec x.\sec x\tan x = -2\sec^2 x\tan x$

Now by Taylor's series

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{x^3}{3!}f'''(x) + ...$$

Substituting the values of $f(x)$, $f'(x)$, $f''(x)$, $f'''(x)$in the above equation, we get

$$\log\cos(x+h) = \log\cos x + h(-\tan x) + \frac{h^2}{2!}\left(-\sec^2 x\right) + \frac{h^3}{3!}\left(-2\sec^2 x\tan x\right).........$$

$$\log\cos(x+h) = \log\cos x - h\tan x - \frac{h^2}{2}\sec^2 - \frac{h^3}{3}\sec^2 x\tan x + ...$$ **. Ans.**

Q.No.12.: Use Taylor's theorem to express the polynomial $2x^3 + 7x^2 + x + 6$ in powers of $(x-2)$**.**

Sol.: Let $f(x) = f[2 + (x-2)]$, using Taylor's theorem, we get

$$f[2+(x-2)] = f(2) + (x-2)f'(2) + \frac{(x-2)^2}{2!}f''(2) + \frac{(x-2)^3}{3!}f'''(2) + ...$$

Now $f\ (x) = 2x^3 + 7x^2 + x + 6$

$f'\ (x) = 6x^2 + 14x + 1$

$f''\ (x) = 12x + 14$

$f'''\ (x) = 12$

$f(x) = 52,\ f'(2) = 53,\ f''(2) = 38,\ f'''(2) = 12,\ f^{iv}(2) = 0$

$$\therefore f(x) = f[2+(x-2)] = f(2) + (x-2)f'(2) + \frac{(x-2)^2}{2!}f''(2) + \frac{(x-2)^3}{3!}f'''(2) + ...$$

$\therefore 2x^3 + 7x^2 + x + 6 = 52 + 53(x-2) + 19(x-2)^2 + 2(x-2)^3$.

This is the required result.

Q.No.13.: Calculate the approximate value of $\sqrt{10}$ to four decimal places using Taylor's expansion.

Sol.: Let $f(x+h) = \sqrt{x+1} = \sqrt{9+1} = \sqrt{10}$

where x = 9 and h = 1

$\therefore f(x) = \sqrt{x} \qquad \therefore f(9) = 3$

$f'(x) = \frac{1}{2}x^{-1/2} \qquad f'(9) = \frac{1}{6}$

$f''(x) = \frac{1}{4}x^{3/2} \qquad f''(9) = \frac{1}{4\times 27}$

$f'''(x) = \frac{3}{8}x^{-5/2} \; f'''(9) = \frac{1}{8\times 81}$

Now Taylor's series is

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \ldots\ldots\ldots\ldots$$

$$\sqrt{10} = f(9) + 1\times f'(9) + \frac{1}{2}f''(9) + \frac{1}{6}f'''(9) + \ldots\ldots\ldots\ldots$$

or $\sqrt{10} = 3 + \frac{1}{6} - \frac{1}{2}\times\frac{1}{4}\times\frac{1}{27} + \frac{1}{6}\times\frac{1}{8\times 81} + \ldots\ldots\ldots\ldots$

$= 3 + .16666 - .00463 + .00025$

$= 3.1623$ (app.).

This is the required result.

Q.No.14.: Expand $\sin(x+h)(y+k)$ in powers of h and k by Taylor's theorem.

Sol.: Here $F(x+h, y+k) = \sin(x+h)(y+k)$

$\therefore F(x, y) = \sin xy$ (i)

By Taylor's theorem

$$F(x+h, y+k) = F(x, y) + (hF_x + kF_y) + \frac{1}{2!}\left(h^2F_{xx} + 2hkF_{xy} + k^2F_{yy}\right) \qquad \text{(ii)}$$

Differentiate (i) partially w. r. t. x and y, we get

$Fx = \frac{\partial}{\partial x}(\sin xy) = y\cos xy$,

$F_{xx} = y(-\sin xy)y = -y^2 \sin xy$

$F_{xy} = \frac{\partial}{\partial x}(x\cos xy) = \cos xy + x(-\sin xy)y = \cos xy - xy\sin xy$.

Putting the values of $F(x+h, y+k)$, F_x, F_y, F_{xx}, F_{xy} in (ii), we get

$$\sin(x+h)(y+k) = \sin xy + hy\cos xy + kx\cos xy$$

$$+\frac{1}{2!}\left[h^2\left(-y^2 \sin xy\right) + \left(2hk\cos xy - xy\sin xy\right) + k^2\left(-x^2 \sin xy\right)\right] + \dots\dots\dots\dots$$

$$= \sin xy + (hy+kx)\cos xy + \frac{1}{2!}\left[2hk\cos xy - (hy+kx)^2 \sin xy\right] + \dots\dots\dots\dots \text{ Ans.}$$

Q.No.15.: Expand $e^x \sin y$ by Taylor's theorem in powers of x and y as for as terms of second degree**.**

Sol.: Here $f(x, y) = e^x \sin y$, $\quad f(0,0) = 0$

$f_x(x, y) = e^x \sin y$, $\quad f_x(0,0) = 0$

$f_y(x, y) = e^x \cos y$, $\quad f_y(0,0) = 1$

$f_{xx}(x, y) = e^x \sin y$, $\quad f_{xx}(0,0) = 0$

$f_{yy}(x, y) = -e^x \sin y$, $\quad f_{yy}(0,0) = 0$

$f_{xy}(x, y) = e^x \cos y$, $\quad f_{xy}(0,0) = 1$

$$f(x, y) = f(0,0) + xf_x(0,0) + yf_y(0,0) + \frac{x^2}{2!}f_{xx}(0,0) + xyf_{xy}(0,0) + \frac{y^2}{2!}f_{yy}(0,0) + \dots\dots$$

$$\therefore e^x \sin y = 0 + x(0) + y(1) + xy(1) + 0$$

$$e^x \sin y = y + xy + \dots\dots\dots\dots \text{ Ans.}$$

Q.No.16.: Expands $\tan^{-1}\frac{y}{x}$ in the neighbourhood of (1, 1) by Taylor's theorem**.**

Sol.: Here $f(x, y) = \tan^{-1}\frac{y}{x}$ (i)

$a = 1,\ b = 1$

$f(1,1)=\tan^{-1}\left(\frac{1}{1}\right)=\tan^{-1}(1)=\frac{\pi}{4}$, putting $x=y=1$ in (i), we get

By Taylor's theorem, we get

$$f(x, y)=f(1,1)+(x-1)f_x(1,1)+(y-1)f_y(1,1)$$
$$+\frac{1}{2!}\left[(x-1)^2 f_{xx}(1,1)+2(x-1)(y-1)f_{xy}(1,1)+(y-1)^2 f_{yy}(1,1)\right]+\ldots\ldots\ldots \quad \text{(ii)}$$

Differentiate (i) partially w. r. t. x and y, we get

$f_x=\frac{\partial}{\partial x}\tan^{-1}\left(\frac{x}{y}\right)=-\frac{y}{x^2+y^2}$ $\quad\therefore f_x(1,1)=\frac{1}{1+1}=\frac{1}{2}\ (\because x=y=1)$

$f_y=\frac{\partial}{\partial x}\tan^{-1}\left(\frac{x}{y}\right)=-\frac{x}{x^2+y^2}$ $\quad\therefore f_y(1,1)=\frac{1}{1+1}=\frac{1}{2}$

$f_{xx}=\frac{\partial}{\partial x}\left(\frac{-y}{x^2+y^2}\right)=-\frac{2xy}{x^2+y^2}$ $\quad\therefore f_{xx}(1,1)=\frac{2}{4}=\frac{1}{2}$

$f_{yy}=\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right)=-\frac{-2xy}{x^2+y^2}$ $\quad\therefore f_{yy}(1,1)=-\frac{2}{4}=-\frac{1}{2}$

$f_{xy}=\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right)$ $\quad\therefore f_{xy}(1,1)=\frac{1-1}{4}=0$

Putting the values of $f(1,1)$, $f_x(1,1)$, $f_y(1,1)$, $f_{xy}(1,1)$ etc. in (ii), we get

$$\tan^{-1}\frac{y}{x}=\frac{\pi}{4}-\frac{1}{2}(x-1)+\frac{1}{2}(y-1)+\frac{1}{2!}$$
$$+\left[(x-1)^2.\frac{1}{2}+2(x-1)(y-1).0+(y-1)^2\left(-\frac{1}{2}\right)\right]+\ldots\ldots\ldots.$$

Hence, $\tan^{-1}\frac{y}{x}=\frac{\pi}{4}-\left(\frac{x-1}{2}\right)+\left(\frac{y-1}{2}\right)+\frac{1}{2!}\left[\frac{(x-1)^2-(y-1)^2}{2}\right]+\ldots\ldots\ldots\ldots$ **Ans.**

Q.No.17.: Expand $x^2y+3y-2$ in powers of $(x-1)$ and $(y+2)$ using Taylor's Theorem.

Sol.: Expansion f(x, y) in powers of $(x-a)$ and $(x-b)$ is given by

$$f(x, y)=f(a,b)+\left[(x-a)f_x(a,b)+(y-b)f_y(a,b)\right]$$

$$+\frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b)+2(x-a)(y-b)f_{xy}(a,b)+(y-b)^2 f_{yy}(a,b)\right]$$

$$+\frac{1}{3!}\left[\begin{array}{l}(x-a)^3 f_{xxx}(a,b)+3(x-a)^2(y-b)f_{xxy}(a,b)\\ +3(x-a)(y-b)^3 f_{xyy}(a,b)+(y-b)^3 f_{yyy}(a,b)\end{array}\right]+\ldots. \qquad \text{(i)}$$

Here $f(x, y) = x^2y + 3y - 2$, $\quad a = 1,\ b = -2$.

$f(1,-2) = 1^2 \times (-2) + 3(-2) - 2 = -10$

$f_x = 2xy$, $\quad f_x(1,-2) = 2(1)(-2) = -4$; $\quad f_y = x^2 + 3$, $\quad f_y(1,-2) = 1^2 + 3 = 4$

$f_{xx} = 2y$, $\quad f_{xx} = (1,-2) = 2(-2) = -4$; $\quad f_{xy} = 2x$, $\quad f_{xy}(1,-2) = 2(1) = 2$

$f_{yy} = 0$, $\quad f_{yy} = (1,-2) = 0$; $\quad f_{xxx} = 0$, $\quad f_{xxx}(1,-2) = 0$

$f_{xxy} = 2$, $\quad f_{xxy} = (1,-2) = 2$; $\quad f_{xyy} = 0$, $\quad f_{xyy}(1,-2) = 0$

$f_{yyy} = 0$, $\quad f_{yyy} = (1,-2) = 0$.

All higher order partial derivatives vanish.

$\therefore$From (i), we get $x^2y + 3y - 1 = f(x, y)$

$$= -10 + [(x-1)(-4) + (y+2)(4)] + \frac{1}{2}\left[(x-1)^2(-4) + 2(x-1)(y+2)(2) + (y+2)^2(0)\right]$$

$$+\frac{1}{6}\left[(x-1)^3(0) + 3(x-1)^2(y+2)(2) + 3(x-1)(y+2)^2(0) + (y+2)^2(0)\right]$$

$$= -10 - 4(x-1) + 4(y+2) - 2(x-1)^2 + 2(x-1)(y+2) + (x-1)^2(y+2).$$

Now let us solve some more problems:

Q.No.1.: Use Maclaurin's theorem to expand $e^{\sin x}$ in a power series up to term containing x^7.

Sol.: Since we know that $\quad e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots\ldots\ldots\ldots$

$$\text{and } \sin x = x - \frac{x^2}{3!} + \frac{x^3}{5!} - \frac{x^7}{7!} + \ldots\ldots\ldots\ldots\ldots\ldots$$

$$\therefore e^{\sin x} = 1 + \sin x + \frac{(\sin x)^2}{2!} + \frac{(\sin x)^3}{3!} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$=1+\left[x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+........\right]+\frac{1}{2!}\left[x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+........\right]^2$$

$$+\frac{1}{3!}\left[x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+........\right]^3+..........$$

$$=1+x\left[1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+.........\right]+\frac{x^2}{2!}\left[1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+.........\right]^2$$

$$+\frac{x^3}{3!}\left[1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+.........\right]^3+...........$$

Using Binomial expansion, we get

$$e^{\sin x}=1+x\left[1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+.........\right]+\frac{x^2}{2!}\left[1-2\left(\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+.........\right)+.....\right]$$

$$+\frac{x^3}{3!}\left[1-3\left(\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+.........\right)+....\right]+.............$$

Collecting terms of the same nature, we get

$$e^{\sin x}=1+x+\frac{x^2}{2}-\frac{1}{8}x^4+...................$$

This is the required expansion.

Q.No.2.: Obtain by Maclaurin's theorem the first five terms in the expansion of $\log(1+\sin x)$.

Sol.: We know $f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+............$ (i)

Here $f(x)=\log(1+\sin x)$ $\therefore f(0)=\log(1+0)=0$ $\Rightarrow f(0)=0$

$$f'(x)=\frac{1}{1+\sin x}.\cos x=\frac{\cos x}{1+\sin x} \qquad \Rightarrow f'(0)=1$$

$$f''(x)=\frac{(1+\sin x)(-\sin x)-\cos^2 x}{(1+\sin x)^2}=\frac{-\sin x-1}{(1+\sin x)^2} \qquad \Rightarrow f''(0)=-1$$

$$f'''(x)=\frac{(1+\sin x)^2(-\cos x)-2(1+\sin x)\cos x(-\sin x-1)}{(1+\sin x)^2} \Rightarrow f'''(0)=1$$

Similarly $f^{iv}(0) = -2$ and $f^{v}(0) = 5$ and so on.

Substituting these values in (i), we get

$$\log(1+\sin x) = 0 + x + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(1) + \frac{x^4}{4!}(-2) + \frac{x^5}{5!}(5) + \ldots\ldots\ldots\ldots$$

$$= x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + \frac{x^5}{24} - \ldots\ldots\ldots\ldots\ldots\text{Ans.}$$

Q.No.3.: Expand $\log[1-\log(1-x)]$ in powers of x by Maclaurin's theorem as for as the term containing x^3.

Sol.: Since we know that

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad \text{(i)}$$

$$\log(1-t) = -x - \frac{t^2}{2} - \frac{t^3}{3} - \frac{t^4}{4} - \ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad \text{(ii)}$$

Putting $t = \log(1-x)$ in (ii), we get

$$\log[1-\log(1-x)] = \left[x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots\ldots\ldots\right] - \frac{1}{2}\left[x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots\ldots\ldots\right]^2$$

$$+ \frac{1}{3}\left[x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots\ldots\right]^3 - \frac{1}{4}\left[x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots\ldots\right]^4 + \ldots\ldots\ldots$$

$$= x\left[1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \ldots\ldots\ldots\right] - \frac{x^2}{2}\left[1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \ldots\ldots\ldots\right]^2$$

$$+ \frac{x^3}{3}\left[1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \ldots\ldots\right]^3 - x^4\left[1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \ldots\ldots\right]^4 + \ldots\ldots$$

Collecting terms of the same nature, we get

$$\log[1-\log(1-x)] = x + \frac{x^3}{3!} + \ldots\ldots\ldots\ldots\ldots\ldots.$$

This is the required expansion.

Q.No.4.: Expand $\cos^3 x$ using Maclaurin's series.

Sol.: $f(x)=\cos^3 x=\frac{1}{4}(\cos 3x+3\cos x)$ $\Rightarrow f(0)=0$

$f'(x)=\frac{1}{4}(-3\sin 3x-3\sin x)$ $\Rightarrow f'(0)=0$

$f''(x)=\frac{3}{4}(-\cos 3x-\cos x)$ $\Rightarrow f''(0)=-3$

$f'''(x)=\frac{3}{4}(9\sin 3x+\sin x)$ $\Rightarrow f'''(0)=0$

$f^{iv}(x)=\frac{3}{4}(27\cos 3x+\cos x)$ $\Rightarrow f^{iv}(0)=21$

Now Maclaurin's series is

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\frac{x^4}{4!}f^{iv}(0)+..........+\frac{x^{2n}}{2n!}f^{2n}(0)........$$

Substituting these values, we get

$$\cos^3 x=1-3\frac{x^2}{2!}+21\frac{x^4}{4!}-..............=1-\frac{3}{2}x^2+\frac{7}{8}x^4-................$$

This is the required expansion.

