1st Topic

Ordinary Differential Equations

Definition of Differential Equation,
Classifications of Differential Equations
Ordinary Differential Equation and Partial Differential Equation,
Order and Degree of a Differential Equation
Geometrical Meaning of Differential Equation of 1st Order & 1st Degree
Formation of a Differential Equation

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Introduction:

To describe, understand and predict the behaviour of a physical process or a system; a **"Mathematical Model"** is constructed by relating the variables by means of one or more equations. Usually these equations describing the system in motion are **"differential equations"** involving derivatives, which measure the rates of change, the behaviour and interaction of components of the system.

Introduction to Mathematical Modeling:

Scientific model is an abstract and simplified description of a given phenomenon and is most often based on mathematical structures.

Historically following the invention of calculus by Newton (1642-1727) and Leibnitz (1646-1716), there is a burst of activity in mathematical sciences. Early mathematical modeling problems in vibration of strings, elastic bars and columns of air due to Taylor (1685-1731). Daniel Bernoulli (1700-1782), Euler (170701783) and D'Alembert (1717-1783).

Modeling is a technique of transforming a physical problem to a "mathematical model". Thus, a mathematical model describes a natural process or a physical system in mathematical terms, representing an idealization by simplifying the reality by ignoring negligible details of the natural process and emphasizing on only its essential manifestations. Such model yielding reproducible results, can be used for prediction. Thus, a mathematical model essentially expresses a physical system in terms of a functional relationship of the kind:

Dependent variable = function of independent variables, parameters and forcing functions

A model should be general enough to explain the phenomenon but not too complicated precluding analysis. Mathematical formulation of problems involving continuously or discretely changing quantities leads to ordinary or partial differential equations, linear or non-linear equations or a combination of these.

Differential Equation:

A differential equation is an equation, which involves derivatives.

In many branches of science and engineering, we come across equations, which contains, besides the dependent and independent variables, different derivatives of the dependent variable w.r.t. the independent variable or variables. These equations are called differential equations.

e.g. **(i)** $e^x dx + e^y dy = 0$, **(ii)** $\frac{d^2x}{dt^2} + n^2 x = 0$, **(iii)** $y = x\frac{dy}{dx} + \frac{x}{\frac{dy}{dx}}$,

(iv) $\frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} = c$, **(v)** $\frac{dx}{dt} - wy = a\cos pt$, $\frac{dy}{dt} - wx = a\sin pt$,

(vi) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u$, **(vii)** $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$,

are all examples of differential equations.

Classifications of Differential equations:

Differential equations are classified into two categories "ordinary and partial" depending on the number of independent variables appearing in the equation.

Ordinary Differential Equation and Partial Differential Equation

Ordinary Differential Equation (ODE):

If there is a single independent variable and the derivatives are ordinary derivatives, then the equation is called an ordinary differential equation.

e.g., The equations (i) $e^x dx + e^y dy = 0$, (ii) $\frac{d^2x}{dt^2} + n^2 x = 0$, (iii) $y = x\frac{dy}{dx} + \frac{x}{\frac{dy}{dx}}$,

(iv) $\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} = c$, (v) $\frac{dx}{dt} - wy = a\cos pt$, $\frac{dy}{dt} - wx = a\sin pt$,

are all ordinary differential equations.

or

Equations, which involve only one independent variable, are called ordinary differential equations (ODE).

Partial Differential Equation (PDE):

If there are two or more independent variables and the derivatives are partial derivatives, then the equation is called a partial differential equation.

e.g., The equations $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u$ and $\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}$ are partial differential equations.

or

Equations, which involve partial differential coefficients w.r.t more than one independent variable, are called partial differential equations (PDE).

Order and Degree of a differential Equation

Order of a differential equation:

The order of a differential equation is the order of the highest derivative, which occurs.

Thus an ordinary differential equation is said to be of order n if n^{th} derivative of y w.r.t. x is the highest derivative of y in that equation. The idea of the order of a differential equation leads to a useful classification of the equations into equations of first order, second order, etc.

Degree of a differential equation:

The degree of a differential equation is the degree of the highest derivative, which occurs.

e.g., (i) $e^x dx + e^y dy = 0$, (ii) $\frac{d^2x}{dt^2} + n^2 x = 0$, (iii) $y = x\frac{dy}{dx} + \frac{x}{\frac{dy}{dx}}$, (iv) $\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} = c$

(i) is the first order and first degree; (ii) is the second order and first degree;

(iii) written as $y\frac{dy}{dx} = x\left(\frac{dy}{dx}\right)^2 + x$ is clearly of the first order but of second degree;

(iv) written as $\left[1+\left(\frac{dy}{dx}\right)^2\right]^3 = c^2\left(\frac{d^2y}{dx^2}\right)^2$ is the second order and second degree.

Linear differential equation:

An n^{th} order ordinary differential equation in the dependent variable y is said to be linear in y if

i. dependent variable y and all its derivatives are of degree one.
ii. No product terms of y and/or any of it derivatives are present.
iii. No transcendental functions of y and/or its derivatives occur.

The general from of an n^{th} order linear O.D.E. in y with variables coefficient is

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \ldots\ldots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = b(x),$$

where RHS b(x) and all the coefficients $a_0(x)$, $a_1(x), \ldots\ldots\ldots, a_n(x)$ are given functions of x and $a_0(x) \neq 0$.

If all the coefficients a_0, a_1, ...a_n are constants then the above equation is known as nth order linear O.D.E. with constant coefficients.

Non-linear differential equation:

An ordinary differential equation is said to be non-linear if

(i) **y and all its derivatives are of degree more than one.**

(ii) **Product terms of y and/or any of it derivatives are present.**

(iii) **Transcendental functions of y and/or its derivatives occur.**

Note: A linear differential equation is of first degree differential equation, but a first degree differential equation need not to be linear, since it may contain nonlinear terms such y^2, $y^{\frac{-1}{2}}$, e^y, $\sin y$, etc.

S.No.	Diff. Equation	Ans. Kind	Order	Degree	linearity
1	$\frac{dy}{dx} = kx^2$	Ordinary	1	1	Yes
2	$\frac{dy}{dx} + P(x)y = y^n Q(x)$	Ordinary	1	1	No (Yes for n = 0, 1)
3	$e^x dx + e^y dy = 0$	Ordinary	1	1	Nonlinear (in x and y)
4	$\left(\frac{d^3y}{dx^2}\right)^4 - 6x^2\left(\frac{dy}{dx}\right)^8 + e^y = \sin xy$	Ordinary	3	4	No
5	$y\frac{d^2y}{dx^2} + \sin x = 0$	Ordinary	2	1	No
6	$x^2 dy + y^2 dx = 0$	Ordinary	1	1	No
7	$\frac{d^4y}{dx^4} + 3\left(\frac{d^2y}{dx^2}\right)^5 + 5y = 0$	Ordinary	4	1	No

8	$y^2dx+(3xy-1)dy=0$	Ordinary	1	1	Nonlinear in y and linear in x
9	$k(y'')^2=[1+(y'')^2]^3$	Ordinary	2	6	No
10	$\frac{\partial u}{\partial t}=k\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$	Partial	2	1	Yes
11	$\frac{\partial^2 Y}{\partial t^2}=a^2\frac{\partial^2 Y}{\partial x^2}$	Partial	2	1	Yes
12	$\left(\frac{dr}{ds}\right)^3=\sqrt{\frac{d^2r}{ds^2}+1}$	Ordinary	2	1	No

Solution of a differential equation:

A solution of a differential equation is a relation free from derivatives, between the variables, which satisfies the given differential equation.

e.g. $x = A\cos(nt+\alpha)$ (1)

is a solution of $\frac{d^2x}{dt^2}+n^2x=0$. (2)

General (or complete) solution of a differential equation:

The general (or complete) solution of a differential equation is that in which the number of independent arbitrary constants is equal to the order of the differential equation.

Thus, $x = A\cos(nt+\alpha)$ **is a general solution of the differential equation** $\frac{d^2x}{dt^2}+n^2x=0$ **as the number of arbitrary constants** $(A,\ \alpha)$ **is the same as the order of differential equation.**

Particular solution of a differential equation:

A particular solution is that which can be obtained from its general solution by giving particular values to the arbitrary constants.

e.g. $x = A\cos\left(nt + \frac{\pi}{4}\right)$ is the particular solution of the equation (2) as it can be derived from the general solution (1) by putting $\alpha = \frac{\pi}{4}$.

Remark: The solution of a differential equation of nth order is its particular solution if it contains less than n arbitrary constants.

Remarks:

A differential equation may have many solutions. Let us illustrate this fact by the following example.

Each of the function $y = \sin x$, $y = \sin x + 3$, $y = \sin x - \frac{4}{5}$

is a solution of the differential equation $\frac{dy}{dx} = \cos x$.

And as we know from calculus that every solution of this equation is of the form

$$y = \sin x + c,$$

where c is constant. If we regard c is arbitrary, then $y = \sin x + c$ represents the totality of all solutions of the differential equation.

This example illustrates that a differential equation may (and, in general, will) have more than one solution, even infinitely many solutions, which can be represented by a single formula involving an arbitrary constant c. Such a function, which contains an arbitrary constant, is called a general solution of the corresponding differential equation of the first order. If we assign a definite value to that constant, then the solution so obtained is called a particular solution.

Singular solution:

A differential equation may sometimes have an additional solution, which cannot be obtained from the general solution by assigning a particular value to the arbitrary constant. Such a solution is called a singular solution and is not of much engineering interest.

Example:

The equation $$\left(\frac{dy}{dx}\right)^2 - x\frac{dy}{dx} + y = 0 \qquad \text{(i)}$$

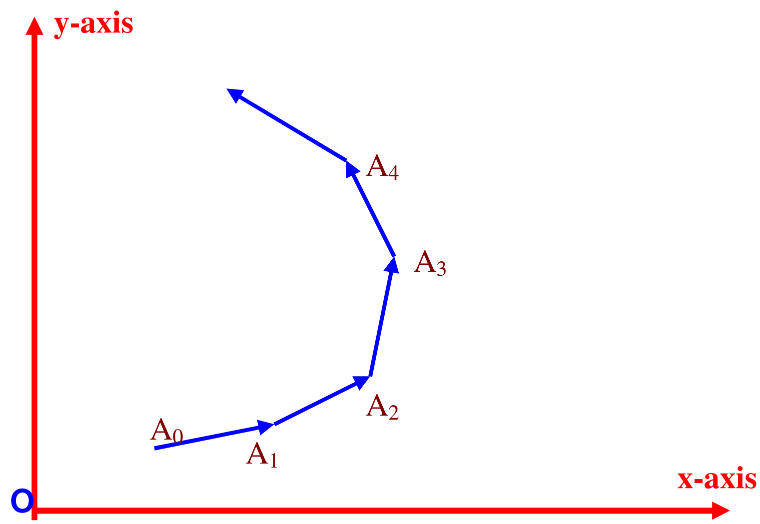
has the general solution $y = cx - c^2$ representing a family of straight lines, where each line corresponds to a definite value of c.

A further solution is $y = \frac{x^2}{4}$. This can be verified by substitution. This is a singular solution of (i), since we cannot obtain it by assigning a definite value to c in the general solution. Obviously, each particular solution represents a tangent to the parabola represented by the singular solution. Singular solution will rarely occur in engineering problems.

Geometrical meaning of a differential equation of the first order and first degree and its solution:

Consider any differential equation of the first order and first degree

$$\frac{dy}{dx} = f(x, y) \text{ or } f\left(x, y, \frac{dy}{dx}\right) = 0. \qquad \text{(i)}$$



We know that the **direction** of a curve at a particular point is determined by drawing a **tangent line** at that point, i.e., its slope is given by $\frac{dy}{dx}$ at that particular point.

Let $A_0(x_0, y_0)$ be any point in the plane. Let $m_0 = \frac{dy_0}{dx_0}$ be the slope of the curve at A_0 derived from (i). Take a neighbouring point $A_1(x_1, y_1)$ such that the slope of

A_0A_1 is m_1. Thus, $m_1 = \dfrac{dy_1}{dx_1}$ be the slope of the curve at A_1 derived from (i). Take a neighbouring point $A_2(x_2, y_2)$ such that the slope of A_1A_2 is m_2. Thus $m_2 = \dfrac{dy_2}{dx_2}$ be the slope of the curve at A_2 derived from (i).

Continuing like this, we get a succession of points. If the points are taken sufficiently close to each other, i.e., if the successive points A_0, A_1, A_3, A_4........... are chosen very near one another, the broken curve $A_0\, A_1\, A_3A_4$........... approximates to a smooth curve $C[y = \phi(x)]$, which is a solution of (i) corresponding to the initial point $A_0 = (x_0, y_0)$. Any point on C and the slope of the tangent at any point to C satisfy (i).