Q.No.5.: Use Maclaurin's theorem or otherwise show that:

(a) $e^{\cos x}=e\left[1-\frac{x^2}{2!}+\frac{x^4}{4!}(4)-..................\right]$

(b) $e^{ax}\cos bx=1+ax+\frac{1}{x}\left(a^2+b^2\right)x^2+\frac{1}{6}a\left(a^2-3b^2\right)x^3+............$

(c) $a^x=1+x\log a+\frac{x^2}{2!}(\log a)^2+\frac{x^3}{3!}(\log a)^3+............$

Sol.: (a) It is required to prove that

$$e^{\cos x}=e\left[1-\frac{x^2}{2!}+\frac{x^4}{4!}(4)-..................\right]$$

$f(x)=e^{\cos x}$ $\Rightarrow f(0)=e$

$f'(x)=e^{\cos x}(-\sin x)=f(x)(-\sin x)$ $\Rightarrow f'(0)=0$

$f''(x)=f'(x)(-\sin x)-f(x)\cos x$ $\Rightarrow f''(0)=-e$

$f'''(x)=f''(x)(-\sin x)-f'(x)\cos x-f'(x)\cos x+f(x)\sin x$ $\Rightarrow f'''(0)=0$

$$f^{iv}(x) = f'''(x)(-\sin x) - f''(x)\cos x - 2[f''(x)\cos x + f'(x)(-\sin x)]$$
$$+ f'(x)\sin x + f(x)\cos x$$

$$= f'''(x)(-\sin x) - f''(x)\cos x - 2f''(x)\cos x - 2f'(x)(-\sin x)$$
$$+ f'(x)\sin x + f(x)\cos x \qquad \Rightarrow f^{iv}(0) = 4e$$

Now Maclaurin's series is

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{iv}(0) + \ldots\ldots\ldots + \frac{x^{2n}}{2n!}f^{2n}(0) + \ldots\ldots\ldots$$

Substituting, we get

$$e^{\cos x} = e + x.0 + \frac{x^2}{2!}(-e) + \frac{x^3}{3!}(0) + \frac{x^4}{4!}(4e) + \ldots\ldots\ldots\ldots\ldots\ldots$$

$$= e - \frac{x^2}{2!}e + \frac{x^4}{4!}(4e)\ldots\ldots\ldots\ldots\ldots\ldots$$

$$= e\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!}(4)\ldots\ldots\ldots\ldots\ldots\ldots\right].$$

This is the required expansion.

(b): It is required to prove that

$$e^{ax}\cos bx = 1 + ax + \frac{1}{x}(a^2 + b^2)x^2 + \frac{1}{6}a(a^2 - 3b^2)x^3 + \ldots\ldots\ldots\ldots$$

$$f(x) = e^{ax}\cos bx \qquad \Rightarrow f(0) = 1$$

$$f'(x) = e^{ax}\cos bx + e^{ax}(-b\sin bx)$$

$$= af(x) + e^{ax}(-b\sin bx) \qquad \Rightarrow f'(0) = a$$

$$f''(x) = af'(x) - b\left[ae^{ax}\sin bx + e^{ax}b\cos bx\right] \qquad \Rightarrow f''(0) = a^2 - b^2$$

$$f'''(x) = af''(x) - b\left[a\left(ae^{ax}\sin bx + be^{ax}\cos x\right) + b\left(ae^{ax}\text{ciosbx} + be^{ax}\sin x\right)\right]$$

$$\Rightarrow f'''(0) = a(a^2 - b^2) - b(ab + ab) = a(a^2 - b^2) - 2b^2a = a(a^2 - 3b^2)$$

Now Maclaurin's series is

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{iv}(0) + \ldots\ldots\ldots + \frac{x^{2n}}{2n!}f^{2n}(0)\ldots\ldots\ldots$$

Substituting, we get

$$e^{ax}\cos bx = 1 + ax + \left(a^2 - b^2\right)\frac{x^2}{2!} + a\left(a^2 - 3b^2\right)\frac{x^3}{3!} + \ldots\ldots\ldots\ldots$$

This is the required expansion.

(c): $f(x) = a^x$

Now Maclaurin's series is

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{iv}(0) + \ldots\ldots\ldots + \frac{x^{2n}}{2n!}f^{2n}(0)\ldots\ldots\ldots$$

Now $f(x) = a^x$, $\qquad f(0) = 1$

$f'(x) = a^x \log a$, $\qquad f'(0) = \log a$

$f''(x) = a^x (\log a)^2$, $\qquad f''(0) = (\log a)^2$

$f'''(x) = a^x (\log a)^3$, $\qquad f'''(0) = (\log a)^3$

Substituting the values, we get

$$a^x = 1 + x\log a + \frac{x^2}{2!}(\log a)^2 + \frac{x^3}{3!}(\log a)^3 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

This is the required expansion.

Q.No.6.: Obtain the Maclaurin's expansion of $\tan\left(\frac{\pi}{4} + x\right)$ and hence find the value of $(46° - 30')$ to four decimal places.

Sol.: Let $f(x) = \tan\left(\frac{\pi}{4} + x\right)$, $\qquad f(0) = \tan\frac{\pi}{4} = 1$

$f'(x) = \sec^2\left(\frac{\pi}{4} + x\right)$, $\qquad f'(0) = \sec^2\frac{\pi}{4} = 2$

$f''(x) = 2\sec^2\left(\frac{\pi}{4} + x\right)\tan\left(\frac{\pi}{4} + x\right)$, $\qquad f''(0) = 2\left(\sqrt{2}\right)^2.1 = 4$

$f'''(x) = 4\sec^2\left(\frac{\pi}{4} + x\right)\tan^2\left(\frac{\pi}{4} + x\right) + 2\sec^4\left(\frac{\pi}{4} + x\right)$, $\quad f'''(0) = 4\left(\sqrt{2}\right)^2.1 + 2\left(\sqrt{2}\right)^4 = 16$

Now Maclaurin's series is

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{iv}(0) + \ldots\ldots\ldots + \frac{x^{2n}}{2n!}f^{2n}(0)\ldots\ldots\ldots$$

$$\therefore \tan\left(\frac{\pi}{4}+x\right)=1+2x+\frac{4x^2}{2}+\frac{16x^3}{3}+\ldots\ldots\ldots\ldots$$

Now $\tan(46°-30')=\tan\left(45°+1.\frac{1}{2}\right)=\tan\left(\frac{\pi}{4}+\frac{14}{120}\right)$

Hence $\tan(46°-30')=1+2.\frac{14}{120}+2.\left(\frac{\pi}{120}\right)^2+\ldots\ldots\ldots=1.0537$. Ans.

Q.No.7.: Expand $\log_e\left(x+\sqrt{x^2+1}\right)$ up to first four terms by Maclaurin's theorem. Hence calculate the value of $\log_e 2$ by putting x = 0.75 in the expansion.

Sol.: Let $f(x)=\log_e\left(x+\sqrt{x^2+1}\right)$, $\qquad f(0)=\log 1=0$

$$f'(x)\frac{1}{x+\left(\sqrt{x^2+1}\right)}\left(1+\frac{x}{\sqrt{x^2+1}}\right)=\frac{1}{\sqrt{x^2+1}}, \qquad f'(0)=1$$

$$f''(x)=\frac{-x}{\left(x^2+1\right)^{3/2}}, \qquad f''(0)=0$$

$$f'''(x)=\frac{-1}{\left(x^2+1\right)^{3/2}}+\frac{3x.2x}{2\left(x^2+1\right)^{5/2}}, \qquad f'''(0)=-1$$

$$f^{iv}(x)=\frac{3.2x}{2\left(x^2+1\right)^{3/2}}+\frac{6x}{\left(x^2+1\right)^{5/2}}-\frac{15x^2.2x}{2\left(x^2+1\right)^{7/2}}, \qquad f^{iv}(0)=0$$

Hence by Maclaurin's series, we get

$$\log_e\left(x+\sqrt{x^2+1}\right)=x-\frac{x^3}{3!}+\frac{9x^5}{5!}+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

To calculate $\log_e 2$, put x = 0.75

$$\log 2=.75-\frac{(.75)^3}{3!}+\frac{(.75)^5}{5!}+\ldots\ldots\ldots\ldots\ldots=0.6974 \text{ .Ans.}$$

Q.No.8.: Expand $\sin^{-1}x$ is a series of power of x.

Sol.: $f(x)=\sin^{-1}x \qquad \therefore f'(x)=\frac{1}{\sqrt{1-x^2}}=\left(1-x^2\right)^{-1/2}$

$$f''(x)=\left(-\frac{1}{2}\right)\left(1-x^2\right)^{-3/2}(-2x)=x\left(1-x^2\right)^{-3/2}$$

$$f'''(x)=\left(1-x^2\right)^{-3/2}+x\left(-\frac{3}{2}\right)\left(1-x^2\right)^{-5/2}(-2x)=\left(1-x^2\right)^{-3/2}+3x^2\left(1-x^2\right)^{-5/2}$$

$$f^{iv}(x)=\left(\frac{-3}{2}\right)\left(1-x^2\right)^{-5/2}(-2x)+6x\left(1-x^2\right)^{-5/2}+3x^2\left(\frac{-5}{2}\right)\left(1-x^2\right)^{-5/2}(-2x)$$

$$=3x\left(1-x^2\right)^{-5/2}+6x\left(1-x^2\right)^{-5/2}+15x^2\left(1-x^2\right)^{-7/2}$$

$$=9x\left(1-x^2\right)^{-5/2}+15x^3\left(1-x^2\right)^{-7/2}$$

$$f^{v}(x)=9\left(1-x^2\right)^{-5/2}+9x\left(-\frac{5}{2}\right)\left(1-x^2\right)^{-7/2}(-2x)+45x^2\left(1-x^2\right)^{-7/2}+15x^3\left(-\frac{7}{2}\right)\left(1-x^2\right)^{-9/2}(-2x)$$

$$=9x\left(1-x^2\right)^{-5/2}+90x^2\left(1-x^2\right)^{-7/2}+105x^4\left(1-x^2\right)^{-9/2}$$

Now put x = 0 in all, we get

$f(0) = 0,\quad f'(0)=1,\quad f''(0)=0,\quad f'''(0)=1,\quad f^{iv}(0)=0,\quad f^{v}(0)=9$

Now by Maclourins theorem

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\frac{x^4}{4!}f^{iv}(0)+\frac{x^5}{5!}f^{v}(0)+\ldots\ldots\ldots\ldots$$

Substituting the values of $f(0) = 0,\quad f'(0)=1,\quad f''(0)=0,\quad f'''(0)=1,$

$f^{iv}(0)=0,\quad f^{v}(0)=9$, we get

$$\sin^{-1}x=0+x+0+\frac{x^3}{3!}+0+x^5\times\frac{9}{5!}+\ldots\ldots\ldots\ldots$$

$$\therefore \sin^{-1}x=x+\frac{1}{2.3}x^3+0+\frac{1.3}{2.4.5}x^5+\ldots\ldots\ldots\ldots \text{ Ans.}$$

Other Method:

$$f(x)=\sin^{-1}x \qquad \therefore f'(x)=\frac{1}{\sqrt{1-x^2}}=\left(1-x^2\right)^{-1/2}$$

By binomial expansion

$$f'(x)=1-\left(-\frac{1}{2}\right)x^2+\frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2!}x^4-\frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)}{3!}x^6+\ldots\ldots\ldots$$

$$=1+\frac{x^2}{2}+\frac{3}{8}x^4+\frac{5}{16}x^6+..................$$

$$f''(x)=\frac{2x}{2}+\frac{12}{8}x^3+\frac{5\times 6}{16}x^5+...............=x+\frac{3}{2}x^3+\frac{15}{8}x^5+..................$$

$$f'''(x)=1+\frac{9}{2}x^2+\frac{75}{8}x^4+.......................$$

$$f^{iv}(x)=9x+\frac{75}{2}x^3+..................$$

$$f^{v}=9+\frac{225}{2}x^2+...................$$

Now put x = 0 in all, we get

$f(0)=0,\quad f'(0)=1,\quad f''(0)=0,\quad f'''(0)=1,\quad f^{iv}(0)=0,\quad f^{v}(0)=9$

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\frac{x^4}{4!}f^{iv}(0)+\frac{x^5}{5!}f^{v}(0)+...................$$

Substituting the values of $f(0)=0$, $f'(0)=1$, $f''(0)=0$, $f'''(0)=1$,

$f^{iv}(0)=0$, $f^{v}(0)=9$, we get

$$\sin^{-1}x=0+x+0+\frac{x^3}{3!}+0+x^5\times\frac{9}{5!}+..................$$

$$\therefore \sin^{-1}x=x+\frac{1}{2.3}x^3+0+\frac{1.3}{2.4.5}x^5+..................\text{ Ans.}$$

Q.No.9.: Prove that $\left(\sin^{-1}x\right)^2=x^2+\frac{x^4}{3}+....................$

Sol.: Let $f(x)=\left(\sin^{-1}x\right)^2$

Now Maclaurin's series is

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\frac{x^4}{4!}f^{iv}(0)+..........+\frac{x^{2n}}{2n!}f^{2n}(0)........$$

$f(x)=\left(\sin^{-1}x\right)^2$, $\qquad f(0)=0$

$f'(x)=\left(\sin^{-1}x\right)\frac{1}{\sqrt{1-x^2}}$, $\qquad f'(0)=0$

$$f''(x)=2\left[\sin^{-1}x\left(-\frac{1}{2}\right)\left(1-x^2\right)^{-3/2}(-2x)+\left(1-x^2\right)^{-1/2}.\left(1-x^2\right)^{-1/2}\right]$$

$$=2\left[x\sin^{-1}x\left(1-x^2\right)^{-3/2}+\left(1-x^2\right)^{-1}\right] \qquad f''(0)=2$$

$$f'''(x)=2\left[\left\{\begin{array}{c} x\left(1-x^2\right)^{-1/2}\left(1-x^2\right)^{-3/2}+x\sin^{-1}x\left(\dfrac{-3}{2}\right)\left(1-x^2\right)^{-5/2} \\ (-2x)+\sin^{-1}x\left(1-x^2\right)^{-3/2}+(-1)\left(1-x^2\right)^{-2}(-2x)\end{array}\right\}\right]$$

$$=2\left[\left\{x\left(1-x^2\right)^{-2}+3x^2\sin^{-1}x\left(1-x^2\right)^{-5/2}+\sin^{-1}x\left(1-x^2\right)^{-3/2}+\left(1-x^2\right)^{-2}(2x)\right\}\right]$$

$$f'''(0)=0$$

Substituting the values, we get

$$f(x)=0+x.0+\frac{x^2}{2!}.2+\frac{x^3}{3!}.0+\frac{x^4}{4!}.8+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\left(\sin^{-1}x\right)^2=x^2+\frac{x^4}{3}+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

This is the required expansion.

Q.No.10.: Prove that $\dfrac{\sin^{-1}x}{\sqrt{1-x^2}}=x+\dfrac{1}{3}x^3+\ldots\ldots\ldots$

Sol.: Let $f(x)=\dfrac{\sin^{-1}x}{\sqrt{1-x^2}}$

Now Maclaurin's series is

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\frac{x^4}{4!}f^{iv}(0)+\ldots\ldots\ldots+\frac{x^{2n}}{2n!}f^{2n}(0)\ldots\ldots.$$

$$f(x)=\frac{\sin^{-1}x}{\sqrt{1-x^2}}, \qquad f(0)=0$$

$$f'(x)=\sin^{-1}(x)\left(-\frac{1}{2}\right)\left(1-x^2\right)^{-3/2}(-2x)+\left(1-x^2\right)^{-1/2}\left(1-x^2\right)^{-1/2}$$

$$=(x)\left(1-x^2\right)^{-3/2}\sin^{-1}x+\left(1-x^2\right)^{-1/2}, \qquad f'(0)=1$$

$$f''(x)=3x^2\sin^{-1}x\left(1-x^2\right)^{-5/2}+\sin^{-1}x\left(1-x^2\right)^{-3/2}+3x\left(1-x^2\right)^{-2}, \quad f''(0)=0$$

$$f'''(x)=3x^2\left(1-x^2\right)^{-1/2}\left(1-x^2\right)^{-5/2}+3x^2\sin^{-1}x\left(-\frac{5}{2}\right)\left(1-x^2\right)^{-7/2}(-2x)$$

$$+3(2x)\sin^{-x}\left(1-x^2\right)^{-5/2}+\left(1-x^2\right)^{-1/2}\left(1-x^2\right)^{-3/2}+\sin^{-1}x\left(-\frac{3}{2}\right)\left(1-x^2\right)^{-6/2}(-2x)$$

$$+3x(-1)\left(1-x^2\right)^{-3}(-2x)+3\left(1-x^2\right)^{-2}, \qquad f'''(0)=4$$

Substituting the values, we get

$$f(x)=\frac{\sin^{-1}x}{\sqrt{1-x^2}}=0+x.1+\frac{x^2}{2!}.0+\frac{x^3}{3!}.4+\ldots\ldots\ldots\ldots\ldots\ldots$$

$$=x+\frac{2}{3}x^3+\ldots\ldots\ldots\ldots\ldots$$

This is the required expansion.