A different choice of the initial point will, in general, give a different curve with the same property, i.e., if the moving point starts at any other point, not on C and move as before, it will describe another curve. The equation of each such curve is thus a **particular solution** of the differential equation (i). The equation of the whole family of such curves is the **general solution** of (i). The slope of the tangent at any point of each member of this family and the co-ordinates of that point satisfy (i).

Concluding remarks:

1. **Particular solution is the equation of one particular curve obtained by the above method.**
2. **General solution of an ordinary differential equation of first order and first degree is the equation of the family of curves obtained by the above method.**
3. **Finally, an ordinary differential equation is a representation of all the tangents of the whole family of the curves.**

Formation of a differential equation:

An ordinary differential equation is formed by elimination of arbitrary constants from a relation in the variables and constants.

To eliminate two arbitrary constants, we require two more equations besides the given relation, leading us to second order derivatives and hence we obtain a differential equation of the second order.

Elimination of n arbitrary constants leads us to n^{th} order derivative and hence we have a differential equation of the n^{th} order.

Now let us formulate ordinary differential equations:

Q.No.1.: Form the differential equation of simple harmonic motion given by

$$x = A\cos(nt + \alpha) .$$

Sol.: To eliminate the constant A and α, differentiating it twice, we get

$$\frac{dx}{dt} = -nA\sin(nt + \alpha) \quad \text{and} \quad \frac{d^2x}{dt^2} = -n^2 A\cos(nt + \alpha) = -n^2 x .$$

Thus $\dfrac{d^2x}{dt^2} + n^2 x = 0$,

which is the required differential equation which states that the acceleration varies as the distance from the origin.

Q.No.2.: Form the differential equation from the equation $x = a\sin(\omega t + b)$.

Sol.: Given $x = a\sin(\omega t + b)$. (i)

Here 'a' and 'b' are constants and to eliminate these constants.

Differentiate (i) w.r.t. x, we get $\dfrac{dx}{dt} = \omega a\cos(\omega t + b)$.

Differentiate again w.r.t. x, we get $\dfrac{d^2x}{dt^2} = -\omega^2 a\sin(\omega t + b) = -\omega^2 x$.

Hence $\dfrac{d^2x}{dt^2} + \omega^2 x = 0$.

This is the required differential equation.

Q.No.3.: Form the differential equation from the equation $y = ax^3 + bx^2$.

Sol.: Given equation is $y = ax^3 + bx^2$. (i)

Differentiate (i) w.r.t. x, we get $\dfrac{dy}{dx} = 3ax^2 + 2bx \Rightarrow \dfrac{1}{x}.\dfrac{dy}{dx} = 3ax + 2b$.

Differentiate again w.r.t. x, we get

$$-\frac{1}{x^2}.\frac{dy}{dx}+\frac{1}{x}\frac{d^2y}{dx^2}=3a \Rightarrow a=\frac{1}{3}\left[\frac{y''}{x}-\frac{y'}{x^2}\right]. \qquad \text{(ii)}$$

Substituting the value of a in (ii), we get

$$\frac{y'}{x}=\left(\frac{y''}{x}-\frac{y'}{x^2}\right)x+2b \Rightarrow \frac{y'}{x}=y''-\frac{y'}{x}+2b \Rightarrow b=\frac{1}{2}\left[\frac{2y'}{x}-y''\right]. \qquad \text{(iii)}$$

Putting (ii) and (iii) in (i), we get

$$y=\frac{x^3}{3}\left[\frac{xy''-y'}{x^2}\right]+\frac{x^2}{2}\left[\frac{2y'-y''x}{x}\right] \Rightarrow 6y=2x^2y''-2xy'+6xy'-3x^2y''$$

$$\Rightarrow x^2y''-4xy'+6y=0 \Rightarrow x^2\frac{d^2y}{dx^2}-4x\frac{dy}{dx}+6y=0,$$

which is the required differential equation.

Q.No.4.: Form the differential equation from the equation $xy=Ae^x+Be^{-x}$.

Sol.: Given equation is $xy=Ae^x+Be^{-x}$. (i)

Differentiate (i) w.r.t. x, we get $\quad y+x\frac{dy}{dx}=Ae^x-Be^{-x}$.

Differentiate again w.r.t. x, we get $\quad \frac{dy}{dx}+\frac{dy}{dx}+x\frac{d^2y}{dx^2}=Ae^x+Be^{-x}$

$$\Rightarrow x\frac{d^2y}{dx^2}+2\frac{dy}{dx}=xy, \qquad \left[\because xy=Ae^x+Be^{-x}\right]$$

which is the required differential equation.

Q.No.5.: Form the differential equation from the equation $y=e^x(A\cos x+B\sin x)$.

Sol.: Given equation is $y=e^x(A\cos x+B\sin x)$. (i)

Differentiate (i) w.r.t. x, we get

$$\frac{dy}{dx}=e^x(A\cos x+B\sin x)+e^x(-A\sin x+B\cos x) \Rightarrow \frac{dy}{dx}=y+e^x(-A\sin x+B\cos x).$$

Differentiate again w.r.t. x, we get

$$\frac{d^2y}{dx^2}=\frac{dy}{dx}+e^x(-A\sin x+B\cos x)+e^x(-A\cos x+B\sin x) \Rightarrow \frac{d^2y}{dx^2}=\frac{dy}{dx}+\left(\frac{dy}{dx}-y\right)-y$$

$$\Rightarrow \frac{d^2y}{dx^2}-2\frac{dy}{dx}+2y,$$

which is the required differential equation.

Q.No.6.: Form the differential equation of the equation $y = ae^{2x} + be^{-3x} + ce^{x}$.

Sol.: Given equation is $y = ae^{2x} + be^{-3x} + ce^{x} \Rightarrow e^{-x}y = ae^{x} + be^{-4x} + c$. (i)

Differentiate (i) w.r.t. x, we get

$$-e^{-x}y + e^{-x}\frac{dy}{dx} = ae^{x} - 4be^{-4x} \Rightarrow -e^{3x}y + e^{3x}\frac{dy}{dx} = ae^{5x} - 4b$$

Differentiate again w.r.t. x, we get

$$-3e^{3x}y - e^{3x}\frac{dy}{dx} + 3e^{3x}\frac{dy}{dx} + e^{3x}\frac{d^2y}{dx^2} = 5ae^{5x} \Rightarrow -3e^{-2x}y + 2e^{-2x}\frac{dy}{dx} + e^{-2x}\frac{d^2y}{dx^2} = 5a .$$

Differentiate again w.r.t. x, we get

$$6e^{-2x}y - 3e^{-2x}\frac{dy}{dx} - 4e^{-2x}\frac{dy}{dx} + 2e^{-2x}\frac{d^2y}{dx^2} - 2e^{-2x}\frac{d^2y}{dx^2} + e^{-2x}\frac{d^3y}{dx^3} = 0$$

$$\Rightarrow \frac{d^3y}{dx^3} - 7\frac{dy}{dx} + 6y = 0.$$

which is the required differential equation.

Q.No.7.: Obtain the differential equation of all circles of radius 'a' and centre (h, k).

Sol.: Since we know that the equation of a circle whose radius is 'a' and centre is (h, k) is

$$(x-h)^2 + (y-k)^2 = a^2. \qquad \text{(i)}$$

Differentiate (i) w.r.t. x, we get

$$2(x-h) + 2(y-k)\frac{dy}{dx} = 0 \Rightarrow (x-h) + (y-k)\frac{dy}{dx} = 0. \qquad \text{(ii)}$$

Differentiate again w.r.t. x, we get

$$1 + (y-k)\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0 \qquad \text{(iii)}$$

$$\Rightarrow y - k = -\frac{1 + \left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}$$

From (ii) $x-h=-(y-k)\frac{dy}{dx}=\frac{\frac{dy}{dx}\left[1+\left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}}$.

Substituting the values of $(x-h)$ and $(y-k)$ in (i), we get

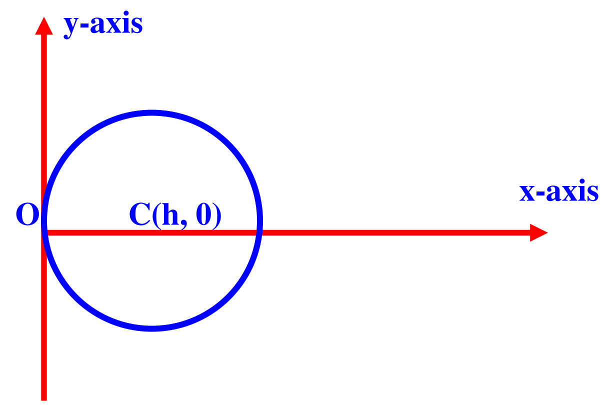
$$\frac{\left(\frac{dy}{dx}\right)^2\left[1+\left(\frac{dy}{dx}\right)^2\right]^2}{\left(\frac{d^2y}{dx^2}\right)^2}+\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^2}{\left(\frac{d^2y}{dx^2}\right)^2}=a^2 \Rightarrow \left[1+\left(\frac{dy}{dx}\right)^2\right]^3=a^2\left(\frac{d^2y}{dx^2}\right)^2,$$

which is the required differential equation.

Q.No.8.: Find the differential equation of a family of the circles passing through the origin and having centres on the x-axis.

Sol.: Since we know that the equation of circle with centre (h, k) and radius 'a' is given by equation $(x-h)^2+(y-k)^2=a^2$. (i)

Here $k=0$ and $a=h$.



$\therefore$ Equation (i) becomes

$$x^2+h^2-2hx+y^2=h^2 \Rightarrow x^2+y^2-2hx=0. \quad \text{(ii)}$$

Here h is the constant and to eliminate it, differentiating (ii) w.r.t. x, we get

$$2x+2y\frac{dy}{dx}-2h=0 \Rightarrow h=-\left(x+y\frac{dy}{dx}\right).$$

Putting the value of h in (ii), we get

$$x^2+y^2-2x\left(x+y\frac{dy}{dx}\right)=0 \Rightarrow x^2+y^2-2x^2-2xy\frac{dy}{dx}=0 \Rightarrow y^2-x^2-2xy\frac{dy}{dx}=0$$

$$\Rightarrow 2xy\frac{dy}{dx} + x^2 - y^2 = 0,$$

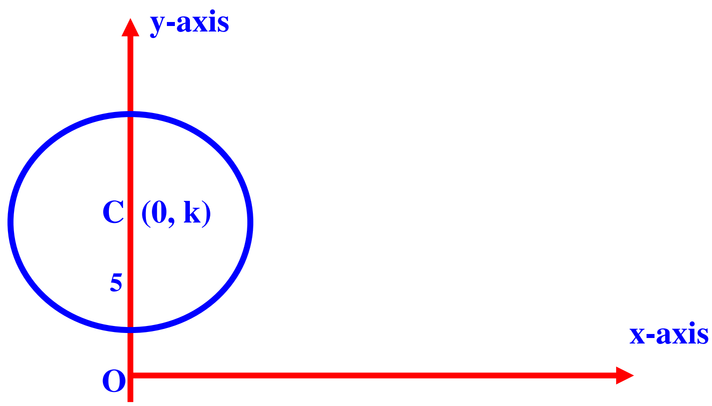
which is the required differential equation.

Q.No.9.: Find the differential equation of all circles of radius 5, with their centers at y-axis.

Sol.: Since we know that the equation of circle with centre (h, k) and radius 'a' is given by equation $(x-h)^2 + (y-k)^2 = a^2$. (i)

Here $h = 0$ and $a = 5$.