Q.No.11.: Prove that $\frac{\cos x}{\sqrt{1+x}}=1-\frac{1}{2}x-\frac{1}{8}x^2-\frac{1}{16}x^3+\ldots\ldots\ldots\ldots\ldots$.

Sol.: Since we know that $\cos x=1-\frac{x^2}{2}+\frac{x^4}{24}-\ldots\ldots\ldots\ldots\ldots\ldots$

and $(1+x)^{-1/2}=1-\frac{1}{2}x+\frac{3}{8}x^2-\frac{5}{16}x^3+\ldots\ldots\ldots\ldots\ldots\ldots$ [by Binomial theorem]

$$\therefore(\cos x)(1-x)^{-1/2}=\left(1-\frac{x^2}{2}+\frac{x^4}{24}-\ldots\ldots\ldots\right)\left(1-\frac{x}{2}+\frac{3}{8}x^2-\frac{5}{16}x^3+\ldots\ldots\ldots\ldots\right)$$

$$=1-\frac{x}{2}+\frac{3}{8}x^2-\frac{5}{16}x^3-\frac{x^2}{2}+\frac{x^3}{4}-\frac{3}{16}x^4+\frac{5}{32}x^5+\ldots\ldots\ldots$$

$$=1-\frac{x}{2}+x^2\left(\frac{3}{8}-\frac{1}{2}\right)+x^3\left(\frac{1}{4}-\frac{5}{16}\right)+\ldots\ldots\ldots$$

$$=1-\frac{x}{2}-\frac{x^2}{8}-\frac{x^3}{16}+\ldots\ldots\ldots\ldots\ldots\ldots.$$

Which is the required expansion.

Q.No.12.: Prove that $e^x\cos x=1+x-\frac{x^3}{3}-\frac{x^4}{6}+\ldots\ldots\ldots\ldots\ldots\ldots$.

Sol.: Since we know that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots\dots\dots\dots\dots$$

and $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\dots\dots\dots\dots$

$$\therefore e^x \cos x = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \dots\dots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\dots\dots\dots\right)$$

$$= 1 - \frac{x^2}{2} + \frac{x^4}{24} + x - \frac{x^3}{2} + \frac{x^5}{24} + \frac{x^2}{2} - \frac{x^4}{4} + \frac{x^6}{48} + \frac{x^3}{6} - \frac{x^5}{12}$$

$$+ \frac{x^7}{24.6} + \frac{x^4}{24} - \frac{x^6}{48} + \frac{x^8}{(24)^2} + \dots\dots\dots\dots\dots\dots\dots$$

$$= 1 - \frac{x^2}{2} + \frac{x^4}{24} + x - \frac{x^3}{2} + \frac{x^2}{2} - \frac{x^4}{4} + \frac{x^3}{6} + \frac{x^4}{24} + \dots\dots\dots\dots\dots$$

$$= 1 + x - \frac{1}{3}.x^3 - \frac{1}{6}x^4 + \dots\dots\dots\dots\dots$$

$$= 1 + x - \frac{x^3}{3} - \frac{x^4}{6} + \dots\dots\dots\dots\dots,$$

which is the required expansion.

Q.No13.: Prove that $\sin\left(e^x - 1\right) = x + \frac{x^2}{2} - \frac{5}{24}x^4 + \dots\dots\dots\dots\dots\dots\dots$.

Sol.: Since we know that $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \dots\dots\dots\dots\dots$

$$\therefore \sin\left(e^x - 1\right) = \sin\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \dots\dots\dots\dots\dots\dots\dots - 1\right)$$

$$= \sin\left(x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \dots\dots\dots\dots\dots\dots\dots\right)$$

$$= \sin\left\{x\left(1 + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \dots\dots\dots\dots\right)\right\}$$

Since we know that

$$\sin z = z - \frac{z^3}{6} + \frac{z^5}{120} + \dots\dots\dots\dots\dots$$

$$\therefore \sin\left(e^x - 1\right) = x\left(1 + \frac{x}{2} + \frac{x^2}{6} + \frac{x^3}{24}\right) - \frac{x^3}{6}\left(1 + \frac{3}{2}x + \frac{3}{6}x^2 + \frac{3}{24}x^3 + \ldots\ldots\ldots\ldots\ldots\right) + \ldots\ldots$$

Applying the Binomial theorem, we obtain

$$= x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^3}{6}\left(1 + \frac{3}{2}x + \frac{3}{6}x^2 + \frac{3}{24}x^3 + \ldots\ldots\ldots\ldots\ldots\ldots\right)$$

$$= x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^3}{6} - \frac{x^4}{4} - \frac{x^5}{12} - \frac{x^6}{48} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$= x + \frac{x^2}{2} - \frac{5}{24}x^4 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

which is the required expansion.

Q.No.14.: Prove that $e^{x\cos x} = 1 + x + x^2 - \frac{x^3}{3} - \frac{11}{24}x^4 + \ldots\ldots\ldots\ldots\ldots$**.**

Sol.: Let $x\cos x = y$

$$\therefore e^y = 1 + y + \frac{y^2}{2} + \frac{y^3}{6} + \frac{y^4}{24} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad \text{(i)}$$

Put $y = x\cos x$ in (i), we get

$$e^{x\cos x} = 1 + x\cos x + \frac{x^2}{2}\cos^2 x + \frac{x^3}{6}\cos^3 x + \frac{x^4}{24}\cos^4 x + \ldots\ldots\ldots\ldots\ldots\ldots \qquad \text{(ii)}$$

Using the expansion of $\cos x$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\therefore e^{x\cos x} = 1 + x\left(1 - \frac{x^7}{2} + \frac{x^4}{24}\ldots\ldots\ldots\right) + \frac{x^2}{2}\left(1 - \frac{x^2}{2} + \frac{x^4}{24} + \ldots\ldots\right)^2 + \frac{x^3}{6}\left(1 - \frac{x^2}{2} + \frac{x^4}{24} + \ldots\ldots\ldots\right)^3$$

$$+ \frac{x^4}{24}\left(1 - \frac{4x^2}{2} + \frac{4x^4}{24} + \ldots\ldots\ldots\ldots\right)^4 + \ldots\ldots\ldots\ldots\ldots\ldots$$

Using Binomial expansion, we get

$$= 1 + x - \frac{x^3}{2} + \frac{x^5}{24} + \ldots\ldots\ldots + \frac{x^2}{2}\left(1 - \frac{2x^2}{2} + \frac{2x^4}{24} + \ldots\ldots\ldots\right)$$

$$+ \frac{x^3}{6}\left(1 - \frac{3x^2}{2} + \frac{3x^4}{24} + \ldots\ldots\ldots\right) + \frac{x^4}{24}\left(1 - \frac{4x^2}{2} + \frac{4x^4}{24} + \ldots\ldots\ldots\ldots\right)$$

$$=1+x-\frac{x^3}{2}+\frac{x^5}{24}+\frac{x^2}{2}-\frac{x^4}{2}+\frac{x^6}{24}+\frac{x^3}{6}-\frac{x^5}{4}+\frac{x^7}{48}+\frac{x^4}{24}-\frac{4x^6}{48}+\frac{4x^8}{(24)^2}+.....$$

$$=1+x-\frac{x^3}{2}+\frac{x^2}{2}-\frac{x^4}{2}+\frac{x^3}{6}+\frac{x^4}{24}+..........$$

$$=1+x+\frac{x^2}{2}-\frac{x^3}{3}-\frac{11x^4}{24}+.............,$$

which is the required expansion.

Q.No.15.: Prove that $\frac{x}{e^x-1}=1-\frac{x}{2}+\frac{x^2}{12}$.................. .

Sol.: Since we know that

$$e^x=1+x+\frac{x^2}{2}+\frac{x^3}{6}+............$$

$$\therefore \frac{x}{e^x-1}=\frac{x}{\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+..............-1\right)}=\frac{x}{x+\frac{x^2}{2}+\frac{x^3}{6}+........}$$

$$=\frac{x}{x\left(1+\frac{x}{2}+\frac{x^2}{6}+..........\right)}=\frac{1}{\left(1+\frac{x}{2}+\frac{x^2}{6}+........\right)}$$

$$=\left(1+\frac{x}{2}+\frac{x^2}{6}+........\right)^{-1} \qquad \text{(i)}$$

Also we know that

$$(1+z)^{-1}=1-z+z^2-.......................$$

Apply this expansion in (i), we get

$$\frac{x}{e^x-1}=1-\left(\frac{x}{2}+\frac{x^2}{6}+.......\right)+\left(\frac{x}{2}+\frac{x^2}{6}+.........\right)^2.....................$$

$$=1-\frac{x}{2}-\frac{x^2}{6}+\frac{x^2}{4}+\frac{x^4}{36}+\frac{x^3}{6}+...................=1-\frac{x}{2}+x^2\left(-\frac{1}{6}+\frac{1}{4}\right).........$$

$$=1-\frac{x}{2}+\frac{x^2}{12}+...............,$$

which is the required expansion.

Q.No.16.: Prove that $\log\left(1+e^x\right)=\log 2+\frac{x}{2}+\frac{x^2}{8}-\frac{x^4}{192}+\ldots\ldots\ldots\ldots$.

Sol.: Let $f(x)=\log\left(1+e^x\right)$ $\qquad \Rightarrow f(0)=\log 2$

$$f'(x)=\frac{1}{1+e^x}.e^x \qquad \Rightarrow f'(0)=\frac{1}{2}$$

$$f''(x)=\frac{\left(1+e^x\right)e^x-e^{2x}}{\left(1+e^x\right)^2}=\frac{\left(e^x+e^{2x}-e^{2x}\right)}{\left(1+e^x\right)^2}$$

$$=\frac{e^x}{\left(1+e^x\right)^2} \qquad \Rightarrow f''(0)=\frac{1}{4}$$

$$f'''(x)=\frac{\left(1+e^x\right)^2e^x-e^{2x}-2\left(1+e^x\right)}{\left(1+e^x\right)^4}=\frac{\left(1+e^{2x}+2e^{2x}\right)e^x-2e^{2x}+2e^{3x}}{\left(1+e^x\right)^4}$$

$$=\frac{e^x+e^{3x}+2e^{2x}-2e^{2x}-2e^{3x}}{\left(1+e^x\right)^4}=\frac{e^x-e^{3x}}{\left(1+e^x\right)^4} \qquad \Rightarrow f'''(0)=0$$

$$f^{iv}(x)=\frac{\left(1+e^x\right)^4\left(e^x-3e^{3x}\right)-\left(e^x-e^{3x}\right)4\left(1+e^x\right)^3e^x}{\left(1+e^x\right)^8} \qquad \Rightarrow f^{iv}(0)=-\frac{1}{8}$$

Putting the values, we get

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\frac{x^4}{4!}f^{iv}(0)+\ldots\ldots\ldots\ldots\ldots\ldots$$

$$=\log 2+\frac{x}{2}+\frac{x^2}{8}-\frac{x^4}{192}+\ldots\ldots\ldots\ldots\ldots\ldots$$

which is the required expansion.

Q.No.17.: Prove that $\log\frac{xe^x}{e^x-1}=\frac{x}{2}-\frac{x^2}{24}+\frac{x^4}{2880}+\ldots\ldots\ldots\ldots\ldots\ldots$.

Sol.: $f(x)=\log\frac{xe^x}{e^x-1}=a \Rightarrow \lim\limits_{x\to 0}\frac{xe^x}{e^x-1}=e^a$

Applying L – Hospital's rule

$$\lim_{x\to 0}\frac{e^x+xe^x}{e^x}=1\Rightarrow e^a=1$$

$\therefore a=0$ $\qquad \Rightarrow f(0)=0$

$$f'(x)=\frac{e^x-1}{xe^x}\cdot\frac{(e^x-1)(xe^x+e^x)-xe.e^x}{(e^x-1)^2}=\frac{e^x-x-1}{x(e^x-1)}$$

Applying L – Hospital's rule

$$\lim_{x\to 0}\frac{e^x-x-1}{x(e^x-1)}=\lim_{x\to 0}\frac{e^x-1}{xe^x+e^x-1}=\lim_{x\to 0}\frac{e^x}{xe^x+2e^x}=\frac{1}{2}$$

$$f'(0)=\frac{1}{2}$$

$$f''(x)=\frac{x(e^x-1)(e^x-1)-(e^x-x-1)(e^x+xe^x-1)}{x^2(xe^x-1)^2}=\frac{-e^{2x}+x^2e^x+2e^{x-1}}{x^2e^{2x}+x^2-2x^2e^x}$$

Applying L – Hospital's rule

$$\lim_{x\to 0}\frac{-e^{2x}+x^2e^x+2e^{x-1}}{x^2e^{2x}+x^2-2x^2e^x}$$

Putting x = 0, we get

$$f''(0)=-\frac{1}{12}$$

Applying Maclaurin's theorem and putting the values, we get

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\frac{x^4}{4!}f^{iv}(0)+\dots\dots\dots\dots$$

$$=0+\frac{x}{2}-\frac{x^2}{24}+0+\frac{x^4}{2880}+\dots\dots\dots$$

$$=\frac{x}{2}-\frac{x^2}{24}+\frac{x^4}{2880}+\dots\dots\dots\dots,$$

which is the required expansion.

Q.No.18.: Prove that $\log\left(\frac{\sin x}{x}\right)=-\frac{x^2}{6}-\frac{x^4}{180}-\frac{x^6}{2835}-\dots\dots\dots$.

Sol.: $\log\left(\frac{\sin x}{x}\right)=\log\left(\frac{x-\frac{x^3}{3!}+\frac{x^5}{5!}+\frac{x^7}{7!}+\dots\dots\dots\infty}{x}\right)$

$$=\log\left(1-\left(\frac{x^2}{3!}-\frac{x^4}{5!}+\frac{x^6}{7!}+\dots\dots\dots\infty\right)\right)$$

$$= -\left(\frac{x^2}{3!} - \frac{x^4}{5!} + \frac{x^6}{7!}\right) - \frac{1}{2}\left(\frac{x^2}{3!} - \frac{x^4}{5!} + \ldots\ldots\ldots\right)^2 - \frac{1}{3}\left(\frac{x^2}{3!} - \frac{x^4}{5!} + \ldots\ldots\ldots\right)^3$$

$$= -\frac{x^2}{6} + \frac{x^4}{120} - \frac{x^6}{5040} - \frac{1}{2}\left(\frac{x^4}{36} + \frac{x^8}{14400} - \frac{x^6}{360}\right)$$

$$= -\frac{x^2}{6} + \frac{x^4}{120} - \frac{x^6}{5040} - \frac{x^4}{72} - \frac{x^6}{720}$$

$$= -\frac{x^2}{6} + x^4\left(\frac{1}{120} - \frac{1}{72}\right) - x^6\left(\frac{1}{5040} + \frac{1}{720}\right)$$

$$= -\frac{x^2}{6} - \frac{x^4}{180} - \frac{x^6}{2835} \ldots\ldots\ldots\ldots$$

which is the required expansion.

Q.No.19.: Prove that $\log\left(\frac{\sinh x}{x}\right) = -\frac{x^2}{6} - \frac{x^4}{180} + \frac{x^6}{2835} - \ldots\ldots\ldots$.

Sol.: $\log\left(\frac{\sinh x}{x}\right) = \log\frac{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^6}{7!} + \ldots\ldots\ldots\ldots\infty\right)}{x}$

$$= \log\left(1 + \left(\frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^6}{7!} + \ldots\ldots\ldots\ldots\infty\right)\right)$$

$$= \left(\frac{x^2}{3!} - \frac{x^4}{5!} + \frac{x^6}{7!}\right) - \frac{1}{2}\left(\frac{x^2}{3!} + \frac{x^4}{5!} + \frac{x^6}{7!}\ldots\ldots\ldots\right)^2$$

$$= +\frac{x^2}{6} + \frac{x^4}{120} - \frac{x^6}{5040} + \frac{1}{2}\left(\frac{x^4}{36} + \frac{x^8}{14400} - \frac{x^6}{360}\right)$$

$$= \frac{x^2}{6} - \frac{x^4}{180} + \frac{x^6}{2835} + \ldots\ldots\ldots\ldots$$

which is the required expansion.