$\therefore$ Equation (i) becomes $x^2 + y^2 - 2yk + k^2 = 25$. (ii)



Here k is the constant and to eliminate it , differentiating (ii) w.r.t. x, we get

$$2x + 2y\frac{dy}{dx} - 2k\frac{dy}{dx} = 0 \Rightarrow 2k\frac{dy}{dx} = 2\left(x + y\frac{dy}{dx}\right) \Rightarrow k = \frac{\left(x + y\frac{dy}{dx}\right)}{\frac{dy}{dx}}.$$

Putting the value of k in (ii), we get

$$x^2 + y^2 - 2y\left[\frac{\left(x + y\frac{dy}{dx}\right)}{\frac{dy}{dx}}\right] + \left[\frac{\left(x + y\frac{dy}{dx}\right)}{\frac{dy}{dx}}\right]^2 = 25$$

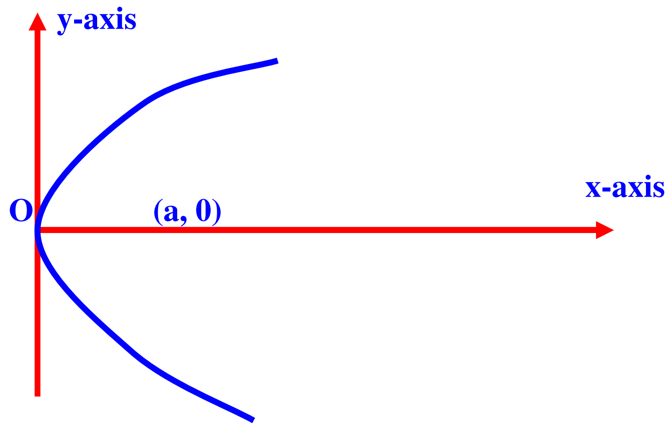
$$\Rightarrow \left(x^2 - 25\right)\left(\frac{dy}{dx}\right)^2 - 2xy\frac{dy}{dx} - 2y^2\left(\frac{dy}{dx}\right)^2 + x^2 + 2xy\frac{dy}{dx} + y^2\left(\frac{dy}{dx}\right)^2 + y^2\left(\frac{dy}{dx}\right)^2 = 0$$

$\Rightarrow \left(x^2 - 25\right)\left(\frac{dy}{dx}\right)^2 + x^2 = 0$, which is the required differential equation.

Q.No.10.: Find the differential equation of all parabolas with x-axis as the axis and (a, 0) as focus.

Sol.: Since we know that the equation of the parabola, whose x-axis as the axis and (a, 0) as focus is $y^2 = 4ax$. (i)

Differentiating (i) both sides, we get



$2y\dfrac{dy}{dx} = 4a \Rightarrow y\dfrac{dy}{dx} = 2a$, which is the required differential equation.

Home Assignments

Q.No.1.: Eliminate the arbitrary constants and obtain the differential equation of

$$y = cx + c^2.$$

Ans.: $x\dfrac{dy}{dx} + \left(\dfrac{dy}{dx}\right)^2 = y$.

Q.No.2.: Eliminate the arbitrary constants and obtain the differential equation of

$$yA + Bx + Cx^2.$$

Ans.: $\dfrac{d^3y}{dx^3} = 0$.

Q.No.3.: Eliminate the arbitrary constants and obtain the differential equation of

$$y = A\cos 2t + B\sin 2t.$$

Ans.: $\dfrac{d^2y}{dt^2} + 4y = 0$.

Q.No.4.: Eliminate the arbitrary constants and obtain the differential equation of

$$y = Ae^{3x} + Be^{2x}.$$

Ans.: $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$.

Q.No.5.: Eliminate the arbitrary constants and obtain the differential equation of

$$y = Ae^{x} + Be^{-x} + C$$

Ans.: $\frac{d^3y}{dx^3} - \frac{dy}{dx} = 0$.

Q.No.6.: Eliminate the arbitrary constants and obtain the differential equation of

$$xy = Ae^{x} + Be^{-x} + x^2.$$

Ans.: $x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + x^2 - xy - 2 = 0$.

Q.No.7.: Eliminate the arbitrary constants and obtain the differential equation of

$$Ax^2 + By^2 = 1$$

Ans.: $xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} = 0$.

Q.No.8.: Eliminate the arbitrary constants and obtain the differential equation of

$$y^2 - 2ay + x^2 = a^2.$$

Ans.: $\left(x^2 - 2y^2\right)\left(\frac{dy}{dx}\right)^2 - 4xy\frac{dy}{dx} - x^2 = 0$.

Q.No.9.: Eliminate the arbitrary constants and obtain the differential equation of

$$e^{2y} + 2axe^{y} + a^2 = 0.$$

Ans.: $\left(1 - x^2\right)\left(\frac{dy}{dx}\right)^2 + 1 = 0$.

Q.No.10.: Obtain the differential equation of all straight lines in a plane.

Ans.: $\frac{d^2y}{dx^2} = 0$.

Q.No.11.: Obtain the differential equation of all circles of radius r whose centres lie on the x-axis.

Ans.: $y^2\left[1+\left(\frac{dy}{dx}\right)^2\right]=r^2$.

Q.No.12.: Obtain the differential equation of all conics whose axis coincide with the axis of co-ordinates.

Ans.: $xy\frac{d^2y}{dx^2}+x\left(\frac{dy}{dx}\right)^2=y\frac{dy}{dx}$.

Q.No.13.: Obtain the differential equation of all circles in a plane.

Ans.: $\left[1+\left(\frac{dy}{dx}\right)^2\right]\frac{d^3y}{dx^3}-3\frac{dy}{dx}\left(\frac{d^2y}{dx^2}\right)^2=0$.

Q.No.14.: Obtain the differential equation of all the circles in the first quadrant which touch the co-ordinate axis.

Ans.: $(x-y)^2\left[1+\left(\frac{dy}{dx}\right)^2\right]=\left(x+y\frac{dy}{dx}\right)^2$.

Q.No.15.: Obtain the differential equation of all parabolas with lotus rectum '4a' and axis parallel to the axis.

Ans.: $2a\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^3=0$.

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2^{nd} Topic

Ordinary Differential Equations of First order

"Variable Separable Form"

Differential equations reducible to variable separable form

Differential equations related with initial value problems

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Differential equations of the first order and first degree:

Normally it is easy to verify that a given function is a solution of a differential equation, while it is very difficult to find the solution of even the first order and first degree ordinary differential equations

$$\frac{dy}{dx} = f(x, y).$$

But on the other hand, there are certain standard types of first order and first degree ordinary differential equations for which solutions can be readily obtained by standard methods.

Now here we shall discuss some special methods of solutions, which are applied to the following types of equations:

1. **Variable Separable Form**
2. **Homogeneous Form**
3. **Linear Differential Equations**
4. **Exact Differential Equations**

In other cases, the particular solution may be determined numerically.

1. EQUATIONS, WHERE VARIABLES ARE SEPARABLE: (VARIABLE SEPARABLE FORM)

If in an equation it is possible to collect all the functions of x and dx on one side and all the functions of y and dy on the other side, then the variables are said to be separable.

Thus, the general form of such an equation is

$$f(y)dy = \phi(x)dx.$$

Method to solve this type of differential equation (i.e. Variable separable form):

Integrating both sides, we get

$$\int f(y)dy = \int \phi(x)dx + c,$$

which is the general solution, c being an arbitrary constant.

Now let us solve some differential equations, where variables are separable:

Q.No.1.: Solve the differential equation $\frac{dy}{dx} = e^{3x-2y} + x^2 e^{-2y}$.

Sol.: Given equation is $\frac{dy}{dx} = e^{3x-2y} + x^2 e^{-2y} \Rightarrow e^{2y}dy = \left(e^{3x} + x^2\right)dx$.

Integrating both sides, we get $\int e^{2y}dy = \int \left(e^{3x} + x^2\right)dx + c$

$$\Rightarrow \frac{e^{2y}}{2} = \frac{e^{3x}}{3} + \frac{x^3}{3} + c \Rightarrow 3e^{2y} = 2\left(e^{3x} + x^3\right) + c', \quad [c' = 6c]$$

which is the required solution.

Q.No.2.: Solve the differential equation $y\sqrt{\left(1-x^2\right)}dy + x\sqrt{\left(1-y^2\right)}dx = 0$.

Sol.: Given equation is $y\sqrt{\left(1-x^2\right)}dy + x\sqrt{\left(1-y^2\right)}dx = 0 \Rightarrow \frac{y}{\sqrt{1-y^2}}dy = -\frac{x}{\sqrt{1-x^2}}dx$ (i)

Put $1 - y^2 = t$ and $1 - x^2 = z$

$\Rightarrow -2ydy = dt$ and $-2xdx = dz$.

Therefore, (i) becomes $-\frac{\frac{dt}{2}}{\sqrt{t}} = \frac{\frac{dz}{2}}{\sqrt{z}} \Rightarrow -\frac{dt}{\sqrt{t}} = \frac{dz}{\sqrt{z}}$.

Integrating both sides, we get $-\int \frac{dt}{\sqrt{t}} = \int \frac{dz}{\sqrt{z}} + c'$

$$\Rightarrow -\frac{t^{1/2}}{\frac{1}{2}} = \frac{z^{1/2}}{\frac{1}{2}} + c' \Rightarrow -\sqrt{t} = \sqrt{z} + c. \quad \left(\text{where } c = \frac{c'}{2}\right)$$

$$\therefore \sqrt{z} + \sqrt{t} + c = 0 \Rightarrow \sqrt{1-x^2} + \sqrt{1-y^2} + c = 0. \text{ Ans.}$$

This is the required solution of this differential equation.

Q.No.3.: Solve the differential equation $\left(x^2 - yx^2\right)\frac{dy}{dx} + y^2 + xy^2 = 0$

Sol.: Given equation is $\left(x^2 - yx^2\right)\frac{dy}{dx} + y^2 + xy^2 = 0$.

$$\Rightarrow x^2(1-y).\frac{dy}{dx} + y^2(1+x) = 0 \Rightarrow x^2(1-y).\frac{dy}{dx} = -y^2(1+x)$$

$$\Rightarrow \left(\frac{y-1}{y^2}\right)dy = \left(\frac{x+1}{x^2}\right)dx \Rightarrow \left(\frac{1}{y} - \frac{1}{y^2}\right)dy = \left(\frac{1}{x} + \frac{1}{x^2}\right)dx$$

Integrating both sides, we get $\int \left(\frac{1}{y} - \frac{1}{y^2}\right)dy = \int \left(\frac{1}{x} + \frac{1}{x^2}\right)dx + c'$

$$\Rightarrow \log y + \frac{1}{y} = \log x - \frac{1}{x} + c' \Rightarrow \log y - \log x + \frac{1}{y} + \frac{1}{x} = c'$$

$$\Rightarrow \log\left(\frac{y}{x}\right) + \frac{1}{y} + \frac{1}{x} = c' \Rightarrow \log\left(\frac{x}{y}\right) - \frac{1}{x} - \frac{1}{y} = c. \text{ Ans.} \qquad (\text{where } c = -c')$$

This is the required solution of this differential equation.

Q.No.4.: Solve the differential equation $xy\frac{dy}{dx} = 1 + x + y + xy$.

Sol.: Given equation is $xy\frac{dy}{dx} = 1 + x + y + xy \Rightarrow xy\frac{dy}{dx} = (1+x)(1+y)$

$$\Rightarrow \left(\frac{y}{1+y}\right)dy = \left(\frac{1+x}{x}\right)dx.$$

Integrating both sides, we get $\int \left(\frac{y}{1+y}\right)dy = \int \left(\frac{1+x}{x}\right)dx + c$

$\Rightarrow \int \left(\frac{1+y-1}{1+y}\right)dy = \int \frac{dx}{x} + \int dx + c \quad \Rightarrow y - \log(1+y) = \log x + x + c$

$\Rightarrow \log x + \log(y+1) + x - y + c = 0 \qquad \Rightarrow y = x + \log[x(1+y)] + c$. Ans.

This is the required solution of this differential equation.

Q.No.5.: Solve the differential equation $3e^x \tan y dx + (1-e^x)\sec^2 y dy = 0$.

Sol.: Given equation is $3e^x \tan y dx + (1-e^x)\sec^2 y dy = 0$.

$\Rightarrow (1-e^x)\sec^2 y dy = -3e^x \tan y dx \Rightarrow \frac{\sec^2 y}{\tan y} dy = -\frac{3e^x}{1-e^x} dx$.

Integrating both sides, we get $\int \left(\frac{\sec^2 y}{\tan y}\right) dy = \int \left(-\frac{3e^x}{1-e^x}\right) dx + \log c$

$\Rightarrow \log \tan y = 3\log(1-e^x) + \log c \Rightarrow \tan y = c(1-e^x)^3$. Ans.

This is the required solution of this differential equation.

Q.No.6.: Solve the differential equation $(xy^2 + x)dx + (yx^2 + y)dy = 0$.

Sol.: Given equation is $(xy^2 + x)dx + (yx^2 + y)dy = 0$

$\Rightarrow x(1+y^2)dx + y(1+x^2)dy \Rightarrow y\frac{dy}{1+y^2} = -x\frac{dx}{1+x^2} \Rightarrow \frac{1}{2}.\frac{2y}{1+y^2}dy = -\frac{1}{2}.\frac{2x}{1+x^2}dx$.