Q.No.20.: Prove that $\sinh^{-1} x = \log\left[x + \sqrt{x^2 + 1}\right]$

Sol.: Let $f(x) = \log\left[x + \sqrt{x^2 + 1}\right]$ $\Rightarrow f(0) = 0$

$$f'(x) = 1 + \frac{1}{2\sqrt{x^2+1}}.2x \qquad \Rightarrow f'(0) = 1$$

$$f''(x) = -\frac{1}{2}\left(x^2+1\right)^{-3/2}.2x = -x\left(x^2+1\right)^{-3/2} \qquad \Rightarrow f''(0) = 0$$

$$f'''(x) = -x\left(-\frac{3}{2}\right)\left(x^2+1\right)^{-5/2}.2x = \left(x^2+1\right)^{-3/2}(-1) \qquad \Rightarrow f'''(0) = 1$$

Applying Maclaurin's theorem and putting the values, we get

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{iv}(0) + \ldots\ldots\ldots\ldots\ldots\ldots$$

$$= 0 + x + 0 + \frac{x^3}{6} + 0 + \frac{3x^5}{40} + \ldots\ldots\ldots\ldots$$

$$= x + \frac{x^3}{6} + \frac{3x^5}{40} + \ldots\ldots\ldots\ldots$$

which is the required expansion.

Q.No.21.: Prove that $\tanh^{-1} x = x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\ldots\ldots\ldots\ldots\ldots$.

Sol.: $\tanh^{-1} x = \frac{1}{2}\left[\log\left(\frac{1+x}{1-x}\right)\right] = \frac{1}{2}\left[\log(1+x) - \log(1-x)\right]$

Since we know that

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \ldots\ldots\ldots\ldots$$

and $\log(1-x) = -\left[x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \ldots\ldots\ldots\ldots\right]$

$$\therefore \tanh^{-1} x = \frac{1}{2}\left[\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{5} + \frac{x^5}{5} - \ldots\ldots\ldots\right) + \left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{5} + \frac{x^5}{5} + \ldots\ldots\ldots\right)\right]$$

$$= x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\ldots\ldots\ldots\ldots\ldots$$

which is the required expansion.

Q.No.22.: Expand $e^x \sec x$.

Sol: $\sec x = \dfrac{1}{\cos x} = \left(1 - \dfrac{x^2}{2!} + \dfrac{x^3}{3!} - \dfrac{x^4}{4!} + ...\right)^{-1}$

Consider y as $\left(\dfrac{x^2}{2!} + \dfrac{x^3}{3!} - \dfrac{x^4}{4!} + ...\right) = (1-y)^{-1}$

By binomial theorem

$= 1 + y - y^2 + y^3 - y^4 + ...$

$$= 1 + \left(\frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + ...\right) - \left(\frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + ...\right)^2 + \left(\frac{x^2}{2!} + ...\right)^3$$

$$= 1 + \left(\frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + ...\right) - \left(\frac{x^2}{2!} + ...\right)^2$$

$$= 1 + \frac{x^2}{2!} + \frac{x3}{3!} + x^4\left(\frac{1}{4!} + \frac{1}{4}\right) = 1 + \frac{x^2}{2!} + \frac{x3}{3!} + \frac{5}{6}x^4$$

$$e^x \sec x = \left(1 + \frac{x^2}{2!} + \frac{x3}{3!} + \frac{5}{6}x^4\right)\left(1 + \frac{x}{1} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ...\right)$$

$$= 1 + x + x^2 + \frac{2}{3}x^3 + ...$$

$\therefore e^x scx = 1 + x + x^2 + \dfrac{2}{3}x^3 + ...$ Hence proves the result.

Another method:

Let $f(x) = e^x \sec x$

$f'(x) = e^x \sec x + e^x \sec x \tan x = e^x \sec x(1 + \tan x)$

$\therefore f'(x) = f(x)(\tan x + 1)$

$f''(x) = f'(x)(1 + \tan x) + f(x)\sec^2 x$

$f'''(x) = f'(x)\sec^2 x + f(x)2\sec^2 x \tan x + f''(x)(1 + \tan x) + f'(x)\sec^2 x$

$= 2f(x)\sec^2 x \tan x + 2f'(x)\sec^2 x + f''(x)(1 + \tan x)$.

Now put $x = 0$ in all

$f(0) = 1$, $f'(0) = 1$, $f''(0) = 2$, $f'''(0) = 4$,

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+...................$$

Substituting the values of $f(0)=1$, $f'(0)=1$, $f''(0)=2$, $f'''(0)=4$, we get

$$e^x\sec x=1+x+x^2\times\frac{2}{2!}+x^3\frac{4}{3!}+.................$$

$$e^x scx=1+x+x^2+\frac{2}{3}x^3+.................$$. Hence proves the result.

Q.No.23.: Prove that $\frac{x}{2}\frac{e^x+1}{e^x-1}=1+\frac{1}{6}\frac{x^2}{2!}-\frac{1}{30}\frac{x^4}{4!}+.................$

Sol: $$f(x)=\frac{x}{2}\frac{\left(1+x+\frac{x^2}{2!}+............+1\right)}{\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+.........-1\right)}=\frac{x}{2}\frac{\left(2+x+\frac{x^2}{2!}+............\right)}{x\left(1+\frac{x}{2!}+\frac{x^2}{3!}+.........\right)}$$

$$=\frac{1}{2}\left(2+x+\frac{x^2}{2!}+\frac{x^3}{3!}+............\right)\left(1+\frac{x^2}{2!}+\frac{x^3}{3!}+............\right)^{-1}$$

$$=\left(1+\frac{x}{2}+\frac{x^2}{4}+\frac{x^3}{12}+......\right)\left[1-\left(\frac{x}{2!}+\frac{x^2}{3!}+............\right)+\left(\frac{x}{2!}+\frac{x^2}{3!}+.......\right)^2\right]$$

$$=\left(1+\frac{x}{2}+\frac{x^2}{4}+\frac{x^3}{12}+......\right)-\left(1+\frac{x}{2}+\frac{x^2}{4}+\frac{x^3}{12}+......\right)\left(\frac{x}{2}+\frac{x^2}{6}+\frac{x^3}{24}+......\right)$$

$$\left(1+\frac{x}{2}+\frac{x^2}{4}+\frac{x^3}{12}+......\right)\left(1+\frac{x}{2}+\frac{x^2}{6}+\frac{x^3}{24}+......\right)^2+.......$$

$$=1+\frac{x}{2}+\frac{x^2}{4}+\frac{x^3}{12}+\frac{x^4}{48}-\frac{x}{2}-\frac{x^2}{4}-\frac{x^3}{8}-\frac{x^4}{12}-\frac{x^2}{6}-\frac{x^3}{12}-\frac{x^4}{48}-\frac{x^3}{8}-\frac{x^4}{24}$$

$$+..........\left(1+\frac{x}{2}+\frac{x^2}{4}+\frac{x^3}{12}\right)\left[\frac{x^2}{4}+\frac{x^3}{12}+2.\frac{x}{2}\left(\frac{x^2}{6}+\frac{x^3}{24}+......\right)\right]$$

$$=1-\frac{x^3}{4}-\frac{x^2}{6}-\frac{3x^4}{24}+.....+\frac{x^4}{4}+\frac{x^3}{8}+\frac{x^4}{16}..............+\left(1+\frac{x}{2}+\frac{x^4}{4}+...........\right)$$

$$\left(\frac{x^4}{36}+\frac{x^3}{24}+........\right)^2+2.\frac{x^2}{36}\left(\frac{x^3}{24}+.........\right)+\frac{x^3}{6}+\frac{x^4}{24}+.........$$

$\frac{x}{2}\frac{e^x+1}{e^x-1}=1+\frac{1}{6}\frac{x^2}{2!}-\frac{1}{30}\frac{x^4}{4!}+..................$ Hence proves the result.

Q.No.24.: Prove that $(1+x)^x=1+x^2-\frac{x^3}{2}+\frac{5}{6}x^4+...........$

Sol.: Let $f(x)=(1+x)^x$

$\log f(x)=x\log(1+x)$

Since $\log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+......................$

$\therefore \log f(x)=x\log(1+x)=x^2-\frac{x^3}{2}+\frac{x^4}{3}-\frac{x^5}{4}+.................$

$\frac{f'(x)}{f(x)}=2x-\frac{3x^2}{2}+\frac{4x^3}{3}-\frac{5x^4}{4}+................$

$f'(x)=x(1+x)^{x-1}$

$f''(x)=(1+x)^{x-1}+x(x-1)(1+x)^{x-2}$

$f'''(x)=(x-1)(1+x)^{x-2}+(x-1)(1+x)^{x-2}+x(1+x)^{x-2}+x(x-1)(x-2)(1+x)^{x-3}$

$f^{iv}(x)=x(x-1)(x-2)(1+x)^{x-2}+.............$

Now Maclaurin's series is

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\frac{x^4}{4!}f^{iv}(0)+..........+\frac{x^{2n}}{2n!}f^{2n}(0)........$$

$$=1+x^2-\frac{x^3}{2}+\frac{5}{6}x^4+...................$$

$$\therefore (1+x)^x=1+x^2-\frac{x^3}{2}+\frac{5}{6}x^4+..................,$$

which is the required expansion.

Q.No.26.: Prove that $\cosh^3 x=1+\frac{3}{2}x^2+\frac{7}{8}x^4+...............=\frac{1}{4}\sum_{n=0}^{\infty}\frac{3^{2n}+3}{2n!}x^{2n}$.

Sol.: We know that $\cosh 3x=4\cosh^3 x-3\cosh x$

$$\therefore \cosh^3 x = \frac{1}{4}[\cosh 3x + 3\cosh x]$$

Let $f(x) = \frac{1}{4}[\cosh 3x + 3\cosh x]$ $\Rightarrow f(0) = 1$

Differentiating f(x), w. r. t. x

$f'(x) = \frac{1}{4}[3\sinh 3x + 3\sinh x]$ $\Rightarrow f'(0) = 0$

Again differentiating w. r. t. x, we get

$f''(x) = \frac{1}{4}[9\cosh 3x + 3\cosh x]$ $\Rightarrow f''(0) = 3$

Again differentiating w. r. t. x, we get

$f'''(x) = \frac{1}{4}[27\sinh 3x + 3\sinh x]$ $\Rightarrow f'''(0) = 0$

Again differentiating w. r. t. x, we get

$f^{iv}(x) = \frac{1}{4}[81\cosh 3x + 3\cosh x]$ $\Rightarrow f^{iv}(0) = 21$

$f^{2n}(x) = \frac{1}{4}\left[3^{2n}\cosh 3x + 3\cosh x\right]$ $\Rightarrow f^{2n}(0) = \frac{1}{4}\left(3^{2n} + 3\right)$

Now Maclaurin's series is

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{iv}(0) + ...+ \frac{x^{2n}}{2n!}f^{2n}(0)........$$

$$= 1 + x.0 + \frac{x^2}{2!}.3 + \frac{x^3}{3!}.0 + \frac{x^4}{4!}.21 + ... + \frac{1}{4}\frac{\left(3^{2n}+3\right)x^{2n}}{2n!}$$

$$= 1 + \frac{3x^2}{2} + \frac{7x^4}{8}.......................$$

$$\frac{1}{4}n = 0\sum^{\infty} \frac{3^{2n}+3}{2n!}x^{2n}\text{. Ans.}$$

Q.No.27.: Prove that $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right) = \frac{1}{2}\left(x - \frac{x^3}{3} + \frac{x^5}{5} - ....\right)$.

Sol.: Let $y = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$, Put $x = \tan\theta$

$$\therefore f(x) = y = \tan^{-1}\left(\frac{\sec\theta - 1}{\tan\theta}\right) = \frac{\theta}{2}$$

$$\therefore \frac{dy}{d\theta} = \frac{1}{2}, \qquad x = \tan\theta, \qquad \therefore \frac{dx}{d\theta} = \sec^2\theta$$

$$\therefore \frac{dy}{dx} = \frac{1}{2\left(1+\tan^2\theta\right)} = \frac{1}{2\left(1+x^2\right)}$$

$$\therefore f'(0) = \frac{1}{2\left(1+x^2\right)}$$

$$f''(0) = \frac{-x}{2\left(1+x^2\right)^2}$$

$$f'''(0) = \frac{6x^2-2}{2\left(1+x^2\right)^3} = \frac{3x^2-1}{\left(1+x^2\right)^2}$$

$$f^{iv}(0) == \frac{12x-12x^3-1}{\left(1+x^2\right)^4}$$

$$f^{v}(0) == \frac{12\left(1-10x^2+26x^4\right)}{\left(1+x^2\right)^5}$$

Now Maclaurin's series is

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{iv}(0) + \frac{x^5}{5!}f^{v}(0)$$

$$= \frac{1}{2}\left(x - \frac{x^3}{3} + \frac{x^5}{5} + \dots\dots\dots\dots\dots\right)$$

This completes the proof.

Q.No.28.: Prove that $\cos^{-1}\left(\frac{x-x^{-1}}{x+x^{-1}}\right) = \pi - 2\left(x - \frac{x^3}{3} + \frac{x^5}{5} + \dots\dots\dots\right)$.

Sol.: Let $f(x) = y = \cos^{-1}\left(\frac{x-x^{-1}}{x+x^{-1}}\right)$, $\quad x = \tan\theta$.

$$y = \cos^{-1}\left(\frac{\tan\theta - \cot\theta}{\tan\theta + \cot\theta}\right) = \cos^{-1}\left(\frac{\sin^2\theta - \cos^2\theta}{1}\right) = \pi - 2\theta = \pi - 2\tan^{-1}x$$

$$\therefore f'(x) = \frac{-2}{1+x^2}$$

$$f''(x) = -2\left[\frac{-2x}{\left(1+x^2\right)^2}\right] = \frac{4x}{\left(1+x^2\right)^2}$$

$$f'''(x) = \frac{(-2)\left(6x^2-2\right)}{\left(1+x^2\right)^3}$$

$$f^{iv}(x) = \frac{(-2)\left(12x+12x^3+36x^3+12x\right)}{\left(1+x^2\right)^4} = \frac{-48x+48x^3}{\left(1+x^2\right)^4}$$

$$f^{v}(x) = \frac{-48\left(1-10x^2+26x^4\right)}{\left(1+x^2\right)^5}$$

Now Maclaurin's series is

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{iv}(0) + \frac{x^5}{5!}f^{v}(0) + \ldots\ldots\ldots\ldots\ldots$$

$$= \pi - 2\left(x - \frac{x^3}{3} + \frac{x^5}{5} + \ldots\ldots\ldots\ldots\ldots\right).$$

This completes the proof.

Q.No.29.: Prove that $\sec^2 = 1 + x^2 + \frac{2}{3}x^4 + \ldots\ldots\ldots\ldots\ldots$

Sol.: Let $f(x) = \sec^2 x$

$$f'(x) = 2\sec^2 x \tan x$$

$$f''(x) = 4\sec^2 x \tan^2 x + 2\sec^4 x = 2\sec^2 x\left(2\tan^2 x + \sec^2 x\right)$$

$$f'''(x) = 4\sec x.\sec x \tan x = 2\tan^2 x + \sec^2 x + 2\sec^2 x\left(4\tan x \sec^2 x + 2\sec^2 x \tan x\right)$$

Now Maclaurin's series is

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{iv}(0) + \frac{x^5}{5!}f^{v}(0)$$

$$= 1 + x^2 + \frac{2}{3}x^4 + \ldots\ldots\ldots\ldots\ldots\ldots$$

This completes the proof.

Q.No.30.: Prove that $e^x \sin^2 x = e^x + x^3 + \dfrac{x^4}{6} + \ldots\ldots\ldots\ldots$

Sol.: Let $f(x) = e^x \sin^2 x$

$f'(x) = e^x\left(\sin^2 x + \sin 2x\right)$

$f''(x) = e^x\left(\sin^2 x + \sin 2x\right) + e^x(\sin 2x + 2\cos 2x) = e^x\left(\sin^2 x + 2\sin 2x + 2\cos 2x\right)$

$f'''(x) = e^x\left(\sin^2 x + 2\sin 2x + 2\cos 2x\right) + e^x(\sin 2x + 4\cos 2x + 4\sin 2x)$

$= e^x\left(\sin^2 x + 3\sin 2x - 4\sin 2x + 6\cos 2x\right) = e^x\left(\sin^2 x - 12\sin 2x + 4\cos 2x\right)$

$f^{iv}(x) = e^x\left(\sin^2 x - 12\sin 2x + 4\cos 2x + 4\cos 2x\right)$

Now Maclaurin's series is

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{iv}(0) + \frac{x^5}{5!}f^{v}(0)$$

$$= x^2 + x^3 + \frac{x^4}{6}\ldots\ldots\ldots\ldots\ldots\ldots$$

This completes the proof.