Integrating both sides, we get $\int \frac{1}{2}.\frac{2y}{1+y^2}dy = -\frac{1}{2}.\int \frac{2x}{1+x^2}dx + \log c'$

$\frac{1}{2}\log(1+y^2) = -\frac{1}{2}\log(1+x^2) + \log c' \Rightarrow \log(1+y^2)^{1/2} = -\log(1+x^2)^{1/2} + \log c'$

$\Rightarrow \log(1+y^2)^{1/2} + \log(1+x^2)^{1/2} = \log c' \Rightarrow \left(\sqrt{1+y^2}\right)\left(\sqrt{1+x^2}\right) = c'$.

Squaring both sides, we get $(1+x^2)(1+y^2) = c$, where $c'^2 = c$.

This is the required solution of this differential equation.

Q.No.7.: Solve the differential equation $\frac{dy}{dx} = e^{2x-3y} + 4x^2e^{-3y}$.

Sol.: Given equation is $\frac{dy}{dx} = e^{2x-3y} + 4x^2e^{-3y} \Rightarrow \frac{dy}{dx} = \frac{e^{2x}}{e^{3y}} + \frac{4x^2}{e^{3y}} \Rightarrow \frac{dy}{dx} = \frac{e^{2x}+4x^2}{e^{3y}}$

$\Rightarrow e^{3y}dy = (e^{2x} + 4x^2)dx$.

Integrating both sides, we get $\int e^{3y}dy = \int \left(e^{2x} + 4x^2\right)dx + c'$

$\Rightarrow \dfrac{e^{3y}}{3} = \dfrac{e^{2x}}{2} + \dfrac{4x^3}{3} + c'$ $\quad\Rightarrow 3e^{3y} - 2e^{3y} + 8x^3 = -6c'$

$\Rightarrow 3e^{3y} - 2e^{3y} + 8x^3 = c$, (where $c = -6c'$).

This is the required solution of this differential equation.

Q.No.8.: Solve the differential equation $\dfrac{dy}{dx} = \dfrac{x(2\log x + 1)}{\sin y + y\cos y}$.

Sol.: Given equation is $\dfrac{dy}{dx} = \dfrac{x(2\log x + 1)}{\sin y + y\cos y}$

$\Rightarrow (\sin y + y\cos y).dy = x(2\log x + 1).dx$.

Integrating both sides, we get

$\int (\sin y + y\cos y).dy = \int x(2\log x + 1).dx + c$

$\Rightarrow -\cos y + y\sin y - \int 1.\sin y.dy = 2\left[\int (\log x).xdx\right] + \int xdx + c$

$\Rightarrow -\cos y + y\sin y + \cos y = 2\left[\log x.\dfrac{x^2}{2} - \int \dfrac{1}{x}.\dfrac{x^2}{2}dx\right] + \dfrac{x^2}{2} + c$

$\Rightarrow y\sin y = x^2\log x + c$. Ans.

This is the required solution of this differential equation.

Q.No.9.: Solve the differential equation $y - x\dfrac{dy}{dx} = a\left(y^2 + \dfrac{dy}{dx}\right)$.

Sol.: Given equation is $y - x\dfrac{dy}{dx} = a\left(y^2 + \dfrac{dy}{dx}\right) \Rightarrow y - ay^2 = (a + x).\dfrac{dy}{dx}$

$\Rightarrow \dfrac{dy}{y - ay^2} = \dfrac{dx}{a + x}$.

Integrating both sides, we get $\int \dfrac{dy}{y - ay^2} = \int \dfrac{dx}{x + a} + \log c$.

Now $\dfrac{1}{y(1 - ay)} = \dfrac{A}{y} + \dfrac{B}{1 - ay} \Rightarrow A(1 - ay) + By = 1$

At $y = 0$, $A = 1$, then $(1 - ay) = 0 \Rightarrow y = \dfrac{1}{a}$.

$\therefore 1 = B \times \frac{1}{a} \Rightarrow B = a$.

Hence $\int \frac{dy}{y - ay^2} = \int \frac{dx}{x + a} + \log c$

$\Rightarrow \int \frac{1}{y} dy + \int \frac{a}{1 - ay} dy = \int \frac{dx}{x + a} + \log c \quad \Rightarrow \log y + \frac{a}{-a} \log(1 - ay) = \log(x + a) + \log c$

$\Rightarrow \log(x + a) - \log y + \log(1 - ay) = \log c \qquad \Rightarrow \log \frac{(x + a)(1 - ay)}{y} = \log c$

$\Rightarrow (x + a)(1 - ay) = cy$,

which is the required general solution of this differential equation.

Q.No.10.: Solve the differential equation $(x + 1)\frac{dy}{dx} + 1 = 2e^{-y}$.

Sol.: Given equation is $(x + 1)\frac{dy}{dx} + 1 = 2e^{-y} \Rightarrow (x + 1)\frac{dy}{dx} = 2e^{-y} - 1$

$\Rightarrow \frac{dx}{x + 1} = \frac{dy}{2e^{-y} - 1}$.

Integrating both sides, we get $\int \frac{dx}{x + 1} = \int \frac{dy}{2e^{-y} - 1} + \log c$

$\Rightarrow \int \frac{dx}{x + 1} = -\int \frac{e^y . dy}{2 - e^y} + \log c \quad \Rightarrow \log(x + 1) = -\log(2 - e^y) + \log c$

$\Rightarrow \log(x + 1) + \log(2 - e^y) = \log c \Rightarrow (x + 1)(2 - e^y) = c$

which is the required general solution of this differential equation.

Q.No.11.: Solve the differential equation $\tan x . \sin^2 y dx + \cos^2 x . \cot y dy = 0$.

Sol.: Given equation is $\tan x . \sin^2 y dx + \cos^2 x . \cot y dy = 0$.

$\Rightarrow \tan x . \sec^2 y . dx + \cot y . \operatorname{cosec}^2 y dy = 0$.

Integrating both sides, we get

$\tan x . d(\tan x) - \cot y d(\cot y) = 0$.

$\therefore \frac{\tan^2 x}{2} - \frac{\cot^2 y}{2} = c$.

This is the required solution of this differential equation.

Q.No.12.: Solve the differential equation $\frac{dy}{dx} = e^{2x-y} + x^3 e^{-y}$.

Sol.: Given equation is $\frac{dy}{dx} = e^{2x-y} + x^3 e^{-y}$.

$\Rightarrow \frac{dy}{dx} = \left(e^{2x} + x^3\right)e^{-y}$.

Separating the variables, we get

$e^y dy = \left(e^{2x} + x^3\right)dx$.

Integrating both sides, we get

$e^y = \frac{e^{2x}}{2} + \frac{x^4}{4} + C$.

This is the required solution of this differential equation.

EQUATIONS REDUCIBLE TO VARIABLE SEPARABLE FORM:

Equations of the form $\frac{dy}{dx} = f(ax + by + c)$ (i)

can be reduced to the **'variable separable form'** by putting $ax + by + c = t$, so that

$a + b\frac{dy}{dx} = \frac{dt}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{b}\left(\frac{dt}{dx} - a\right)$.

$\therefore$ Equation (i) becomes $\frac{1}{b}\left(\frac{dt}{dx} - a\right) = f(t) \Rightarrow \frac{dt}{dx} = a + bf(t) \Rightarrow \frac{dt}{a + bf(t)} = dx$.

After integrating both sides, t is to be replaced by its value, we get the required solution.

Q.No.1.: Solve $\frac{y}{x}\frac{dy}{dx} + \frac{2\left(x^2 + y^2\right) - 1}{x^2 + y^2 + 1} = 0$.

Sol.: Given equation is $\frac{y}{x}\frac{dy}{dx} + \frac{2\left(x^2 + y^2\right) - 1}{x^2 + y^2 + 1} = 0$. (i)

Putting $x^2 + y^2 = t$, we get $2x + 2y\frac{dy}{dx} = \frac{dt}{dx} \Rightarrow \frac{y}{x}\frac{dy}{dx} = \frac{1}{2x}\frac{dt}{dx} - 1$.

Therefore (i) becomes $\left(\frac{1}{2x}\frac{dt}{dx} - 1\right) + \frac{2t - 1}{t + 1} = 0$

$$\Rightarrow \frac{1}{2x}\frac{dt}{dx} = 1 - \frac{2t-1}{t+1} = \frac{2-t}{t+1} \Rightarrow 2xdx = \frac{t+1}{2-t}dt \Rightarrow 2xdx + \left(1 + \frac{3}{t-2}\right)dt = 0.$$

Integrating both sides, we get $x^2 + t + 3\log(t-2) = c$

$$\Rightarrow 2x^2 + y^2 + 3\log\left(x^2 + y^2 - 2\right) = c, \quad \left[\because t = x^2 + y^2\right]$$

which is the required solution.

Q.No.2.: Solve the differential equation $(x-y)^2\frac{dy}{dx} = a^2$.

Sol.: Given equation is $(x-y)^2\frac{dy}{dx} = a^2$.

Put $x - y = t \Rightarrow 1 - \frac{dy}{dx} = \frac{dt}{dx} \Rightarrow \frac{dy}{dx} = 1 - \frac{dt}{dx}$.

$\therefore$ The given equation becomes

$$t^2\left(1 - \frac{dt}{dx}\right) = a^2 \Rightarrow 1 - \frac{dt}{dx} = \frac{a^2}{t^2} \Rightarrow \frac{dt}{dx} = \frac{t^2 - a^2}{t^2} \Rightarrow \frac{t^2}{t^2 - a^2}dt = dx.$$

Integrating both sides, we get

$$\int \frac{t^2}{t^2 - a^2}dt = \int dx + c \Rightarrow \int \frac{t^2 - a^2 + a^2}{t^2 - a^2}dt = \int dx + c$$

$$\Rightarrow \int dt + \int \frac{a^2}{t^2 - a^2}dt = \int dx + c \Rightarrow t + a^2\frac{1}{2a}\log\left|\frac{t-a}{t+a}\right| = x + c$$

$$\Rightarrow x - y + \frac{a}{2}\log\left|\frac{x-y-a}{x-y+a}\right| = x + c \quad \Rightarrow 2y = a\log\left|\frac{x-y-a}{x-y+a}\right| + c,$$

which is the required general solution of this differential equation.

Q.No.3.: Solve the differential equation $(x+y+1)^2\frac{dy}{dx} = 1$.

Sol.: Given equation is $(x+y+1)^2\frac{dy}{dx} = 1$ (i)

Put $x + y + 1 = t \Rightarrow 1 + \frac{dy}{dx} = \frac{dt}{dx} \Rightarrow \frac{dy}{dx} = \frac{dt}{dx} - 1$.

Then the given equation becomes

$$t^2\left(\frac{dt}{dx} - 1\right) = 1 \Rightarrow \frac{dt}{dx} = \frac{1+t^2}{t^2} \Rightarrow \frac{t^2}{1+t^2}dt = dx \Rightarrow \left(1 - \frac{1}{1+t^2}\right)dt = dx.$$

Integrating on both sides, we get $\int \left(1-\frac{1}{1+t^2}\right)dt = \int dx + c'$

$\Rightarrow t - \tan^{-1} t = x + c' \Rightarrow (x+y+1) - \tan^{-1}(x+y+1) = x + c'$

$y = \tan^{-1}(x+y+1) + c$, where $c = c' - 1$.

This is the required solution of this differential equation.

Q.No.4.: Solve the differential equation $\cos(x+y)dy = dx$.

Sol.: Given equation is $\cos(x+y)dy = dx \Rightarrow \frac{dy}{dx} = \sec(x+y)$.

Put $x + y = t \Rightarrow 1 + \frac{dy}{dx} = \frac{dt}{dx} \Rightarrow \frac{dy}{dx} = \frac{dt}{dx} - 1$.

$\therefore$ The given equation becomes $\frac{dt}{dx} - 1 = \sec t \Rightarrow \frac{dt}{1+\sec t} = dx$.

Integrating both sides, we get $\int \frac{\cos t}{1+\cos t} dt = \int dx + c \Rightarrow \int \frac{1+\cos t - 1}{1+\cos t} dt = x + c$

$\Rightarrow t - \int \frac{1}{1+\cos t} dt = x + c \Rightarrow t - \int \frac{1}{2\cos^2 \frac{t}{2}} dt = x + c \Rightarrow t - \frac{1}{2}\int \sec^2 \frac{t}{2} dt = x + c$

$\Rightarrow t - \frac{2}{2}\tan\frac{t}{2} = x + c \Rightarrow t - \tan\frac{t}{2} = x + c \Rightarrow x + y - \tan\frac{x+y}{2} = x + c$

$\Rightarrow y = \tan\left(\frac{x+y}{2}\right) + c$,

which is the required general solution of this differential equation.