Q.No.32.: If $x = y - \dfrac{y^2}{2} + \dfrac{y^3}{3} - \dfrac{y^4}{4} + \ldots\ldots\ldots\ldots\ldots\ldots,$

Prove that $y = x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \ldots\ldots\ldots\ldots\ldots\ldots$

Sol.: Given $x = y - \dfrac{y^2}{2} + \dfrac{y^3}{3} - \dfrac{y^4}{4} + \ldots\ldots\ldots \Rightarrow \log(1+y) \Rightarrow 1 + y = e^x \Rightarrow y = e^x - 1$

Let $f(x) = y = e^x - 1$

Then $f'(x) = e^x$, $f''(x) = e^x$, $f'''(x) = e^x$, $f^{iv}(x) = e^x$, $f^{v}(x) = e^x$.

Now put x = 0, we get

$f'(0) = 0$, $f''(0) = 1$, $f'''(0) = 1$, $f^{iv}(0) = 1$, $f^{v}(0) = 1$

Now by Maclaurin's theorem

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{iv}(0) + \frac{x^5}{5!}f^{v}(0)$$

$$= x + \frac{x^2}{2!} + \frac{x3}{3!} + \frac{x^4}{4!}\ldots\ldots\ldots\ldots\ldots\ldots$$

This completes the proof.

Q.No.33.: If $x^3+y^3+xy-1=0$, Prove that $y=1-\frac{x}{2}-\frac{26}{81}x^2+\ldots\ldots\ldots\ldots\ldots$.

Sol.: Here $x^3+y^3+xy-1=0$ (i)

Putting x = 0, we get $y^3=1$ $(y)0=1$ (ii)

Differentiating (i) w. r. t. x, we get

$$3x^2+3y^2y'+xy_1'+y=0 \qquad \text{(iii)}$$

Put $x=0$ and $(y)0=1$ in (iii), we get

$$3(y')0=-1 \Rightarrow (y')0=-\frac{1}{3} \qquad \text{(iv)}$$

Differentiating (iii) w. r. t. x, we get

$$6x+6y(y')^2+3y^2y''+y'+xy''+y'=0 \qquad \text{(v)}$$

Putting x = 0, $(y)0=1$, $(y')0=-\frac{1}{3}$ in (v), we get

$$6.1.\frac{1}{9}+3(y'')_0-\frac{1}{3}+0.y''-\frac{1}{3}=0$$

$$(y'')0=0 \qquad \text{(vi)}$$

Differentiating (v) w. r. t. x, we get

$$6+12yy'y''+6(y')^3+6yy'y''+3y^2y'''+y''+xy'''+y''+y''=0 \qquad \text{(vii)}$$

Putting x = 0, $(y)0=1$, $(y')0=-\frac{1}{3}$, $(y'')0=0$ in (vii), we get

$$(y''')_0=-\frac{52}{27} \qquad \text{(viii)}$$

Now by Maclaurin's theorem

$$y=(y)0+x(y')0+\frac{x^2}{2!}(y'')0+\frac{x^3}{3!}(y''')0+\frac{x^4}{4!}\left(y^{iv}\right)+\frac{x^5}{5!}\left(y^{v}\right)0+\ldots\ldots\ldots\ldots$$

$$=1-\frac{x}{3}+\frac{x^2}{2!}.0-\frac{x^3}{1.2..3}.\frac{52}{27}-\ldots\ldots\ldots$$

$$=1-\frac{x}{3}-\frac{26x^3}{81}\ldots\ldots\ldots\ldots\ldots.$$

This completes the proof.

Q.No.34.: Expand $\log\sin(x+h)$ in powers of h by Taylor's theorem.

Sol.: Let $f(x+h)=\log\sin(x+h)$

$$f'(x)=\frac{1}{\sin x}.\cos x=\cot x$$

$$f''(x)=-\operatorname{cosec}^2 x$$

$$f'''(x)=2\operatorname{cosec}x\operatorname{cosec}x\cot x=2\operatorname{cosec}^2 x\cot x$$

Now by Maclaurin's theorem

$$f(x+h)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\frac{x^4}{4!}f^{iv}(0)+\frac{x^5}{5!}f^{v}(0)..................$$

$$\log\sin(x+h)=\log\sin x+h\cot x+\frac{h^2}{2!}\left(-\operatorname{cosec}^2 x\right)+\frac{h^3}{3!}2\operatorname{cosec}^2 x\cot x+............$$

$$=\log\sin x+h\cot x-\frac{h^2\operatorname{cosec}^2 x}{2!}+\frac{2h^3}{3!}\operatorname{cosec}^2 x\cot x+..............$$

This completes the proof.

Q.No.35.: Prove that

$$\sin^{-1}(x+h)=\sin^{-x}+\frac{h}{\sqrt{1-x^2}}+\frac{h^2}{2!}\frac{x}{\left(1-x^2\right)^{3/2}}+\frac{h^3}{3!}\frac{1+2x^2}{\left(1-x^2\right)^{5/2}}+............$$

Sol.: Let $y=\sin^{-1}(x+h)$

$$y'=\frac{1}{\sqrt{1-(x+h)^2}}$$

$$y''=\frac{x+h}{\sqrt{\left[1-(x+h)^2\right]^{3/2}}}$$

$$y'''=\frac{\left[1-(x+h)^2\right]^{3/2}-(x+h).\frac{3}{2}\left[1-(x+h)^2\right]^{1/2}\left[-2(x+h)\right]}{\sqrt{\left[1-(x+h)^2\right]^{3/2}}}$$

$$=\frac{\left[1-(x+h)^2\right]^{3/2}-3(x+h)^2.\left[1-(x+h)^2\right]^{1/2}}{\sqrt{\left[1-(x+h)^2\right]^{3}}}$$

$$=\frac{\left[1-(x+h)^2\right]-3(x+h)^2.}{\sqrt{\left[1-(x+h)^2\right]^{5/2}}}$$

Put x = 0 in all, we get

$$(y)_0=\sin^{-1}h,\ \ (y_1)_0=\frac{1}{\sqrt{1-x^2}},\quad (y'')_0=\frac{x}{\left(1-x^2\right)^{3/2}},\ (y''')_0=\frac{2x^2-1}{\left(1-x^2\right)^{5/2}}$$

Now by Maclaurin's theorem

$$y=(y)_0+x(y')_0+\frac{x^2}{2!}(y'')_0+\frac{x^3}{3!}(y''')_0+\ldots\ldots\ldots\ldots$$

$$=\sin^{-1}h+\frac{h}{\sqrt{1-x^2}}+\frac{h^2}{2!}\frac{x}{\left(1-x^2\right)^{3/2}}+\frac{h^3}{3!}\frac{2x^2}{\left(1-x^2\right)^{5/2}}.$$

This completes the proof.

Q.No.36.: (i) Expand $4x^2+5x+3$ in power of $(x-1)$,

(ii) Expand x^3-2x^2+5x-7 in power of $(x-1)$.

Sol.: $f(x)=4x^2+5x+3$, $f'(x)=8x+5$, $f''(x)=8$

$f(x)=f(1+(x-1))$

Using Taylor's theorem, we get

$$f[1+(x-1)]=f(1)+(x-1)f'(1)+\frac{(x-1)^2}{2!}f''(1)+\frac{(x-1)^3}{3!}f'''(1)+\ldots\ldots..$$

$f(1)=4(1)^2+5(1)+3=12$

$f'(1)=8+5=13$, $f''(1)=8$,

$$f[1+(x-1)]=12+13(x-1)+\frac{(x-1)^2.8}{2}=12+13(x-1)+(x-1)^2.4.$$

which is the required expansion.

(ii): $f(x)=x^3-2x^2+5x-7$

$f'(x)=3x^2+4x+5$, $f''(x)=6x-5$, $f'''(x)=6$

Using Taylor's theorem, we get

$$f[1+(x-1)]=f(1)+(x-1)f'(1)+\frac{(x-1)^2}{2!}f''(1)+\frac{(x-1)^3}{3!}f'''(1)+\ldots\ldots..$$

$f(1) = (1)^3 - 2 + 5 - 7 = -3$

$f'(1) = 3 - 4 + 5 = 4$, $f''(1) = 2$, $f'''(1) = 6$

$$f[1 + (x-1)] = -3 + (x-1).4 + \frac{(x-1)^2}{2!}.2 + \frac{(x-1)^3}{3!}.6$$

$$= -3 + 4(x-1) + (x-1)^2 + (x-3)^3.$$

which is the required expansion.

Q.No.37.: Arrange $7 + (x+2) + 3(x+2)^3 + (x+2)^4 - (x+2)^5$ in power of x using Taylor's theorem.

Sol.: $f(x) = 7 + (x+2) + 3(x+2)^3 + (x+2)^4 - (x+2)^5$, $f(0) = 7 + 2 + 3.8 + 16 - 32 = 17$

$f'(x) = 1 + 3.3(x+2)^2 + 4(x+2)^3 - 5(x+2)^4$,

$= 1 + 9(x^2 + 4x + 4) + 4(x+2)^3 - 5(x+2)^4$, $\quad f'(0) = 1 + 36 + 32 - 80 = -11$

$f''(x) = 18x + 36 + 12(x^2 + 4x + 4) - 20(x+2)^3$, $\quad f''(0) = 36 + 48 - 160 = -76$

$f'''(x) = 18 + 24x + 48 - 60(x+2)^2$, $\quad f'''(0) = 18 + 48 - 249 = -174$

$f^{iv}(x) = 24 - 120x - 240$, $\quad f^{iv}(0) = 24 - 240 = -216$

$f^{v}(x) = -120$ $\quad f^{v}(0) = -120$

Now by Maclaurin's theorem

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{iv}(0) + \frac{x^5}{5!}f^{v}(0) + \ldots\ldots\ldots\ldots\ldots$$

Putting these values, we get

$$= 17 + \frac{x}{1!}.-11 + \frac{x^2}{2}.-76 + \frac{x^3}{6}.-174 + \frac{x^4}{24}.-216 + \frac{x^5}{120}.-120$$

$$= 17 - 11x - 38x^2 - 29x^3 - 9x^4 - x^5$$

which is the required expansion.

Q.No.38.: Prove that $\log(1 - x + x^2) = -x + \frac{x^2}{2} + \frac{2}{3}x^3 + \frac{1}{4}x^4 - \frac{1}{5}x^5 - \frac{1}{3}x^6$.

Sol.: $f(x) = \log(1 - x + x^2)$

$f'(x) = \dfrac{2x-1}{1-x+x^2}$, $\qquad f'(0) = -1$

$f''(x) = \dfrac{(1-x+x^2)2-(2x-1)^2}{(1-x+x^2)^2}$, $\qquad f''(0) = 1$

$f'''(x) = \dfrac{10x^4+2x^2+4+28x^3-12x-8x^5}{(1-x+x^2)^4}$, $\qquad f'''(0) = 4$

Similarly $f^{iv}(0) = 6$, $f^{v}(0) = -24$, $f^{vi}(0) = -240$

Now by Maclaurin's theorem

$$f(\log(1-x+x^2)) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{iv}(0) + \frac{x^5}{5!}f^{v}(0) + \ldots\ldots\ldots\ldots\ldots\ldots$$

$$= 0 - x + \frac{x^2}{2!}.1 + \frac{x^3}{3!}.4 + \frac{x^4}{24}.6 + \frac{x^5}{120}.-24 + \frac{x^6}{6.120}.-240$$

$$= -x + \frac{x^2}{2} + \frac{2}{3}x^3 + \frac{1}{4}x^4 - \frac{1}{5}x^5 - \frac{1}{3}x^6.$$

which is the required expansion.

Q.No.39.: Use Taylor's theorem to prove that

$$\tanh^{-1}(x+h) = \tan^{-1}x + (h\sin z).\frac{\sin z}{1} - (h\sin z)^2\frac{\sin 2z}{2} + (h\sin z)^3\frac{\sin 3z}{3} + \ldots\ldots,$$

where $z = \cot^{-1}x$.

Sol.: Given $f(x+h) = \tan^{-1}(x+h) \Rightarrow f(x) = \tan^{-1}x$

Now $f'(x) = \dfrac{1}{1+x^2} = \dfrac{1}{1+\cot^2 z} = \sin^2 z \Rightarrow f'(x) = \cot z \qquad [\because z = \cot^{-1}x]$

$f''(x) = \dfrac{-1.2x}{(1+x^2)^2} = -\dfrac{2x}{(1+x^2)^2} = \dfrac{-2\cot z}{(1+\cot^2 z)^2} = -2\sin^3 z\cos z$ and so on

Putting these value in the expansion of Taylor's theorem

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \ldots\ldots\ldots\ldots$$

$$= \tan^{-1}x + h\sin^2 z + \frac{h^2}{2!}(-2\sin^3 z.\cos z) + \ldots\ldots\ldots\ldots\ldots$$

$$= \tan^{-1}x + (h\sin z).\frac{\sin z}{1} - (h\sin z)^2.\frac{\sin 2z}{2} + \ldots\ldots\ldots\ldots.$$

This completes the proof.

Q.No.40.: Expand $\cos x$ in powers of $\left(x-\frac{\pi}{2}\right)$.

Sol.: $f(x)=\cos x$

$$f\left(\frac{\pi}{2}+\left(x-\frac{\pi}{2}\right)\right)=f\left(\frac{\pi}{2}\right)+\left(x-\frac{\pi}{2}\right)f'\left(\frac{\pi}{2}\right)+\frac{\left(x-\frac{\pi}{2}\right)^2}{2!}f''\left(\frac{\pi}{2}\right)+\frac{\left(x-\frac{\pi}{2}\right)^3}{3!}f'''\left(\frac{\pi}{2}\right)+\ldots\ldots\ldots\ldots$$

$f'(x)=-\sin x$, $f''(x)=-\cos x$, $f'''(x)=\sin x$

$f\left(\frac{\pi}{2}\right)=0$, $f'\left(\frac{\pi}{2}\right)=-1$, $f''\left(\frac{\pi}{2}\right)=0$, $f'''\left(\frac{\pi}{2}\right)=1$

$$\therefore f\left(\frac{\pi}{2}+\left(x-\frac{\pi}{2}\right)\right)=-\left(x-\frac{\pi}{2}\right)+\frac{\left(x-\frac{\pi}{2}\right)^3}{3!}-\frac{\left(x-\frac{\pi}{2}\right)^5}{5!}+\ldots\ldots\ldots\ldots\ldots\ldots\ldots.$$

which is the required expansion.

Q.No.41.: Obtain the expansion of $\tan^{-1}x$ in the powers of $\left(x-\frac{\pi}{4}\right)$.

Sol.: $f(x)=\tan^{-1}x=f\left[\frac{\pi}{4}+\left(x-\frac{\pi}{4}\right)\right]$

Using Taylor's theorem

$$f\left(\frac{\pi}{4}+\left(x-\frac{\pi}{4}\right)\right)=f\left(\frac{\pi}{4}\right)+\left(x-\frac{\pi}{4}\right)f'\left(\frac{\pi}{2}\right)+\frac{\left(x-\frac{\pi}{4}\right)^2}{2!}f''\left(\frac{\pi}{4}\right)+\frac{\left(x-\frac{\pi}{4}\right)^3}{3!}f'''\left(\frac{\pi}{4}\right)+\ldots\ldots\ldots\ldots$$

$f(x)=\tan^{-1}x$

$f'(x)=\frac{1}{1+x^2}$ $\qquad\Rightarrow f'\left(\frac{\pi}{4}\right)=\frac{1}{1+\frac{\pi^2}{16}}$

$f''(x)=\frac{-2x}{\left(1+x^2\right)^2}$ $\qquad\Rightarrow f''\left(\frac{\pi}{4}\right)=\frac{-\frac{\pi}{2}}{\left(1+\frac{\pi^2}{16}\right)^2}$

$$f'''(x)=\frac{2(3x^2-1)}{(1+x^2)^3} \qquad \Rightarrow f'''\left(\frac{\pi}{4}\right)=\frac{2\left(\frac{3\pi^2}{16}-1\right)}{\left(1+\frac{\pi^2}{16}\right)^3}$$

Substitute the values, we get

$$\therefore f(x)=f\left[\frac{\pi}{4}+\left(x-\frac{\pi}{4}\right)\right]=\tan^{-1}\left(\frac{\pi}{4}\right)+\frac{\left(x-\frac{\pi}{4}\right)}{\left(1+\frac{\pi^2}{16}\right)}-\frac{\pi\left(x-\frac{\pi}{4}\right)}{4\left(1+\frac{\pi^2}{16}\right)^2}+\frac{2\left(x-\frac{\pi}{4}\right)\left(\frac{3\pi^2}{16}-1\right)}{3\left(1+\frac{\pi^2}{16}\right)^3}+\dots\dots$$

Q.No.42.: Find the value of a and b such that the expansion of $\log_e(1+x)-\frac{x(1+ax)}{1+bx}$ in ascending powers of x may begins with terms containing x^4 and show that this is $-\frac{1}{36}x^4$.