Q.No.5.: Solve the differential equation $\frac{dy}{dx} = \cos(x+y+1)$.

Sol.: Given equation is $\frac{dy}{dx} = \cos(x+y+1)$.

Put $x + y + 1 = t$ so that $\frac{dy}{dx} + 1 = \frac{dt}{dx} \Rightarrow \frac{dy}{dx} = \frac{dt}{dx} - 1$.

$\therefore$ The given equation becomes $\frac{dt}{dx} - 1 = \cos t \Rightarrow \frac{dt}{dx} = \cos t + 1 \Rightarrow \frac{dt}{\cos t + 1} = dx$.

Integrating both sides, we get $\int \frac{dt}{\cos t + 1} = \int dx + c$

$$\Rightarrow \int \frac{dt}{2\cos^2\frac{t}{2}} = \int dx + c \Rightarrow \int \frac{\sec^2\frac{t}{2}}{2}dt = \int dx + c \Rightarrow \frac{\tan\frac{t}{2}}{2.\frac{1}{2}} = x + c \Rightarrow \frac{\sin\frac{t}{2}}{\cos\frac{t}{2}} = x + c$$

$$\Rightarrow \frac{2\sin^2\frac{t}{2}}{2\sin\frac{t}{2}\cos\frac{t}{2}} = x + c \Rightarrow \frac{1-\cos t}{\sin t} = x + c \Rightarrow \frac{1}{\sin t} - \frac{\cos t}{\sin t} = x + c$$

$$\Rightarrow \operatorname{cosec} t - \cot t = x + c$$

$$\Rightarrow x = \operatorname{cosec}(x+y+1) - \cot(x+y+1) + c',$$

which is the required general solution of this differential equation.

Q.No.6.: Solve the differential equation $\frac{dy}{dx} - x\tan(y-x) = 1$.

Sol.: Given equation is $\frac{dy}{dx} - x\tan(y-x) = 1 \Rightarrow \frac{dy}{dx} = 1 + x\tan(y-x)$.

Put $y - x = t \Rightarrow \frac{dy}{dx} = \frac{dt}{dx} + 1$.

$\therefore$ The given equation becomes $\frac{dt}{dx} + 1 = 1 + x\tan t \Rightarrow \frac{dt}{dx} = x\tan t \Rightarrow \frac{dt}{\tan t} = xdx$.

Integrating both sides, we get $\int \frac{dt}{\tan t} = \int x.dx + c$

$$\Rightarrow \log\sin t = \frac{x^2}{2} + c \Rightarrow \log\sin(y-x) = \frac{1}{2}.x^2 + c,$$

which is the required general solution of this differential equation.

Q.No.7.: Solve the differential equation $x^4\frac{dy}{dx} + x^3y + \operatorname{cosec}(xy) = 0$.

Sol.: Given equation is $x^4\frac{dy}{dx} + x^3y + \operatorname{cosec}(xy) = 0$.

Putting $xy = t \Rightarrow y = \frac{t}{x}$ so that $x\frac{dy}{dx} + y = \frac{dt}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{x}\left(\frac{dt}{dx} - y\right)$.

$\therefore$ Equation (i) becomes

$$\Rightarrow x^4\left[\frac{1}{x}\left(\frac{dt}{dx} - y\right)\right] + x^3\left(\frac{t}{x}\right) + \operatorname{cosec} t = 0 \Rightarrow x^3\frac{dt}{dx} - x^3y + x^2t + \operatorname{cosec} t = 0$$

$\Rightarrow x^3\dfrac{dt}{dx} - x^2t + x^2t + \operatorname{cosec} t = 0 \Rightarrow x^3\dfrac{dt}{dx} = -\operatorname{cosec} t$.

Separating the variable, we get $\dfrac{1}{\operatorname{cosec} t}dt = -\dfrac{dx}{x^3}$

Integrating both sides, we get

$$\int \frac{1}{\operatorname{cosec} t}dt = -\int \frac{dx}{x^3} + c \Rightarrow \int \sin t dt = -\int x^{-3}dt + c \Rightarrow -\cos t = -\frac{x^{-2}}{-2} + c$$

$\Rightarrow -\cos t = -\dfrac{1}{2x^2} + c \;\Rightarrow \cos xy + \dfrac{1}{2x^2} = c$. Ans.

Q.No.8.: Solve the differential equation $y' = \sin^2(x - y + 1)$.

Sol.: Given equation is $y' = \sin^2(x - y + 1)$.

Put $z = x - y + 1$, so $\dfrac{dz}{dx} = 1 - \dfrac{dy}{dx} + 0$.

Substituting z and $\dfrac{dy}{dx}$, we get a separable equation as

$1 - \dfrac{dz}{dx} = \sin^2 z \Rightarrow \dfrac{dz}{dx} = 1 - \sin^2 z = \cos^2 z$.

Separating the variables, we have

$\sec^2 z dz = dx$.

Integrating both sides, we get

$\tan z = x + c \Rightarrow \tan(x - y + 1) = x + c$.

This is the required solution of this differential equation.

Q.No.9.: Solve the differential equation $\dfrac{y}{x}\dfrac{dy}{dx} + \dfrac{x^2 + y^2 - 1}{2(x^2 + y^2) + 1} = 0$.

Sol.: Given equation is $\dfrac{y}{x}\dfrac{dy}{dx} + \dfrac{x^2 + y^2 - 1}{2(x^2 + y^2) + 1} = 0$.

Putting $x^2 + y^2 = t$, we get $2x + 2y\dfrac{dy}{dx} = \dfrac{dt}{dx} \Rightarrow \dfrac{y}{x}\dfrac{dy}{dx} = \dfrac{1}{2x}\dfrac{dt}{dx} - 1$.

Therefore, the given equation becomes

$$\frac{1}{2x}\frac{dt}{dx} - 1 + \frac{t-1}{2t+1} = 0 \Rightarrow \frac{1}{2x}\frac{dt}{dx} = 1 - \frac{t-1}{2t+1} = \frac{t+2}{2t+1} \Rightarrow 2xdx = \frac{2t+1}{t+2}dt$$

$\Rightarrow 2xdx = \left(2 - \frac{3}{t+2}\right)dt$.

Integrating both sides, we get $x^2 = 2t - 3\log(t+2) + c$

$\Rightarrow x^2 + 2y^2 - 3\log\left(x^2 + y^2 + 2\right) + c$, $\left[\because t = x^2 + y^2\right]$.

This is the required solution of this differential equation.

INITIAL VALUE PROBLEM:

A differential equation together with an initial condition is called an initial value problem. In this type of problems, we can determine the value of the arbitrary constant in the general solution by using initial condition.

Q.No.1.: Solve the differential equation $\frac{dy}{dx} = xe^{y-x^2}$, if $y = 0$ when $x = 0$.

Sol.: Given equation is $\frac{dy}{dx} = xe^{y-x^2} \Rightarrow \frac{dy}{dx} = xe^{y}.e^{-x^2} \Rightarrow \frac{dy}{e^y} = x.e^{-x^2}.dx$.

Integrating both sides, we get $\int \frac{dy}{e^y} = \int x.e^{-x^2}.dx + c'$. (i)

Put $x^2 = t \Rightarrow xdx = \frac{1}{2}dt$.

Then (i) becomes $-e^{-y} = \int e^{-t}\frac{dt}{2} + c'$

$\Rightarrow -2e^{-y} = -e^{-t} + 2c' \Rightarrow -2e^{-y} = -e^{-x^2} + c$, $[\text{where} \quad c = 2c']$. (ii)

When $x = 0, \ y = 0$. Putting in (ii), we get $c = -1$

$-2e^{-y} = -e^{-x^2} - 1 \Rightarrow 2e^{-y} = e^{-x^2} + 1$. Ans.

This is the required solution of this differential equation.

Q.No.2.: Solve the differential equation $x\frac{dy}{dx} + \cot y = 0$, if $y = \frac{\pi}{4}$ when $x = \sqrt{2}$.

Sol.: Given equation is $x\frac{dy}{dx} + \cot y = 0 \Rightarrow -\frac{dy}{\cot y} = \frac{dx}{x}$.

Integrating both sides, we get $-\int \tan y dy = \int \frac{dx}{x} + c$

$\Rightarrow \log\cos y = \log x + c \;\Rightarrow \log\left(\dfrac{\cos y}{x}\right) = c$.

Now putting $y = \dfrac{\pi}{4}$ and $x = \sqrt{2}$, then $\log\left(\dfrac{1/\sqrt{2}}{\sqrt{2}}\right) = c \Rightarrow c = \log\left(\dfrac{1}{2}\right)$

Hence $\log\left(\dfrac{\cos y}{x}\right) = \log\dfrac{1}{2} \Rightarrow \dfrac{\cos y}{x} = \dfrac{1}{2} \Rightarrow x = 2\cos y$. Ans.

This is the required solution of this differential equation.

Q.No.3.: Solve $\dfrac{dy}{dx} = (4x + y + 1)^2$, if $y(0) = 1$.

Sol.: Given equation is $\dfrac{dy}{dx} = (4x + y + 1)^2$.

Putting $4x + y + 1 = t$, we get $\dfrac{dy}{dx} = \dfrac{dt}{dx} - 4$.

Then the given equation becomes $\dfrac{dt}{dx} - 4 = t^2 \Rightarrow \dfrac{dt}{dx} = 4 + t^2 \Rightarrow \dfrac{dt}{4 + t^2} = dx$.

Integrating both sides, we get $\displaystyle\int \dfrac{dt}{4 + t^2} = \int dx + c \Rightarrow \dfrac{1}{2}\tan^{-1}\dfrac{t}{2} = x + c$

$\Rightarrow \dfrac{1}{2}\tan^{-1}\left[\dfrac{1}{2}(4x + y + 1)\right] = x + c \;\Rightarrow 4x + y + 1 = 2\tan 2(x + c)$.

When $x = 0,\ y = 1,$ we get

$0 + 1 + 1 = 2\tan 2c \;\Rightarrow 1 = \tan 2c \;\Rightarrow \dfrac{1}{2}\tan^{-1}(1) = c \;\Rightarrow c = \dfrac{\pi}{8}$.

Hence, the required solution is

$4x + y + 1 = 2\tan\left(2x + \dfrac{\pi}{4}\right)$. Ans.

Q.No.4.: Solve the differential equation

$$3e^x \tan y dx + (1 + e^x)\sec^2 y dy = 0, \text{ given } y = \frac{\pi}{4} \text{ when } x = 0.$$

Sol.: The given equation can be written as $\dfrac{3e^x}{1 + e^x}dx + \dfrac{\sec^2 y}{\tan y}dy = 0$.

Integrating, we have $3\log(1 + e^x) + \log\tan y = \log c$

$\Rightarrow \log\left(1+e^x\right)^3 \tan y = \log c$

$\Rightarrow \left(1+e^x\right)^3 \tan y = c$, (i)

which is the general solution of the given equation.

Since $y = \frac{\pi}{4}$ when x = 0, we have from (i) $(1+1)^3 \times 1 = c \Rightarrow c = 8$.

$\therefore$ The required particular solution is $\left(1+e^x\right)^3 \tan y = 8$.

Q.No.5.: Show that the particular solution of $\left(x^2+1\right)\frac{dy}{dx} + \left(y^2+1\right) = 0$, y(0) = 1, is $\frac{1-x}{1+x}$.

Sol.: Given equation is $\left(x^2+1\right)\frac{dy}{dx} + \left(y^2+1\right) = 0$.

Separating the variables, we get

$$\frac{dy}{y^2+1} + \frac{dx}{x^2+1} = 0.$$

Integrating both sides, we get $\left(\tan^{-1} y + \tan^{-1} x\right) = \tan c$.

Using $\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a.\tan b}$

$$\frac{y+x}{1-xy} = \tan c.$$

When x = 0, y = 1, then $\frac{1+0}{1-0} = \tan c$

$$\therefore \frac{y+x}{1-xy} = 1$$

Solving, we get $y = \frac{1-x}{1+x}$.

This is the required solution of this differential equation.

Home Assignments

Q.No.1.: Solve the differential equation $\frac{dy}{dx} = e^{2x+3y}$.

Ans.: $3e^{2x} + 2e^{-3y} = c$.

Q.No.2.: Solve the differential equation $xy\frac{dy}{dx}=1+x+y+xy$.

Ans.: $y=x+\log[x(1+y)]+c$.

Q.No.3.: Solve the differential equation $(x+y)(dx-dy)=dx+dy$.

Ans.: $x+y=ce^{x-y}$.

Q.No.4.: Solve the differential equation $x\frac{dy}{dx}+\cot y=0$, if $y=\frac{\pi}{4}$ when $x=\sqrt{2}$.