Sol.: $\log_e(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-\frac{x^6}{6}+\dots\dots\dots$

$$(1+bx)^{-1}=1-bx+(bx)^2-(bx)^3+(bx)^4-(bx)^5+(bx)^6-\dots\dots$$

$$x(1+ax)(1+bx)^{-1}=(x+ax^2)\left(1-bx+(bx)^2-(bx)^3+\dots\dots\right)$$

$$=x-bx^2+b^2x^3-b^3x^4+ax^2-abx^3+ab^2x^4-ab^3x^5-\dots\dots$$

Now $\log_e(1+x)-\frac{x(1+ax)}{1+bx}=\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\dots\dots\right)-x+x^2(b-a)$

$$+x^3(ab-b^2)+x^4\left(b^3-ab^2-\frac{1}{4}\right)+\dots\dots$$

$$=x^2\left(b-a-\frac{1}{2}\right)+x^3\left(\frac{1}{3}+ab-b^2\right)+x^4\left(b^2-ab^2-\frac{1}{4}\right)+\dots\dots$$

Applying the given conditions, we get

$$b-a-\frac{1}{2}=0\Rightarrow b-a=\frac{1}{2}$$

$$\frac{1}{3}+ab-b^2=a\Rightarrow ab-b^2=-\frac{1}{3}$$

$a = b - \frac{1}{2}$

$b\left(b - \frac{1}{2}\right) - b^2 = -\frac{1}{3} \Rightarrow b^2 - \frac{b}{2} - b^2 = -\frac{1}{3} \Rightarrow b = \frac{2}{3}, \quad a = \frac{1}{6}$

Terms containing $x^4 = x^4\left(b - ab^2 - \frac{1}{4}\right)$

$$= x^4\left(\frac{2}{3} - \frac{1}{6}.\frac{4}{9} - \frac{1}{4}\right) = -\frac{x^4}{36}. \text{ Ans.}$$

Q.No.43.: Prove that

$$\frac{1}{2}(f(a+2x) - f(a)) = xf'(a+x) + \frac{x^3}{3!}f'''(a+x) + \frac{x^5}{5!}f^{v}(a+x) + \ldots\ldots\ldots\ldots$$

Sol.: Let us first expand $f(a+2x)$

$y = f(a+2x) = f(x + (a+x))$

$$= f(a+x) + xf'(a+x) + \frac{x^3}{3!}f'''(a+x) + \frac{x^5}{5!}f^{v}(a+x) + \ldots\ldots\ldots\ldots \qquad \text{(i)}$$

Similarly expanding f(a) by Taylor's theorem, we get

$y = f(a) = f(-x + (a+x))$

$$= f(a+x) + xf'(a+x) + \frac{x^2}{2!}f''(a+x) - \frac{x^3}{3!}f'''(a+x) + \frac{x^4}{4!}f^{iv}(a+x) + \ldots\ldots\ldots\ldots \quad \text{(ii)}$$

Adding (i) and (ii) and dividing by 2, we get

$$f[(a+2x) - f(a)] = f(a+x) + xf'(a+x) + \frac{x^2}{2!}f''(a+x) + \frac{x^3}{3!}f'''(a+x) + \frac{x^4}{4!}f^{iv}(a+x) + \ldots\ldots$$

$$- f(a+x) + xf'(a+x) - \frac{x^2}{2!}f''(a+x) + \frac{x^3}{3!}f'''(a+x) - \frac{x^4}{4!}f^{iv}(a+x) + \ldots\ldots$$

$$\therefore \frac{1}{2}[f(a+2x) - f(a)] = \frac{1}{2}\left[2xf'(a+x) + 2\frac{x^3}{3!}f'''(a+x) + 2\frac{x^5}{5!}f^{v}(a+x) + \ldots\ldots\right]$$

$$= xf'(a+x) + \frac{x^3}{3!}f'''(a+x) + \frac{x^5}{5!}f^{v}(a+x) + \ldots\ldots$$

which is the required expansion.

Q.No.44.: Prove that $\log x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 + \ldots\ldots\ldots\ldots$

$$=\frac{(x-1)}{x}-\frac{1}{2}\left(\frac{x-1}{x}\right)^2+\frac{1}{3}\left(\frac{x-1}{x}\right)^3+\ldots\ldots\ldots.$$

Sol.: Let $f(x)=\log x=f(1+(x-1))$

By Taylor's theorem, we get

$$=f(1)+(x-1)f'(x)+\frac{(x-1)^2}{2!}f''(x)+\frac{(x-1)^3}{3!}f'''(x)+\ldots\ldots\ldots$$

$$=0+(x-1).1+\frac{(x-1)^2}{2!}.-1+\frac{(x-1)^3}{3!}.2+\frac{(x-1)^4}{4!}.-6+\ldots\ldots\ldots$$

$$=(x-1)-\frac{(x-1)^2}{2}+\frac{(x-1)^3}{3}-\frac{(x-1)^4}{4}+\ldots\ldots\ldots.$$

Now $\log x=f(x)\Rightarrow\left(1-\frac{x-1}{x}\right)=-f(x)$

$$-f(x)=f(1)-\left(\frac{x-1}{x}\right)f'(1)-\left(\frac{x-1}{x}\right)^2f''(1)-\left(\frac{x-1}{x}\right)^3f'''(1)-\ldots\ldots\ldots$$

$$\therefore f(x)=\left(\frac{x-1}{x}\right)+\left(\frac{x-1}{x}\right)^2.\frac{1}{2}+\left(\frac{x-1}{x}\right)^3.\frac{1}{3}+\ldots\ldots\ldots.$$

which is the required expansion.

Q. No.45.: If $y=\left[\log\left(x+\sqrt{a^2+x^2}\right)\right]^2$, Expand y up to four terms by Maclaurin's theorem.

Sol.: $y=\left[\log\left(x+\sqrt{a^2+x^2}\right)\right]^2$

$$y_i=\frac{2}{\sqrt{a^2+x^2}}\left(\frac{\sqrt{a^2+x^2}+x}{x+\sqrt{a^2+x^2}}\right).\log\left(x+\sqrt{a^2+x^2}\right)=\frac{2}{\sqrt{a^2+x^2}}\log\left(x+\sqrt{a^2+x^2}\right)$$

$$y_{ii}=\frac{2-2\log\left(x+\sqrt{a^2+x^2}\right).\left(x^2+a^2\right)^{-1/2}.x}{\left(a^2+x^2\right)}$$

$$y_{iii}=\frac{2\left(x^2+a^2\right)^{3/2}\left[x\left(x^2+a^2\right)^{-1/2}+\log\left(x+\sqrt{a^2+x^2}\right)\right]-2x\log\left(x+\sqrt{a^2+x^2}\right).3\left(x^2+a^2\right)^{1/2}.x}{\left(x^2+a^2\right)^3}$$

Putting x = 0, we get

$$y(0)=(\log_e a)^2$$

$$y_i(0)=\frac{2}{a}(\log_e a)$$

$$y_{ii}(0)=\frac{2}{a^2}$$

$$y_{iii}(0)=\frac{2\log a}{a^3}$$

Now by Maclaurin's Theorem, we get

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\frac{x^4}{4!}f^{iv}(0)+\frac{x^5}{5!}f^{v}(0)+\ldots\ldots\ldots\ldots\ldots$$

$$\log\left(x+\sqrt{a^2+x^2}\right)=(\log_e a)^2+\frac{2}{a}(\log_2 a)x+\frac{2}{a^2}.x^2+\left(\frac{1}{3a^2}\log_e a\right)x^3+\ldots\ldots\ldots.$$

This completes the proof.

Q.No.46.: Apply Maclaurin's theorem to prove that

$$\log\sec x=\frac{x^2}{2}+\frac{x^4}{12}+\frac{x^6}{45}.$$

Sol.: Here $f(x)=\log\sec x$

$$f'(x)=\tan x$$

$$f''(x)=1+\tan^2 x$$

$$f'''(x)=2\tan x\left(1+\tan^2 x\right)$$

$$f^{iv}(x)=2\sec^2 x\left(1+\tan^2 x\right)+2\tan x\left(2\tan x\sec^2 x\right)$$

$$f^{v}(x)=4\tan x\sec^2 x\left(4+6\tan^2 x\right)$$

$$f^{vi}(x)=16\sec^2 x+48\tan^2 x\sec^2 x+72\tan^2 x\sec^2 x+120\tan^4 x\sec^2 x$$

$$f(0)=0:\ f'(0)=0;\quad f''(0)=1;\ f'''(0)=0;\ f^{iv}(0)=2;\ f^{v}(0)=0;\quad f^{vi}(0)=2.$$

Now by Maclaurin's Theorem, we get

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\frac{x^4}{4!}f^{iv}(0)+\frac{x^5}{5!}f^{v}(0)+\frac{x^6}{6!}f^{vi}(0)+\ldots\ldots\ldots\ldots$$

Substituting the values from above, we get

$$\log\sec x=\frac{x^2}{2}+\frac{x^4}{12}+\frac{x^6}{45}+\ldots\ldots\ldots$$

This completes the proof.

Q.No.47.: Expand $\cos(x+h)(y+k)$ by Taylor's theorem.

Sol.: $f(x+h)(y+k)=\cos(x+h)(y+k)$

$f(x)=\cos xy$

By Taylor's Theorem, we get

$$F(x+h,\, y+k)=F(x,\, y)+\left(hF_x+kF_y\right)+\frac{1}{2!}\left(h^2F_{xx}+2hk+F_{xy}+k^2F_{yy}\right)$$

$F_x=-\sin xy.y$, $F_y=-\sin xy.x$, $F_{xx}=-y^2\cos xy$, $F_{yy}=-x^2\cos xy$,

$F_{xy}=-[xy\cos xy+\sin xy]$

Putting the values, we get

$$F(x+h,\, y+k)=\cos xy-\sin xy(hy+kx)-\frac{1}{2!}\left(h^2y^2\cos xy+k^2x^2\cos xy\right)$$

$$+2hk(xy\cos xy)+2hk\sin xy$$

$$=\cos xy-\sin xy(hy+kx)-\frac{1}{2!}\left(2hk\sin xy+(hy+kx)^2\cos xy\right).$$

This completes the proof.

Q.No.48.: Expand $e^x\cos y$ in the neighbourhood at $f\left(1,\frac{\pi}{4}\right)$.

Sol.: $e^x\cos y$

Differentiating, we get

$f_x=e^x\cos y$, $\quad f_{xx}=e^x\cos y$,

$f_{xy}=\frac{\partial}{\partial y}\left(e^x\cos y\right)=-e^x\sin y$, $\quad f_x=\frac{\partial}{\partial y}\left(e^x\cos y\right)=-e^x\sin y$

$f_{yy}=\frac{\partial}{\partial y}\left(-e^x\sin y\right)=-e^x\cos y$

$$f(a,b)=f(a,b)+(x-1)f_x(a,b)+(y-1)f_y(a,b)$$
$$+\frac{1}{2!}\left[(x-1)^2 f_{xx}(a,b)+2(x-1)(y-1)f_{xy}(a,b)+(y-1)^2 f_{yy}(a,b)\right]+\ldots\ldots$$

$$f\left(1,\frac{\pi}{4}\right)=\frac{e}{\sqrt{2}}+(x-1)\frac{e}{\sqrt{2}}+\left(y-\frac{\pi}{4}\right)\left(\frac{-e}{\sqrt{2}}\right)$$
$$+\frac{1}{2!}\left[(x-1)^2\frac{e}{\sqrt{2}}+2(x-1)\left(y-\frac{\pi}{4}\right)\left(\frac{-e}{\sqrt{2}}\right)+\left(y-\frac{\pi}{4}\right)^2\left(\frac{-e}{\sqrt{2}}\right)\right]+\ldots\ldots\ldots$$

$$=\frac{e}{\sqrt{2}}\left[1+(x-1)-\left(y-\frac{\pi}{4}\right)+\frac{1}{2!}\left\{(x-1)^2-2(x-1)\left(y-\frac{\pi}{4}\right)+\left(y-\frac{\pi}{4}\right)^2\right\}+\ldots\ldots\right].$$

This completes the proof.

Q.No.49.: Expand $e^x\log(1+y)$ by Maclaurin's theorem, in powers of x and y upto terms of third degree

Sol.: Let $f(x,y)=e^x\log(1+y) \quad \Rightarrow f(0,\ 0)=0$

$$f_x(x,y)=e^x\log(1+y) \quad \Rightarrow f_x(0,\ 0)=0$$

$$f_y(x,\ y)=\frac{e^x}{1+y} \quad \Rightarrow f_y(0,\ 0)=1$$

$$f_{xx}(x,y)=e^x\log(1+y) \quad \Rightarrow f_{xx}(0,\ 0)=0$$

$$f_{xy}(x,y)=\frac{e^x}{(1+y)} \quad \Rightarrow f_{xy}(0,\ 0)=1$$

$$f_{yy}(x,y)=\frac{(1+y)(0)-e^x(1)}{(1+y)^2}=\frac{-e^x}{(1+y)^2} \quad \Rightarrow f_{yy}(0,\ 0)=-1$$

$$f_{xxx}(x,\ y)=e^x\log(1+y) \quad \Rightarrow f_{xxx}(0,\ 0)=0$$

$$f_{xxy}(x,\ y)=\frac{e^x}{(1+y)} \quad \Rightarrow f_{xxy}(0,\ 0)=1$$

$$f_{xyy}(x,\ y)=\frac{(1+y)(0)-e^x(1)}{(1+y)^2}=\frac{-e^x}{(1+y)^2} \quad \Rightarrow f_{xyy}(0,\ 0)=-1$$

$$f_{yyy}(x,\ y)=\frac{2e^x}{(1+y)^3} \quad \Rightarrow f_{yyy}(0,\ 0)=2$$

Now Maclaurin's theorem of f(x, y) gives

$$f(x,y)=f(0,0)+xf_x(0,0)+yf_y(0,0)+\frac{1}{2!}\left[x^2f_{xx}(0,0)+2xyf_{xy}(0,0)+y^2f_{yy}(0,0)\right]$$

$$+\frac{1}{3!}\left[x^3f_{xxx}(0,0)+3x^2yf_{xxy}(0,0)+3xy^2f_{xyy}(0,0)+y^3f_{yyy}(0,0)\right]+....$$

$$=0+x(0)+y(1)+\frac{1}{2!}\left[x^2(0)+2xy(1)+y^2(-1)\right]$$

$$+\frac{1}{3!}\left[x^3(0)+3x^2y(1)+3xy^2(-1)+y^3(2)\right]+....$$

$$=y+xy-\frac{y^2}{2}+\frac{1}{2}\left(x^2y-xy^2\right)+\frac{1}{3}y^3+....\ \text{Ans.}$$

Q.No.50.: Find Taylor's series for $\log\cos x$ about the point $\frac{\pi}{3}$.

Sol.: We know that by Taylor's expansion

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+\frac{h^3}{3!}f''(x)+...............$$

If $x=a$, and $h=x-a$ then,

$$f(x)=f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\frac{(x-a)^3}{3!}f''(a)+...............$$

Now $f(x)=\log\cos x$ (i)

$$f(a)=f\left(\frac{\pi}{3}\right)=\log\cos\frac{\pi}{3}=-\log 2 \qquad \text{(ii)}$$

$$f\,'(a)=f\,'\left(\frac{\pi}{3}\right)=\tan\frac{\pi}{3}=-\sqrt{3} \qquad \text{(iii)}$$

$$f\,''(a)=f\,''\left(\frac{\pi}{3}\right)=-\sec^2\frac{\pi}{3}=-4 \qquad \text{(iv)}$$

$\therefore$From (i), (ii), (iii), (iv), we get

$$\log\cos x=-\log 2-(x-a)\sqrt{3}-\frac{4}{2!}(x-a)^2-...........$$

$$= -\left[\log 2 + (x-a)\sqrt{3} + \frac{4}{2!}(x-a)^2 + \dots\dots\dots\right].$$

$$\log\cos x = -\log 2 - (x-a)\sqrt{3} - \frac{4}{2!}(x-a)^2 \dots\dots\dots$$

This completes the proof.