Ans.: $x\sec y=2$.

Q.No.5.: Solve the differential equation $\frac{dy}{dx}+\sqrt{\frac{1-y^2}{1-x^2}}=0$.

Ans.: $y\sqrt{1-x^2}+x\sqrt{1-y^2}=c$.

Q.No.6.: Solve the differential equation $\frac{y}{x}.\frac{dy}{dx}=\sqrt{1+x^2+y^2+x^2y^2}$.

Ans.: $\sqrt{1+y^2}=\frac{2}{3}\left(1+x^2\right)^{3/2}+c$.

Q.No.7.: Solve the differential equation $e^y\left(1+x^2\right)\frac{dy}{dx}-2x\left(1+e^y\right)=0$.

Ans.: $1+e^y=c\left(1+x^2\right)$.

Q.No.8.: Solve the differential equation $\sec^2 x\tan y dx+\sec^2 y\tan x dy=0$.

Ans.: $\tan x\tan y=c$.

Q.No.9.: Solve the differential equation $\left(1+x^3\right)dy-x^2ydx=0$, if $y=2$ when $x=1$.

Ans.: $y^3=4\left(x^3+1\right)$.

Q.No.10.: Solve the differential equation $a(xdy+ydx)=xydy$.

Ans.: $y=a\log(xy)+c$.

Q.No.11.: Solve the differential equation $\frac{dy}{dx}=e^{x-y}+x^2e^{-y}$.

Ans.: $e^y=e^x+\frac{x^3}{3}+c$.

Q.No.12.: Solve the differential equation $(x+y)^2\dfrac{dy}{dx}=a^2$.

Ans.: $x+y=a\tan\left(\dfrac{y-c}{a}\right)$.

Q.No.13.: Solve the differential equation $\sin(x+y)dy=dx$.

Ans.: $\tan(x+y)-\sec(x+y)=y+c$.

Q.No.14.: Solve the differential equation $\dfrac{dy}{dx}=\cos(x+y)$.

Ans.: $x+c=\tan\left(\dfrac{x+y}{2}\right)$.

Q.No.15.: Solve the differential equation $\dfrac{dy}{dx}=\sin(x+y)+\cos(x+y)$.

Ans.: $\log\left[1+\tan\left(\dfrac{x+y}{2}\right)\right]=x+c$.

Q.No.16.: Solve the differential equation $\tan y\dfrac{dy}{dx}=\sin(x+y)+\sin(x-y)$.

Ans.:. $2\cos x+\sec y=c$.

Q.No.17.: Solve the differential equation $4xydx+\left(x^2+1\right)dy=0$.

Ans.: $y(x+1)^2=c$.

Q.No.18.: Solve the differential equation $(x+4)\left(y^2+1\right)dx+y\left(x^2+3x+2\right)dy=0$.

Ans.: $3\left(x^2+x\right)y=x^3-3x+c$.

Q.No.19.: Solve the differential equation $(xy+x)dx=\left(x^2y^2+x^2+y^2+1\right)dy$.

Ans.: $\log\left(x^2+1\right)=y^2-2y+4\ \log|c(y+1)|$.

Q.No.20.: Obtain particular solution $2xyy'=1+y^2$; $y(2)=3$.

Ans.: $y^2=5x-1$.

Q.No.21.: Solve the differential equation $y'=(x+y)^2$.

Ans.: $(x+y)=\tan(x+c)$.

Q.No.22.: Solve the differential equation $(2x-4y+5)y'+x-2y+3=0$.

Ans.: $4x - 8y + \log|4x - 8y + 11| = c$.

Q.No.23.: Solve the differential equation $xyy' = \left(\dfrac{1+y^2}{1+x^2}\right)\left(1+x+x^2\right)$.

Ans.: $\dfrac{1}{2}\log\left(1+y^2\right) = \log x + \tan^{-1}x + c$.

Q.No.24.: Solve the differential equation $x^3 + e^{2x^2 3y^2}dx - y^3e^{-x^2-2y^2}dy = 0$.

Ans.: $25\left(3x^2 - 1\right)e^{3x^2} + 9\left(5y^2 + 1\right)e^{-5y^2} = c$.

Q.No.25.: Solve the differential equation $y' = \dfrac{(y-1)(x-2)(y+3)}{(x-1)(y-2)(x+3)}$.

Ans.: $(x+1)(y+3)^5 = c(y-1)(x+3)^5$.

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3^{rd} Topic

Ordinary Differential Equations of First order

Part-I:

"Homogeneous Form of Differential Equations"

(Reduction to a Variable Separable Form)

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Homogeneous Form of Differential Equation:

A differential equation of the form

$$\frac{dy}{dx} = \frac{f(x,y)}{\phi(x,y)},$$

is called a homogeneous form of differential equation if $f(x,y)$ and $\phi(x,y)$ are homogeneous functions of the same degree in x and y.

Now, since $f(x,y)$ and $\phi(x,y)$ are homogeneous functions of the same degree in x and y, then a homogeneous form of differential equation can also be written as

$$\frac{dy}{dx} = g\left(\frac{y}{x}\right).$$

Method to solve this type of differential equation :

Given differential equation is $\frac{dy}{dx} = g\left(\frac{y}{x}\right)$.

(i) Put $y = vx$, and then $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in the given differential equation, we get

$v + x\frac{dv}{dx} = g(v)$**, which reduces to a separable equation.**

(ii) Separating the variables v and x, we get $\frac{dv}{g(v) - v} = \frac{dx}{x}$.

(iii) Integrating on both sides, we get the required solution.

$$\int \frac{dv}{g(v) - v} = \int \frac{dx}{x} + c$$

(iv) At last, replace v by $\frac{y}{x}$, in the solution obtained in (iii).

Now let us solve some differential equations

Q.No.1.: Solve $\left(x^2 - y^2\right)dx - xydy = 0$.

Sol.: Given equation is $\frac{dy}{dx} = \frac{x^2 - y^2}{xy} = \frac{1 - \frac{y^2}{x^2}}{\frac{y}{x}}$, (i)

which is homogeneous form of differential equation in x and y.

Putting $y = vx$, then $\frac{dy}{dx} = v + x\frac{dv}{dx}$.

$\therefore$ Equation (i) becomes $v + x\frac{dv}{dx} = \frac{1 - v^2}{v} \Rightarrow x\frac{dv}{dx} = \frac{1 - v^2}{v} - v = \frac{1 - 2v^2}{v}$.

Separating the variables, we get $\frac{v}{1 - 2v^2}dv = \frac{dx}{x}$.

Integrating both sides, we get

$$\int \frac{v}{1 - 2v^2}dv = \int \frac{dx}{x} + c \Rightarrow -\frac{1}{4}\int \frac{-4v}{1 - 2v^2}dv = \int \frac{dx}{x} + c \Rightarrow -\frac{1}{4}\log\left(1 - 2v^2\right) = \log x + c$$

$$\Rightarrow 4\log x + \log\left(1 - 2v^2\right) = -4c \Rightarrow \log x^4\left(1 - 2v^2\right) = -4c$$

$$\Rightarrow x^4\left(1-\frac{2y^2}{x^2}\right)=e^{-3c}=c' \quad (\text{where } c'=-4c)$$

$$\Rightarrow x^2\left(x^2-2y^2\right)=c'.$$

This is the required solution of this differential equation.

Q.No.2.: Solve $\left(1+e^{x/y}\right)dx+e^{x/y}\left(1-\frac{x}{y}\right)dy=0$.

Sol.: The given equation can be written as $\frac{dx}{dy}=-\frac{e^{x/y}\left(1-\frac{x}{y}\right)}{1+e^{x/y}}$, (i)

which is homogeneous form of differential equation in x and y.

Putting $x=vy$, then $\frac{dx}{dy}=v+y\frac{dv}{dy}$.

$\therefore$ Equation (i) becomes $v+y\frac{dv}{dy}=-\frac{e^v(1-v)}{1+e^v}$

$$\Rightarrow y\frac{dv}{dy}=-\frac{e^v(1-v)}{1+e^v}-v=\frac{-e^v(1-v)-v\left(1+e^v\right)}{1+e^v}=-\frac{v+e^v}{1+e^v}.$$

Separating the variables, we get $-\frac{dy}{y}=\frac{1+e^v}{v+e^v}dv=\frac{d\left(v+e^v\right)}{v+e^v}$.

Integrating both sides, we get $-\int\frac{dy}{y}=\int\frac{d\left(v+e^v\right)}{v+e^v}+c$

$$\Rightarrow -\log y=\log\left(v+e^v\right)+c \Rightarrow y\left(v+e^v\right)=e^{-c}\Rightarrow x+ye^{x/y}=c' \text{ [say]}.$$

This is the required solution of this differential equation.

Q.No.3.: Solve the differential equation $2xy.\frac{dy}{dx}=3y^2+x^2$.

Sol.: Here the given equation is $2xy.\frac{dy}{dx}=3y^2+x^2\Rightarrow\frac{dy}{dx}=\frac{3y^2+x^2}{2xy}$, (i)

which is homogeneous form of differential equation in x and y.

Putting $y=vx$, then $\frac{dy}{dx}=v+x\frac{dv}{dx}$.

$\therefore$ Equation (i) becomes $\quad v + x.\dfrac{dv}{dx} = \dfrac{3v^2x^2 + x^2}{2vx^2} = \dfrac{3v^2+1}{2v}$

$\Rightarrow x\dfrac{dv}{dx} = \dfrac{3v^2+1-2v^2}{2v} = \dfrac{v^2+1}{2v}$.

Separating the variables, we get $\Rightarrow \dfrac{2vdv}{1+v^2} = \dfrac{dx}{x}$

Integrating both sides, we get

$$\int \frac{2vdv}{1+v^2} = \int \frac{dx}{x} + \log c \Rightarrow \log\left(1+v^2\right) = \log x + \log c \Rightarrow \log\left(\frac{1+v^2}{x}\right) = \log c$$

$$\Rightarrow \frac{x^2+y^2}{x^3} = c$$

$\therefore x^2 + y^2 = cx^3$.

This is the required solution of this differential equation.

Q.No.4.: Solve the differential equation $\left(x^2 + 2y^2\right)dx - xydy = 0$.

Sol.: The given equation is $\dfrac{dy}{dx} = \dfrac{x^2+2y}{xy}$, (i)

which is homogeneous form of differential equation in x and y.

Put $y = vx$, then $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$.

$\therefore$ Equation (i) becomes $\quad v + x\dfrac{dv}{dx} = \dfrac{x^2+2v^2x^2}{vx^2} \Rightarrow x\dfrac{dv}{dx} = \dfrac{1+2v^2}{v} - v = \dfrac{1+2v^2-v^2}{v}$.

Separating the variables, we get $\dfrac{v}{1+v^2}dv = \dfrac{dx}{x}$.

Integrating both sides, we get

$$\int \frac{1}{2}\frac{2v}{1+v^2}.dv = \int \frac{dx}{x} + \log c \Rightarrow \frac{1}{2}\log\left(1+v^2\right) = \log x + \log c$$

$$\Rightarrow \log\left(1+v^2\right) = \log x^2 + \log c^2 \Rightarrow \left(1+v^2\right) = x^2c_1 \Rightarrow \left[1+\left(\frac{y}{x}\right)^2\right] = x^2c_1$$

$$\Rightarrow x^2 + y^2 = cx^4.$$

This is the required solution of this differential equation.

Q.No.5.: Solve the differential equation $xdy - ydx = \sqrt{x^2 + y^2}dx$.

Sol.: The given equation is $xdy - ydx = \sqrt{x^2 + y^2}dx$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{\sqrt{x^2 + y^2} + y}{x}\right). \qquad \text{(i)}$$

Since given equation is homogeneous form of differential equation in x and y.

Putting $y = vx$, then $\frac{dy}{dx} = v + x\frac{dv}{dx}$.

$\therefore$ Equation (i) becomes

$$\frac{dy}{dx} = \frac{\sqrt{x^2 + y^2} + y}{x} \Rightarrow v + x.\frac{dv}{dx} = \sqrt{1 + v^2} + v \Rightarrow x.\frac{dv}{dx} = \sqrt{1 + v^2}.$$

Separating the variables, we get $\frac{dv}{\sqrt{1 + v^2}} = \frac{dx}{x}$.

Integrating both sides, we get

$$\int \frac{dv}{\sqrt{1 + v^2}} = \int \frac{dx}{x} + \log c \Rightarrow \log\left|v + \sqrt{1 + v^2}\right| = \log x + \log c \Rightarrow v + \sqrt{1 + v^2} = xc.$$

Putting $v = \frac{y}{x}$, we get $\frac{y}{x} + \sqrt{1 + \frac{y^2}{x^2}} = xc \Rightarrow y + \sqrt{x^2 + y^2} = cx^2$.