Q.No.51.: Expand $\frac{(x+h)(y+k)}{x+h+y+k}$ in powers of h, k up to second degree terms.

Sol.: Here $f(x+h, y+k) = \frac{(x+h)(y+k)}{x+h+y+k}$

Put h = k = 0, we have $f(x, y) = \frac{xy}{x+y}$

$$f_x = \frac{(x+y).y - xy.1}{(x+y)^2} = \frac{y^2}{(x+y)^2}, \qquad f_y = \frac{x^2}{(x+y)^2}, \quad \text{by symmetry}$$

$$f_{xx} = -\frac{2y^2}{(x+y)^3}, \qquad f_{yy} = -\frac{2x^2}{(x+y)^3}$$

$$f_{xy} = \frac{(x+y)^2.2x - x^2.2(x+y)}{(x+y)^4} = \frac{2x(x+y) - 2x^2}{(x+y)^3} = \frac{2xy}{(x+y)^3}$$

$$\therefore \frac{(x+h)(y+k)}{x+h+y+k} = f(x+h, y+k) = f(x,y) + \left[hf_x + kf_y\right] + \frac{1}{2!}\left[h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy}\right] + \dots\dots$$

$$= \frac{xy}{x+y} + \left[h.\frac{y^2}{(x+y)^2} + k\frac{x^2}{(x+y)^2}\right]$$

$$+ \frac{1}{2}\left[h^2.\frac{-2y^2}{(x+y)^3} + 2hk\frac{2xy}{(x+y)^3} + k^2.\frac{-2x^2}{(x+y)^3}\right] + \dots\dots\dots\dots$$

$$= \frac{xy}{x+y} + \frac{y^2}{(x+y)^2}.h + \frac{x^2}{(x+y)^2}.k - \frac{y^2}{(x+y)^3}.h^2$$

$$+ \frac{2xy}{(x+y)^3}.hk - \frac{x^2}{(x+y)^3}.k^2 + \dots\dots\dots\dots\dots\dots\dots\dots .$$

This completes the proof.

Q.No.52.: Expand $f(x, y) = \sin xy$ in powers of $x - 1$ and $\left(y - \frac{\pi}{2}\right)$ as for as terms of 2nd degree.

Sol.: $\sin xy = \sin[(x-1)+(1)]\left[\left(y - \frac{\pi}{2}\right) + \left(\frac{\pi}{2}\right)\right]$

Here h =1, and $y = \frac{\pi}{2}$

We know that

$$f[(x+h),(y+k)] = f(x,y) + [hf_x + kf_y] + \frac{1}{2!}\left[h^2 f_{xx} + 2hk_{xy} + k^2 f_{yy}\right] \qquad \text{(i)}$$

$$f_x = \frac{\partial}{\partial x}\left[\sin(x-1)\left(y - \frac{\pi}{2}\right)\right] = \left(y - \frac{\pi}{2}\right)\cos(x-1)\left(y - \frac{\pi}{2}\right)$$

$$f_y = \frac{\partial}{\partial x}\left[\sin(x-1)\left(y - \frac{\pi}{2}\right)\right] = (x-1)\cos(x-1)\left(y - \frac{\pi}{2}\right)$$

$$f_{xy} = \frac{\partial}{\partial x}\left[\left(y - \frac{\pi}{2}\right)\cos(x-1)\left(y - \frac{\pi}{2}\right)\right]$$

$$= \cos(x-1)\left(y - \frac{x}{2}\right) + \left(y - \frac{\pi}{2}\right)\left[-(x-1)\sin(x-1)\left(y - \frac{\pi}{2}\right)\right]$$

$$f_{xx} = -\left(y - \frac{\pi}{2}\right)^2 \sin(x-1)\left(y - \frac{\pi}{2}\right)$$

$$f_{yy} = -(x-1)^2 \sin(x-1)\left(y - \frac{\pi}{2}\right)$$

$$\therefore f[(x+h),(y+k)] = \sin[(x-1)+1]\left[\left(y - \frac{\pi}{2}\right) + \frac{\pi}{2}\right]$$

Substituting the values in (i), we get

$$\sin xy = \sin(x-1)\left(y - \frac{\pi}{2}\right) + \left[\left(y - \frac{\pi}{2}\right).\cos(x-1)\left(y - \frac{\pi}{2}\right) + \frac{\pi}{2}(x-1)\cos(x-1)\left(y - \frac{\pi}{2}\right)\right]$$

$$+ \frac{1}{2!}\left[-\left(y - \frac{\pi}{2}\right)^2 \sin(x-1)\left(y - \frac{\pi}{2}\right)\right]$$

$$+\pi\left[\cos(x-1)\left(y-\frac{\pi}{2}\right)-\left(y-\frac{\pi}{2}\right)(x-1)\sin(x-1)\left(y-\frac{\pi}{2}\right)\right]$$

$$-\left[\frac{\pi^2}{4}(x-1)^2\sin(x-1)\left(y-\frac{\pi}{2}\right)\right].$$

This completes the proof.

Q.No.53.: State and prove Taylor's theorem. Use it to expand $f(x)=\log\sin x$ in powers of $(x-2)$.

Sol.: Taylor's Theorem:

Statement: If f (x + h) can be expanded in ascending powers of x , then

$$f(x+h)=f(h)+xf'(h)+\frac{x^2}{2!}f''(h)+\frac{x^3}{3!}f'''(h)+.......+\frac{x^n}{n!}f^n(h)+...........\ .$$

Proof: Suppose $f(x)=a_0+a_1x+a_2x^2+a_3x^3+......+a_nx^n+......$ (i)

where a_0, a_1, a_2, a_3 ,...... are constants to be evaluated.

Differentiating (i) w.r.t. x, we get

$$f'(x+h)=a_1+2a_2x+3a_3x^2+4a_4x^3+......+na_nx^{n-1}+...... \quad \text{(ii)}$$

Differentiating (ii) w.r.t. x, we get

$$f''(x+h)=2a_2+3.2a_3x+4.3a_4x^2+......+n(n-1)a_nx^{n-2}+...... \quad \text{(iii)}$$

Differentiating (iii) w.r.t. x, we get

$$f'''(x+h)=3.2.1a_3+4.3.2a_4x+......+n(n-1)(n-2)a_nx^{n-3}+...... \quad \text{(iv)}$$

Similarly, if we go on differentiating, we get

$$f^n(x+h)=n(n-1)(n-2)......3.2.1.a_n+\text{ terms containing x} \quad \text{(v)}$$

Putting x = 0 in (i) to (v) , we get

$$a_0=f(h),\ a_1=f'(h),\ a_2=\frac{f''(h)}{2!},\ a_3=\frac{f'''(h)}{3!},........,\ a_n=\frac{f^n(h)}{n!}.$$

Putting these values of constants in (i), we get

$$f(x+h)=f(h)+xf'(h)+\frac{x^2}{2!}f''(h)+\frac{x^3}{3!}f'''(h)+.......+\frac{x^n}{n!}f^n(h)+...........\ .$$

This completes the proof.

(b): $f(x)=\log\sin x=\log\sin[(x-2)+2]$

Now by Taylor's expansion, we get

$$\log\sin[(x-2)+2] = f(2) + (x-2)f'(2) + \frac{(x-2)^2}{2!}f''(2) + \ldots\ldots\ldots\ldots\ldots$$

$$= \log\sin 2 + (x-2)\cot 2 - \frac{(x-2)^2}{2!}\operatorname{cosec}^2(2) + \ldots\ldots\ldots$$

$$\therefore \log\sin x = \log\sin 2 + (x-2)\cot 2 - \frac{(x-2)}{2!}\operatorname{cosec}^2 2 + \ldots\ldots\ldots\ldots\ldots$$

This completes the proof.

Q.No.54.: Prove that $e^{a\sin^{-1}x} = 1 + ax + \frac{(ax)^2}{2!} + \frac{a(1^2-a^2)}{3!} + x^3 + \ldots\ldots\ldots\ldots\ldots\ldots$,

Hence show that $e^{\theta} = 1 + \sin\theta + \frac{\sin^2\theta}{2!} + \frac{2}{3!}\sin^3\theta + \ldots\ldots\ldots\ldots\ldots$.

Sol.: We will first expand $\sin^{-1}x$ by Maclaurin's theorem

$$f(x) = \sin^{-1}x,\quad f(0) = 0,\quad f'(0) = \frac{1}{\sqrt{1-(0)^2}} = 1,\quad f''(0) = 0,\quad f'''(0) = 1$$

According to Maclaurin's theorem

$$f(x) = f(0) + f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots\ldots\ldots\ldots\ldots\ldots$$

$$\therefore \sin^{-1}x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\ldots\ldots\ldots$$

$$a\sin^{-1}x = ax + \frac{ax^3}{3!} + \frac{ax^5}{5!} + \ldots\ldots\ldots\ldots$$

We know that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\therefore e^{a\sin^{-1}x} = 1 + (a\sin^{-1}x) + \frac{(a\sin^{-1})^2}{2!} + \ldots\ldots\ldots\ldots\ldots\ldots$$

$$= 1 + a\left[x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\ldots\right] + \frac{a^2}{2!}\left[x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\ldots\right]^2 + \frac{a^3}{3!}\left[x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\ldots\right]^3$$

Using Binomial theorem

$$= 1 + a\left[x + \frac{x^3}{3!} + \frac{x^5}{5!} + ...\right] + \frac{a^2x^2}{2!}\left[1 + 2\left(\frac{x^2}{3!} + \frac{x^4}{5!} + ...\right)\right] + \frac{a^3x^3}{3!}\left[1 + \left(3\frac{x^2}{3!} + \frac{x^4}{5!} + ...\right)\right]$$

Combining the terms of similar powers of x

$$e^{a\sin^{-1}x} = 1 + ax + \frac{(ax)^2}{2!} + \frac{a\left(1^2 + a^2\right)}{3!}x^3 + \frac{a^2\left(2^2 + a^2\right)}{4!}x^4 + ... \qquad \text{(i)}$$

Hence this proves the result.

(b): In equation (i) put $a = 1$, and $\sin^{-1}x = \theta$, so that $x = \sin\theta$

Substituting the values in (i), we get

$$e^{\theta} = 1 + \sin\theta + \frac{\sin^2\theta}{2!} + \frac{2}{3!}\sin^3\theta + \frac{5}{4!}\sin^4\theta + ...$$

Hence this prove the result.

QNo.55.: Expand $\sin\left(m\sin^{-1}x\right)$ in ascending powers of x up to x^5.

Sol.: Let $f(x) = \sin\left(m\sin^{-1}x\right)$

$$f'(x) = \frac{m\cos\left(m\sin^{-1}x\right)}{\sqrt{1-x^2}}$$

$$f''(x) = \frac{m\left[\frac{\sqrt{1-x^2}(-\sin m)\left(\sin^{-1}x\right)m}{\sqrt{1-x^2}} - \frac{m\cos\left(m\sin^{-1}x\right)(-2x)}{\sqrt{1-x^2}}\right]}{1-x^2}$$

$$= \frac{-m\left[-m\sin m\left(\sin^{-1}x\right) + \frac{\cos\left(m\sin^{-1}x\right)x}{\sqrt{1-x^2}}\right]}{(1-x)^2} = \frac{-m[mf(x) - f'(x)x]}{\left(1-x^2\right)}$$

$$= \frac{-\left[m^2f(x) - f'(x)x\right]}{\left(1-x^2\right)}$$

$$f'''(x) = \frac{-\left[\left(1-x^2\right)\left(m^2f'(x)\right) - \left(f'(x) + xf''(x)\right)\right] - \left[\left(m^2f(x) - f'(x).x. - 2x\right)\right]}{\left(1-x^2\right)^2}$$

$$= \frac{-\left[\left(1-x^2\right)\left(f'(x)\left(m^2-1\right) + f''(x)x\right)\right] + \left[\left(m^2f(x) - f'(x).x. - 2x\right)\right]}{\left(1-x^2\right)^2}$$

Now $f(0)=0$, $f'(0)=m$, $f''(0)=0$, $f'''(0)=-\frac{m(m^2-1)}{1}$

According to Maclaurin's theorem

$$f(x)=f(0)+f'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+.................$$

Substituting the values, we get

$$f(x)=mx-\frac{m(m^2-1)}{3!}x^3+\frac{m(m^2-1)m^2 3^2}{5!}x^5-.................$$

Hence this prove the result**.**

Q.No.56.: If $y=\sin\log(x^2+2x+1)$**,** prove that

$$y=2x-x^2-\frac{2}{3}x^3+\frac{3}{2}x^4-..........$$

Sol.: $y=\sin\log(x+1)^2=\sin 2\log(x+1)=f(x)$

$$f'(x)=\frac{2\cos 2\log(x+1)}{x+1}$$

$$f''(x)=\frac{-(x+1)\frac{4\sin 2\log(x+1)}{x+1}-2\cos 2\log(x+1)}{(x+1)^2}=\frac{-4f(x)}{(x+1)^2}-\frac{f'(x)}{(x+1)}$$

$$f'''(x)=\frac{-4\left[(x+1)^2f'(x)-2(x+1)f(x)\right]}{(x+1)^4}-\frac{\left[(x+1)^2f''(x)-f'(x)\right]}{(x+1)^2}$$

Now $f(0)=0$, $f'(0)=2$, $f''(0)=-2$, $f'''(0)=-4$

According to Maclaurin's theorem

$$f(x)=f(0)+f'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+.................$$

$$=2x-x^2-\frac{2}{3}x^3+\frac{3}{2}x^4-.............$$

Hence this prove the result**.**

Q.No.57.: Expand $\sqrt{x}$ in powers of $x-1$ up to involving $(x-1)^4$.

Sol.: $f(x)=\sqrt{x}$

Let $f(x) = f[1+(x-1)]$

According to Maclaurin's theorem

$$f[(1+x-1)] = f(1) + (x-1)f'(1) + f''(1)\frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!}f'''(1) + \ldots\ldots\ldots\ldots\ldots\ldots \qquad \text{(i)}$$

Now $f(x) = \sqrt{x}$, $\quad f(1) = 1$

$f'(x) = \frac{1}{2\sqrt{x}}$, $\quad f'(1) = \frac{1}{2}$

$f''(x) = -\frac{1}{4x^{3/2}}$, $\quad f''(1) = -\frac{1}{4}$

$f'''(x) = -\frac{1}{4}\left(-\frac{3}{2}\right)\frac{1}{x^{5/2}}$, $\quad f'''(1) = \frac{3}{8}$

Substituting the values in (i), we get

$$f(x) = 1 + \frac{(x-1)}{2} - \frac{1}{4}\frac{(x-1)^2}{2!} + \frac{3}{8}\frac{(x-1)^3}{3!} - \frac{15}{16}\frac{(x-1)^4}{4!} - \ldots\ldots\ldots$$

Hence this proves the result**.**

Q.No.58.: Expand $\tan^{-1}\left(x + \frac{\pi}{2}\right)$ about the point $x = 0$.

Sol.: Let $f(x+h) = \tan^{-1}\left(x + \frac{x}{2}\right)$, where $h = \frac{\pi}{2}$

According to Maclaurin's theorem

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \ldots\ldots\ldots\ldots\ldots\ldots \qquad \text{(i)}$$

Now $f(x) = \tan^{-1}x$, $\quad f(0) = 0$

$f'(x) = \frac{1}{1+x^2}$, $\quad f'(0) = 1$

$f''(x) = -\frac{2x}{(1+x^2)^2}$ $\quad f''(0) = 0$

$f'''(x) = -\frac{\left[(1+x^2)^2.2 - 8x(1+x^2)\right]}{(1+x^2)^2}$, $\quad f'''(0) = -2$

$$f^{iv}(x)=\frac{8x}{\left(1+x^2\right)^3}+\frac{8}{\left(1+x^2\right)^3}-\frac{6x^2}{\left(1+x^2\right)^4},\quad f^{iv}(0)=8$$

Substituting these values in (i), we get

$$f(x+h)=\tan^{-1}\left(x+\frac{\pi}{2}\right)=\frac{\pi}{2}-\frac{\pi^3}{2^2.3!}+\frac{\pi^5}{2^2.5!}-\text{..................}\text{ at } x=0.$$

Hence this prove the result**.**

Q.No.59.: Expand y^x in the neighborhood of point (1, 1) up to the second degree term.

Sol.: Let $f(x,y)=y^x$

$\therefore f(1,1)=1^1=1$

$f_x(x,y)=y^x\log y$, $\qquad f_x(1,1)=0$

$f_y(x,y)=xy^{x-1}$, $\qquad f_y(1,1)=1$

$f_{xx}(x,y)=y^x(\log y)^2$, $\qquad f_{xx}(1,1)=0$

$f_{yy}(x,y)=x(x-1)y^{x-2}$, $\qquad f_{yy}(1,1)=0$

$f_{xy}(x,y)=xy^{x-1}\log y+\frac{y^x}{y}$, $\qquad f_{xy}(1,1)=1$

By Taylor's theorem

$$f(x,y)=f(a,b)+(x-a)f_x(a,b)+(y-b)f_y(a,b)+\frac{(x-a)^2}{2!}f_{xx}(a,b)+(x-a)(y-b)f_{xy}(a,b)$$
$$+\frac{(y-b)^2}{2!}f_{yy}(a,b)+\text{....................}$$

Substituting a = 1, and b = 1, we get

$$f(x,y)=f(1,1)+(x-1)f_x(1,1)+(y-1)f_y(1,1)+\frac{(x-1)^2}{2!}f_{xx}(1,1)+(x-a)(y-1)f_{xy}(1,1)$$
$$+\frac{(y-1)^2}{2!}f_{yy}(1,1)+\text{..........}$$

$$f(x,y)=1+(x-1).0+(y-1).1+\frac{(x-1)^2}{2!}.0+(x-1)(y-1).1+0+0$$

$\therefore y^x=1+xy-x$.

Hence this proves the result**.**

Q.No.60.: Expand $\sin(x+b)$ in a series in ascending power of b and complete the value

when $x = \dfrac{\pi}{3}$ and $b = 0.01$radian.

Sol.: Let $f(x+b) = \sin(x+b)$

From Taylor's series

$$f(x+b) = f(x) + bf'(x) + \frac{b^2}{2!}f''(x) + \frac{b^3}{3!}f'''(x) + \dots\dots\dots\dots\dots\dots$$

Now $f(x) = \sin x$, $f'(x) = \cos x$, $f''(x) = -\sin x$

Substituting the values, we get

$$\sin(x+b) = \sin x + b\cos x - \frac{b^2}{2!}\sin x - \frac{b^3}{3!}\cos x + \frac{b^4}{4!}\sin x + \dots\dots\dots\dots\dots\dots\dots$$

Putting $x = \dfrac{\pi}{3}$, and $b = 0.01$radian

$$\sin\left(\frac{\pi}{3} + 0.01\right) = \sin\frac{\pi}{3} + 0.01\cos\frac{\pi}{3} - \frac{(0.01)^2}{2!}\sin\frac{\pi}{3} - \dots\dots\dots\dots\dots\dots\dots$$

$$\therefore \sin(0.01) = \frac{\sqrt{3}}{2} + 0.01.\frac{1}{2} - \frac{0.0001}{2}.\frac{\sqrt{3}}{2} - \dots\dots\dots\dots = 0.8710\text{ . Ans.}$$

Q.No.61.: Is Maclaurin's expansion of e^{-1/x^2} valid in any interval ? Give reason.

Sol.: Let $f(x) = e^{-1/x^2}$, $f(0) = \dfrac{1}{e^{1/x^2}} = 0$

According to Maclaurin's theorem

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \dots\dots\dots\dots\dots\dots$$

Now $f'(x) = e^{-1/x^2}.\dfrac{2}{x^3}$, $\qquad f'(0) = \infty$

$$f''(x) = e^{-1/x^2}.\frac{4}{x^6} - e^{-1/x^2}.\frac{6}{x^4} = \frac{4-6x2}{e^{1/x^2}x^6}, \qquad f''(0) = \infty$$

$$f'''(x) = e^{-1/x^2}.\frac{8}{x^9} - e^{-1/x^2}.\frac{24}{x^7} - e^{-1/x^2}\frac{12}{x^7} + e^{-1/x^2}\frac{24}{x^5}, \qquad f'''(0) = \infty$$

Hence $f(x)$ becomes equal to infinity when we apply Maclaurin's series to e^{-1/x^2} for any interval of x. Hence the Maclaurin's expansion of e^{-1/x^2} is not valid in any interval.

Q.No.62.: Use Taylor's theorem to expand $f(x,y)=x^2+xy+y^2$ in powers of $(x-1)$ and $(y-2)$.

Sol.: Differentiating $f(x,y)=x^2+xy+y^2$ partially w.r.t. x and y, we get

$f_x=2x+y$, $f_y=x+2y$, $f_{xy}=1$, $f_{xx}=2$, $f_{yy}=2$, $f_{xxx}=0$, $f_{xxy}=0$, $f_{yyx}=0$, $f_{yyy}=0$.

The Taylor's series expansion of f(x, y) in powers of $(x-1)$ and $(y-2)$ is

$$f(x,y)=f(1,2)+\left[(x-1)f_x(1,2)+(y-2)f_y(1,2)\right]+\frac{1}{2!}\left[(x-1)^2f_{xx}(1,2)+2(x-2)(y-2)f_{xy}(1,2)\right.$$
$$\left.+(y-2)^2f_{yy}(1,2)\right]+\frac{1}{3!}\left[(x-1)^3f_{xxx}(1,2)+3(x-1)^2(y-2)f_{xxy}(1,2)\right.$$
$$\left.+3(x-1)(y-2)^2f_{yyx}(1,2)+(y-2)^3f_{yyy}(1,2)\right]+\ldots\ldots\ldots$$

Here f(1, 2) = 7, $f_x(1,2)=4$, $f_y(1,2)=5$, $f_{xy}(1,2)=1$, $f_{xx}=f_{yy}=2$, etc.

Substituting these values

$$f(x,y)=7+4(x-1)+5(y-2)+\frac{1}{2!}\left[2(x-1)^2+2(x-1)(y-2)+2(y-2)+2(y-2)^2\right]+0+\ldots\ldots.$$

Q.No.63.: Expand $f(x,y)=e^{x+y}$ in Taylor's series up to terms up to terms of second degree in the form $a_0+b_1x+b_2y+c_1x^2+c_2xy+c_3y^2+\ldots\ldots\ldots$

(a). by direct use of Taylor's theorem.

(b). by expanding e^{x+y} in a series of powers of x + y.

(c). by multiplying together the separate expansion of e^x and e^y.

Sol.: a. $f=e^{x+y}$, $f_x=e^{x+y}$, $f_y=e^{x+y}$, $f_{xx}=e^{x+y}$, $f_{yy}=e^{x+y}$, $f_{xy}=e^{x+y}$ etc.

Since the series in powers of x and y, the expansion is about (0, 0) (i.e. Maclaurin's series) so $f=f_x=f_y=f_{xx}=f_{yy}=f_{xy}$ at (0, 0) = 1.

By Taylor's theorem

$$e^{x+y}=1+x+y+\frac{x^2+2xy+y^2}{2!}+\ldots\ldots\ldots$$

b. Expanding in powers of x + y

$$e^{x+y}=\sum_{n=0}^{\infty}\frac{(x+y)^n}{n!}=1+(x+y)+\frac{(x+y)^2}{2!}+\frac{(x+y)^3}{!}+\ldots\ldots\ldots$$

c. Term wise series multiplication

$$e^{x+y}=e^x.e^y=\left(\sum_{n=0}^{\infty}\frac{x^n}{n!}\right)\left(\sum_{n=0}^{\infty}\frac{y^n}{n!}\right)$$

$$=\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\ldots\ldots\right)\left(1+y+\frac{y^2}{2!}+\frac{y^3}{3!}+\ldots\ldots\ldots\right)$$

$$=1+(x+y)+\frac{x^2}{2!}+xy+\frac{y^2}{2!}+\ldots\ldots\ldots$$

Q.No.64.: Expand $f(x,y)=e^y\text{In}(1+x)$ in powers of x and y and verify the result by direct expansion.

Sol.: $f=e^y\text{In}(1+x)$

$$f_x=e^y\frac{1}{1+x},\quad f_y=e^y\text{In}(1+x),\quad f_{xy}=\frac{e^y}{1+x},\quad f_{xx}=\frac{-e^y}{(1+x)^2},\quad f_{yy}=e^y\text{In}(1+x),$$

$$f_{xxx}=\frac{2e^y}{(1+x)^3},\quad f_{yyy}=e^y\text{In}(1+x),\quad f_{xxy}=\frac{-e^y}{(1+x)^2},\quad f_{yyx}=\frac{e^y}{1+x}$$

Evaluating these derivatives at $x = 0,\ y = 0$,

$f(0, 0) = 0,\quad f_x(0, 0) = 1,\quad f_y(0, 0) = 0,\quad f_{xy} = 1,\quad f_{xx}(0,0)=-1,\quad f_{yy}(0, 0) = 0,$

$f_{xxx}(0, 0) = 2,\quad f_{yyy}(0, 0) = 0,\quad f_{xxy}(0,0)=-1,\quad f_{yyx}(0, 0) = 1.$

The Taylor's series expansion up to third degree terms is

$$e^y\text{In}(1+x)=f(0,0)+xf_x(0,0)+yf_y(0,0)+\frac{1}{2!}\left[x^2f_{xx}(0,0)+2xyf_{xy}(0,0)+y^2f_{yy}(0,0)\right]$$

$$+\frac{1}{3!}\left[x^3f_{xxx}(0,0)+3x^2yf_{xxy}(0,0)+3xy^2f_{yyx}(0,0)+y^3f_{yyy}(0,0)\right]+\ldots\ldots\ldots$$

$$=0+x.1+0+\frac{1}{2!}\left[-x^2+2.1xy+0\right]+\frac{1}{3!}\left[2x^3-3x^2y+3xy^2+0\right]$$

$$=x-\frac{x^2}{2}+xy+\frac{x^3}{3}-\frac{x^2y}{2}+\frac{xy^2}{2}+\ldots\ldots\ldots\ldots$$

Verification by series multiplication:

We know that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ and

$$\text{In}(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots\ldots\ldots\ldots$$

so $$e^y \text{In}(1+x) = \left(1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} = \ldots\ldots\ldots\right) \times \left(x - \frac{x^2}{2} + \frac{x^3}{3} \ldots\ldots\ldots\ldots\right)$$

Multiplication term by term up to third degree

$$e^y \text{In}(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + xy - \frac{x^2 y}{2} + \frac{xy^2}{2} + \ldots\ldots\ldots$$

Q.No.65.: Find Taylor's expansion of $f(x, y) = \cot^{-1} xy$ in powers of (x + 0.5) and $(y-2)$ up to second degree terms. Hence compute $f(-0.4,\ \ 2.2)$ approximately.

Sol.: Here $f(x, y) = \cot^{-1} xy$

$$f_x = \frac{-y}{1 + x^2 y^2}, \qquad f_y = \frac{-x}{1 + x^2 y^2}, \qquad f_{xx} = \frac{2xy^3}{\left(1 + x^2 y^2\right)^2}, \qquad f_{yy} = \frac{2x^3 y}{\left(1 + x^2 y^2\right)^2},$$

$$f_{xy} = \frac{\left(x^2 y^2 - 1\right)}{\left(1 + x^2 y^2\right)^2}$$

Evaluating these derivatives at the point $x = -\frac{1}{2},\quad y = 2$

$f(x, y) = \cot^{-1}(x, y)$ at $x = -\frac{1}{2}.y = 2$

$$f\left(-\frac{1}{2}.2\right) = \cot^{-1}\left(-\frac{1}{2}.2\right) = \cot^{-1}(-1) = \frac{3\pi}{4}$$

$$f_x = -1, \quad f_y = \frac{1}{4}, \quad f_{xy} = 0, \quad f_{xx} = -2, \quad f_{yy} = -\frac{1}{8}$$

Expanding $\cot^{-1} xy$ in Taylor's series in powers of (x + 0.5) and $(y-2)$, we get

$$f(x, y) = \cot^{-1} xy = f(-0.5{,}2) + (x + 0.5) f_x(-0.5{,}2) + (y-2) f_y(-0.5) + \frac{1}{2!}\left[(x + 0.5)^2\right.$$

$$\times f_{xx}(-0.5) + 2(x+0.5)(y-2)\times f_{xy}(-0.5,2) + (y-2)^2 f_{yy}(-0.5,2)\Big] + \ldots\ldots\ldots$$

$$= \frac{3\pi}{4} - (x+0.5) + \frac{y-2}{4} + \frac{1}{2}\left[-2(x+0.5)^2 - \frac{1}{8}(y-2)^2\right] + \ldots\ldots\ldots$$

Put $x = -0.4$ and $y = 2.2$ to compute

$$\cot^{-1}[(-0.4),(2.2)] = f(-0.4,2.2) = \frac{3\pi}{4} - (0.1) + \frac{.2}{4} - (0.1)^2 - \frac{1}{16}(.2)^2 = 2.29369.$$

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Home Assignments

Q.No.1.: Expand $f(x,y) = x^3 + y^3 + xy^2$ in powers of $(x-1)$ and $(y-2)$ using Taylor's series.

Ans.: $13+7(x-1)+16(y-2)+3(x-1)^2+4(x-1)(y-2)+7(y-2)^2+(x-1)^3$

$+(x-1).(y-2)^2+(y-2)^3$.

Q.No.2.: Obtain Taylor's expansion of $(1+x-y)^{-1}$ in powers of $(x-1)$ and $(y-1)$.

Ans.: $1-x+y+x^2-2xy+y^2+..........$

Q.No.3.: Expand $\cos x \cos y$ in powers of x and y up to fourth degree terms.

Ans.: $1-\frac{1}{2}\left(x^2+y^2\right)+\frac{1}{24}\left(x^4+6x^2y^2+y^4\right)+..........$

Q.No.4.: Obtain the expansion of e^{xy} in powers of $(x-1)$ and $(y-1)$.

Ans.: $e\left[1+(x-1)+(y-1)+\frac{(x-1)^2}{2!}+(x-1)(y-1)+\frac{(y-1)^2}{2!}+..........\right]$

Q.No.5.: Find the Taylor's expansion of $e^x \cos y$ about the point $x = 1, \;\; y=\frac{\pi}{4}$.

Ans.: $\frac{e}{\sqrt{2}}\left[1+(x-1)-\left(y-\frac{\pi}{4}\right)+\frac{(x-1)^2}{2!}-(x-1)\left(y-\frac{\pi}{4}\right)-\frac{\left(y-\frac{\pi}{4}\right)^2}{2!}+...........\right]$.

Q.No.6.: Find the Maclaurin's expansion of $e^x \text{In}(1+y)$ up to terms of 3rd degree.

Ans.: $y+xy-\frac{y^2}{2}+\frac{\left(x^2y-xy^2\right)}{2}+\frac{y^3}{3}+...........$

Q.No.7.: Expand $e^{ax}\sin by$ about origin up to 3rd degree term.

Ans.: $(by+abxy)+\frac{1}{6}\left(3a^2bx^2y-b^3y^3\right)+.........$

Q.No.8.: Find Taylor's expansion x^y about (1, 1).

Ans.: $1+(x-1)+(x-1)(y-1)+\frac{1}{2}(x-1)^2+...........$

Q.No.9.: Expand $\frac{(xy+hk+hy+xk)}{(x+y+h+k)}$ in powers of h and k up to second degree terms.

Ans.: $\frac{xy}{x+y}+\frac{y^2}{(x+y)^3}h+\frac{x^2}{(x+y)^2}k-\frac{y^2}{(x+y)^3}h^2+\frac{2xy}{(x+y)^3}hk-\frac{x^2}{(x+y)^3}k^2+..........$

Q.No.10.: Calculate $\ln\left[(1.03)^{1/3}+(0.98)^{1/4}-1\right]$ approximately by using Taylor's expansion up to first order terms.

Ans.: 0.005.

Q.No.11.: Compute $\tan^{-1}\left(\frac{0.9}{1.1}\right)$ approximately.

Ans.: 0.6904.

Q.No.12.: Find Taylor's expansion of $\sqrt{1+x+y^2}$ in powers of $(x-1)$ and $(y-0)$.

Ans.: $\sqrt{2}\left[1+\frac{x-1}{4}-\frac{(x-1)^2}{32}+\frac{y^2}{4}+..........\right]$.

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