This is the required solution of this differential equation.

Q.No.6.: Solve the differential equation $\left(x^2 - y^2\right)dx = 2xydy$.

Sol.: The given equation is $\left(x^2 - y^2\right)dx = 2xydy$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{x^2 - y^2}{2xy}\right). \qquad \text{(i)}$$

which is a homogeneous form of differential equation in x and y.

Putting $y = vx$, so that $\frac{dy}{dx} = v + x\frac{dv}{dx}$.

$\therefore$ Equation (i) becomes $t + x\frac{dv}{dx} = \frac{x^2 - x^2v^2}{2x^2v} = \frac{\left(1 - v^2\right)}{2v}$

$$\Rightarrow x\frac{dv}{dx} = \frac{1 - v^2}{2v} - v = \left(\frac{1 - 3v^2}{2v}\right).$$

Separating the variables, we get $\frac{2vdv}{1-3v^2}=\frac{dx}{x}$.

Integrating both sides, we get $\int \frac{2vdv}{1-3v^2}=\int \frac{dx}{x}+\log c$ (ii)

Putting $y=1-3v^2$ $\therefore dy=-6vdv \Rightarrow vdv=-\frac{1}{6}dy$, we get

$-\frac{1}{3}\int \frac{dy}{y}=\log x+\log c=\log xc \Rightarrow -\frac{1}{3}\log y=\log xc \Rightarrow y^{-1/3}=xc$.

Putting $y=1-3t^2$ in above equation, we get $\left(1-3t^2\right)^{-1/3}=xc$.

Putting $t=\left(\frac{y}{x}\right)$ in above equation, we get

$$\left(1-\frac{3y^2}{x^2}\right)^{-1/3}=xc \Rightarrow \left(\frac{x^2-3y^2}{x^2}\right)^{-1}=x^3c \Rightarrow \frac{x^2}{x^2-3y^2}=x^3c \Rightarrow x\left(x^2-3y^2\right)^{-1}=\frac{1}{c}$$

$\Rightarrow x\left(x^2-3y^2\right)=c$.

This is the required solution of this differential equation.

Q.No.7.: Solve the differential equation $\left(y^2-2xy\right)dx=\left(x^2-2xy\right)dy$

Sol.: Here the given equation is $\frac{dy}{dx}=\frac{y^2-2xy}{x^2-2xy}$, (i)

which is homogeneous form of differential equation in x and y.

Put $y=vx$, then $\frac{dy}{dx}=v+x\frac{dv}{dx}$.

$\therefore$ Equation (i) becomes $v+x\frac{dv}{dx}=\frac{v^2x^2-2vx^2}{x^2-2xy}=\frac{v^2-2v}{1-2v} \Rightarrow x\frac{dv}{dx}=\frac{v^2-2v}{1-2v}-v$

$\Rightarrow x\frac{dv}{dx}=\frac{v^2-2v-v+2v^2}{1-2v}$.

Separating the variables, we get $\frac{1-2v}{3\left(v^2-v\right)}dv=\frac{dx}{x}$.

Integrating both sides, we get

$$\int -\frac{1}{3}\frac{2v-1}{\left(v^2-v\right)}dv=\int \frac{dx}{x}+\log c \Rightarrow -\frac{1}{3}\log\left(v^2-v\right)=\log x+\log c$$

$$\Rightarrow -\log\left(v^2 - v\right) = \log x^3 + \log c^3 \Rightarrow \log\left(v^2 - v\right)x^3 = \log c^{-3} \Rightarrow \left(\frac{y^2}{x^2} - \frac{y}{x}\right)x^3 = c^{-3}$$

$$\Rightarrow \frac{\left(y^2 - yx\right)x^3}{x^2} = c^{-3} \Rightarrow \frac{x^3\left(y^2 - yx\right)}{x^2} = c_1 \Rightarrow xy(y - x) = c.$$

This is the required solution of this differential equation.

Q.No.8.: Solve the differential equation $x^2ydx - \left(x^3 + y^3\right)dy = 0$.

Sol.: The given equation is $x^2ydx - \left(x^3 + y^3\right)dy = 0$

$$\Rightarrow \frac{dy}{dx} = \frac{x^2y}{x^3 + y^3}, \tag{i}$$

which is a homogeneous form of differential equation in x and y.

Putting $y = vx$, so that $\frac{dy}{dx} = v + x\frac{dv}{dx}$.

$\therefore$ Equation (i) becomes $\quad v + x\frac{dv}{dx} = \frac{x^3v}{x^3\left(1 + v^3\right)} = \frac{v}{1 + v^3}$

$$\Rightarrow x\frac{dv}{dx} = \frac{v}{1 + v^3} - v = \frac{-v^4}{1 + v^3}.$$

Separating the variables, we get $\left(\frac{1 + v^3}{v^4}\right)dy = -\frac{dx}{x}$.

Integrating both sides, we get $\int \left(\frac{1}{v^4} + \frac{1}{v}\right)dv = -\int \frac{dx}{x} - \log c$

$$\Rightarrow \int v^{-4}dv + \int \frac{1}{v}dv + \int \frac{dx}{x} = -\log c \Rightarrow \frac{v^{-3}}{-3} + \log v + \log x = -\log c$$

$$\Rightarrow -\frac{v^{-3}}{3} + \log vx = -\log c \Rightarrow \log c + \log y = \frac{1}{3}\frac{1}{v^3} \Rightarrow \log(cy) = \frac{1}{3}\frac{x^3}{y^3}$$

$$\Rightarrow \left(\frac{x}{y}\right)^3 = 3\log cy.$$

This is the required solution of this differential equation.

Q.No.9.: Solve the differential equation $ydx - xdy = \sqrt{x^2 + y^2}dx$.

Sol.: The given equation is $xdy = \left(y - \sqrt{x^2 + y^2}\right)dx$

$$\Rightarrow \frac{dy}{dx} = \frac{y - \sqrt{x^2 + y^2}}{x}. \qquad \text{(i)}$$

Since the given equation is homogeneous form of differential equation in x and y.

$\therefore$ Putting $y = vx$ $\therefore \frac{dy}{dx} = v + \frac{dv}{dx}$.

$\therefore$ Equation (i) becomes $v + x.\frac{dv}{dx} = v - \sqrt{1 + v^2} \Rightarrow x.\frac{dv}{dx} = -\sqrt{1 + v^2}$.

Separating the variables, we get $\frac{dv}{1 + v^2} = -\frac{dx}{x}$.

Integrating both sides, we get

$$\int \frac{dv}{1 + v^2} = -\int \frac{dx}{x} + c \Rightarrow \log\left(v + \sqrt{1 + v^2}\right) = -\log x + \log c$$

$$\Rightarrow \log\left(v + \sqrt{1 + v^2}\right) = \log\left(\frac{c}{x}\right) \Rightarrow \left(v + \sqrt{1 + v^2}\right) = \left(\frac{c}{x}\right).$$

Putting $v = \frac{y}{x}$, we get

$$\Rightarrow \frac{y}{x} + \frac{\sqrt{x^2 + y^2}}{x} = \frac{c}{x} \Rightarrow y + \sqrt{x^2 + y^2} = c.$$

This is the required solution of this differential equation.

Q.No.10.: Solve the differential equation $y^2 + x^2\frac{dy}{dx} = xy\frac{dy}{dx}$.

Sol.: The given equation is $y^2 + x^2\frac{dy}{dx} = xy\frac{dy}{dx}$

$$\Rightarrow y^2 = x^2\left[\frac{y}{x} - 1\right].\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{\frac{y^2}{x^2}}{\frac{y}{x} - 1}. \qquad \text{(i)}$$

Since given equation is homogeneous form of differential equation in x and y.

Put $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$.

$\therefore$ Equation (i) becomes $v + x\frac{dv}{dx} = \frac{v^2}{v-1} \Rightarrow x\frac{dv}{dx} = \frac{v^2 - v^2 + v}{v-1}$.

Separating the variables, we get $\frac{v-1}{v}dv = \frac{dx}{x}$.

Integrating both sides, we get $\int \frac{v-1}{v}dv = \int \frac{dx}{x} + \log c$

$$\Rightarrow \int \left(1 - \frac{1}{v}\right)dv = \int \frac{dx}{x} + \log c \Rightarrow v - \log v = \log x + \log c$$

By putting $v = \frac{y}{x}$, we get

$$\frac{y}{x} - \log\frac{y}{x} = \log(xc).$$

This is the required solution of this differential equation.

Q.No.11.: Solve the differential equation $x^3dx - y^3dy = 3xy(ydx - xdy)$

Sol.: The given equation is $x^3dx - y^3dy = 3xy^2dx - 3x^2ydy$

$$\Rightarrow \left(x^3 - 3xy^2\right)dx = \left(y^3 - 3x^2y\right)dy$$

$$\Rightarrow \frac{dy}{dx} = \frac{x^3 - 3xy^2}{y^3 - 3x^2y} = \frac{1 - 3\left(\frac{y}{x}\right)^2}{\left(\frac{y}{x}\right)^3 - 3\left(\frac{y}{x}\right)}. \qquad \text{(i)}$$

Since given equation is homogeneous form of differential equation in x and y.

Put $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$.

$\therefore$ Equation (i) becomes $v + x\frac{dv}{dx} = \frac{1 - 3v^2}{v^3 - 3v} \Rightarrow x\frac{dv}{dx} = \frac{1 - 3v^2 - v^4 + 3v^2}{v^3 - 3v}$

$$\Rightarrow \frac{v^3 - 3v}{1 - v^4}dv = \frac{dx}{x}.$$

Integrating both sides, we get $\int \frac{v^3 - 3v}{1 - v^4}dv = \int \frac{dx}{x} + \log c$

$$\Rightarrow \int \frac{v^3}{1-v^4}dv - 3\int \frac{v}{1-v^4}dv = \log x + \log c$$

$$\Rightarrow -\frac{1}{4}\int \frac{4v^3}{1-v^4}dv - \frac{3}{2}\int \frac{2v}{1-v^4}dv = \log(xc)$$

Put $v^2 = u \Rightarrow 2vdv = du$, we get

$$\Rightarrow -\frac{1}{4}\int \frac{4v^3}{1-v^4}dv - \frac{3}{2}\int \frac{1}{1-u^2}du = \log(xc)$$

$$\Rightarrow -\frac{1}{4}\log(1-v^4) - \frac{3}{4}\log\left|\frac{1+u}{1-u}\right| = \log(xc) \qquad \left[\because \int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + c\right]$$

$$\Rightarrow -\frac{1}{4}\log(1-v^4) - \frac{1}{4}\log\left|\frac{1+v^2}{1-v^2}\right|^3 = \log(xc) \Rightarrow -\frac{1}{4}\log\frac{(1+v^2)^4}{(1-v^2)^2} = \log(xc)$$

$$\Rightarrow \log\frac{(1+v^2)^{-1}}{(1-v^2)^{-1/2}} = \log(xc) \Rightarrow \frac{\sqrt{(1-v^2)}}{(1+v^2)} = xc \Rightarrow (1-v^2) = x^2c^2(1+v^2)^2$$

$$\Rightarrow \left[1-\left(\frac{y}{x}\right)^2\right] = x^2c^2\left[1+\left(\frac{y}{x}\right)^2\right]^2 \Rightarrow \left[x^2-y^2\right] = c^2\left[x^2+y^2\right]^2$$

$$\Rightarrow (x^2+y^2)^2 = c'(y^2-x^2).$$

This is the required solution of this differential equation.

Q.No.12.: Solve the differential equation $\frac{dy}{dx} = \frac{y}{x} + \sin\frac{y}{x}$

Sol.: The given equation is $\frac{dy}{dx} = \frac{y}{x} + \sin\frac{y}{x}$. (i)

Since given equation is homogeneous form of differential equation in x and y.

Put $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$.

$\therefore$ Equation (i) becomes $v + x\frac{dv}{dx} = v + \sin v \Rightarrow \frac{dv}{\sin x} = \frac{dx}{x}$.

Integrating both sides, we get $\int \operatorname{cosec} v dv = \int \frac{dx}{x} + \log c$

$$\Rightarrow \log\left|\operatorname{cosec} v - \cot v\right| = \log x + \log c \Rightarrow \log\left|\frac{1-\cos v}{\sin v}\right| = \log xc$$

$$\Rightarrow \log\left|\frac{2\sin^2 \frac{v}{x}}{2\sin\frac{v}{2}\cos\frac{v}{2}}\right| = \log(xc)$$

$$\Rightarrow \tan\frac{v}{2} = xc \Rightarrow \tan\frac{y}{2x} = xc \Rightarrow y = 2x\tan^{-1}(cx).$$

This is the required solution of this differential equation.

Q.No.13.: Solve the differential equation $\left[x\tan\frac{y}{x} - y\sec^2\frac{y}{x}\right]dx + x\sec^2\frac{y}{x}dy = 0$.

Sol.: The given equation is $\left[x\tan\frac{y}{x} - y\sec^2\frac{y}{x}\right]dx + x\sec^2\frac{y}{x}dy = 0$

$$\Rightarrow \frac{dy}{dx} = \frac{y\sec^2\frac{y}{x} - x\tan\frac{y}{x}}{x\sec^2\frac{y}{x}} = \frac{y}{x} - \frac{\tan\frac{y}{x}}{\sec^2\frac{y}{x}}. \qquad \text{(i)}$$

Since given equation is homogeneous form of differential equation in x and y.

Put $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$.

$\therefore$ Equation (i) becomes $\quad v + x\frac{dv}{dx} = v - \frac{\tan v}{\sec^2 v} \Rightarrow \frac{\sec^2 v}{\tan v}dv = -\frac{dx}{x}$.

Integrating both sides, we get $\int\frac{\sec^2 v}{\tan v}dv = -\int\frac{dx}{x} + \log c$

$$\Rightarrow \log\tan v + \log x = \log c \Rightarrow \log(x\tan v) = \log c$$

$$\Rightarrow (x\tan v) = c \Rightarrow x\tan\frac{y}{x} = c.$$

This is the required solution of this differential equation.

Q.No.14.: Solve the differential equation $ye^{x/y}dx = \left(xe^{x/y} + y^2\right)dy$

Sol.: The given equation is $ye^{x/y}dx = \left(xe^{x/y} + y^2\right)dy$

$$\Rightarrow \frac{dx}{dy} = \frac{xe^{x/y} + y^2}{ye^{x/y}} \Rightarrow \frac{dx}{dy} = \frac{x}{y} + \frac{y}{x^{x/y}}, \qquad \text{(i)}$$

which is a homogeneous form of differential equation in x and y.

Putting $x = vy$ $\therefore$ $\frac{dx}{dy} = v + y\frac{dv}{dy}$.

$\therefore$ Equation (i) becomes $v + y\frac{dv}{dy} = v + \frac{y}{e^v} \Rightarrow y\left(\frac{dv}{dy} - e^{-v}\right) = 0 \Rightarrow \frac{dv}{dy} = e^{-v}$.

Separating the variable, we get $\frac{dv}{e^{-v}} = dy$.

Integrating both sides, we get $\int \frac{dv}{e^{-v}} = \int dy + c$

$\Rightarrow e^v = y + c \Rightarrow e^{x/y} = y + c$.

This is the required solution of this differential equation.

Q.No.15.: Solve the differential equation $xy\log\left(\frac{x}{y}\right)dx + \left[y^2 - x^2\log\left(\frac{x}{y}\right)\right]dy = 0$

Sol.: The given equation is $xy\log\left(\frac{x}{y}\right)dx + \left[y^2 - x^2\log\left(\frac{x}{y}\right)\right]dy = 0$

$$\Rightarrow \frac{dx}{dy} = \frac{\left[\left(\frac{x}{y}\right)^2\log\frac{x}{y} - 1\right]}{\frac{x}{y}\log\frac{x}{y}}, \qquad \text{(i)}$$

which is a homogeneous form of differential equation in x and y.

Putting $x = vy$ $\therefore$ $\frac{dx}{dy} = v + y\frac{dv}{dy}$.

$\therefore$ Equation (i) becomes $v + y\frac{dv}{dy} = \frac{\left[v^2\log v - 1\right]}{v\log v} \Rightarrow y\frac{dv}{dy} = \frac{\left[v^2\log v - 1 - v^2\log v\right]}{v\log v}$

$\Rightarrow y\frac{dv}{dy} = \frac{-1}{v\log v}$.

Separating the variable, we get $\frac{dy}{y} = -v\log v dv$

Integrating both sides, we get $\int \frac{dy}{y} = \int -v\log v dv + c$

$$\Rightarrow \log y = -\left[\log v.\frac{v^2}{2} - \int \frac{v^2}{2}.\frac{1}{v}dv\right] + c \Rightarrow \log y == -\left[\log v.\frac{v^2}{2} - \frac{1}{2}\int vdv\right] + c$$

$$\Rightarrow \log y == -\left[\log v.\frac{v^2}{2} - \frac{v^2}{4}\right] + c \Rightarrow \log y == -\frac{x^2}{2y^2}\log\frac{x}{y} + \frac{x^2}{4y^2} + c$$

$$\Rightarrow \log y == -\frac{x^2}{2y^2}\log\left(\frac{y}{x}\right)^{-1} + \frac{x^2}{4y^2} + c \Rightarrow \log y = \frac{x^2}{2y^2}\log\left(\frac{y}{x}\right) + \frac{x^2}{4y^2} + c$$

$$\Rightarrow \log y = \frac{x^2}{4y^2}\left[2\log\left(\frac{y}{x}\right) + 1\right] + c\,.$$

This is the required solution of this differential equation.

Q.No.16.: Solve the differential equation $xdx + \sin^2\frac{y}{x}(ydx - xdy) = 0$.

Sol.: The given equation is $xdx + \sin^2\frac{y}{x}(ydx - xdy) = 0$

$$\Rightarrow x + \sin^2\frac{y}{x}\left[y - x\frac{dy}{dx}\right] = 0 \Rightarrow \frac{dy}{dx} = \frac{1}{\sin^2\frac{y}{x}} + \frac{y}{x}\,.$$

Since given equation is homogeneous form of differential equation in x and y.

Putting $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$.

$\therefore$ Equation (i) becomes $\quad v + x\frac{dv}{dx} = \frac{1}{\sin^2 v} + v \Rightarrow \sin^2 vdv = \frac{dx}{x}$.

Integrating both sides, we get

$$\int \sin^2 vdv = \int \frac{dx}{x} + c' \Rightarrow \int \frac{1-\cos 2v}{2}dv = \log x + c'$$

$$\Rightarrow \frac{1}{2}\left[\frac{y}{x} - \frac{\sin^2\frac{y}{x}}{2}\right] = \log x + c' \Rightarrow \frac{1}{2}\left[v - \frac{\sin^2 v}{2}\right] = \log x + c'$$

$$\Rightarrow \log x = \frac{1}{2}\left[\frac{y}{x} - \frac{1}{2}\sin\left(\frac{2y}{x}\right)\right] + c, \quad \text{where } c = -c'.$$

This is the required solution of this differential equation.

Q.No.17.: Solve the differential equation $(x+2y)dx+(2x+y)dy=0$.

Sol.: Given equation is $(x+2y)dx+(2x+y)dy=0$

$$\Rightarrow \frac{dy}{dx}=-\frac{(x+2y)}{2x+y}=0, \tag{i}$$

which is homogeneous form of differential equation in x and y.

Put $u=\frac{y}{x}$, then $\frac{xdu}{dx}+u=\frac{dy}{dx}$.

$\therefore$ Equation (i) becomes $u+\frac{xdu}{dx}=\frac{-(1+2u)}{(2+u)}$.

$$\Rightarrow \frac{xdu}{dx}=\frac{-(1+2u)}{2+u}-u=\frac{-\left(u^2+4u+1\right)}{u+2}.$$

Separating the variables, we get $-\frac{dx}{x}=\frac{(u+2)du}{\left(u^2+4u+1\right)}=\frac{1}{2}\frac{d\left(u^2+4u+1\right)}{\left(u^2+4u+1\right)}$.

Integrating both sides, we get

$$-2\log x=\log\left(u^2+4u+1\right)+c_0 \Rightarrow x^2\left(u^2+4u+1\right)=c \Rightarrow x^2\left(\frac{y^2}{x^2}+4\frac{y}{x}+1\right)=c$$

$$\Rightarrow y^2+4xy+x^2=c.$$

This is the required solution of this differential equation.

Q.No.18.: Solve the differential equation $\left(1+2e^{x/y}\right)+2e^{x/y}.\left(1-\frac{x}{y}\right)y'=0$.

Sol.: Given equation is $\left(1+2e^{x/y}\right)+2e^{x/y}.\left(1-\frac{x}{y}\right)y'=0$

$$\Rightarrow \left(1+2e^{x/y}\right)dx+2e^{x/y}\left(1-\frac{x}{y}\right)dy=0, \tag{i}$$

which is homogeneous form of differential equation in x and y.

Put $\frac{x}{y}=u$, so that $dx=ydu+udy$.

$\therefore$ Equation (i) becomes $\left(1+2e^u\right)(udy+ydu)+2e^u(1-u)dy=0$

$$\Rightarrow \left(u+2e^u\right)dy+y\left(1+2e^u\right)du=0.$$

Separating the variables, we get $\frac{dy}{y}+\frac{1+2e^u}{u+2e^u}du=0$.

Integrating both sides, we get

$\log y+\log\left(u+2e^u\right)=\log c \Rightarrow y\left(u+2e^u\right)=c$.

Replacing u, we get

$y\left(\frac{x}{y}+2e^{x/y}\right)=c$.

This is the required solution of this differential equation.

Q.No.19.: Solve the differential equation $\left(y+\sqrt{x^2+y^2}\right)dx-xdy=0$, $y(1)=0$.

Sol.: Given equation is $\left(y+\sqrt{x^2+y^2}\right)dx-xdy=0$

$\Rightarrow \frac{dy}{dx}=\frac{y+\sqrt{x^2+y^2}}{x}=\frac{y}{x}+\sqrt{1+\left(\frac{y}{x}\right)^2}$,

which is homogeneous form of differential equation in x and y.

Put $y=ux$, then $\frac{dy}{dx}=u+x\frac{du}{dx}$.

$\therefore$ Equation (i) becomes $u+x\frac{du}{dx}=u+\sqrt{1+u^2} \Rightarrow x\frac{du}{dx}=\sqrt{1+u^2}$

Separating the variables, we get $\frac{dx}{x}=\frac{du}{\sqrt{u^2+1}}$.

Integrating both sides, we get

$\log|x|+\log|c|=\log\left|u+\sqrt{u^2+1}\right| \Rightarrow u+\sqrt{u^2+1}=cx$

Replacing u, we have

$\frac{y}{x}+\sqrt{\frac{y^2}{x^2}+1}=cx \Rightarrow y+\sqrt{x^2+y^2}=cx^2$.

Put $y=0$, when $x=1$, then $c=1$.

So, the required solution is

$y+\sqrt{x^2+y^2}=x^2$.

Home Assignments

Q.No.1.: Solve the differential equation $\left(x^2y-2xy^2\right)dx-\left(x^3-3x^2y\right)dy=0$.

Ans.: $cy^3=x^2e^{-x/y}$.

Q.No.2.: Solve the differential equation $(x+y)dx+(y-x)dy=0$.

Ans.: $\log\left(x^2+y^2\right)=2\tan^{-1}\dfrac{y}{x}+c$.

Q.No.3.: Solve the differential equation $x\dfrac{dy}{dx}+\dfrac{y^2}{x}=y$.

Ans.: $cx=e^{x/y}$.

Q.No.4.: Solve the differential equation $x(x-y)\dfrac{dy}{dx}=y(x+y)$.

Ans.: $\dfrac{x}{y}+\log\dfrac{x}{y}=c$.

Q.No.5.: Solve the differential equation $\left(\sqrt{xy}-x\right)dy+ydx=0$.

Ans.: $2\sqrt{\dfrac{x}{y}}+\log y=c$.

Q.No.6.: Solve the differential equation $\left(y^2+2xy\right)dx+\left(2x^2+3xy\right)dy=0$.

Ans.: $xy^2(x+y)=c$.

Q.No.7.: Solve the differential equation $x\dfrac{dy}{dx}=y(\log y-\log x)$.

Ans.: $y=xe^{1+cx}$.

Q.No.8.: Solve the differential equation $x\dfrac{dy}{dx}=y+x\cos^2\dfrac{y}{x}$.

Ans.: $\tan\dfrac{y}{x}=\log(cx)$.

Q.No.9.: Solve the differential equation

$$\left(x\cos\frac{y}{x}+y\sin\frac{y}{x}\right)y-\left(y\sin\frac{y}{x}-x\cos\frac{y}{x}\right)x\frac{dy}{dx}=0.$$

Ans.: $xy\cos\dfrac{y}{x} = c$.

Q.No.10.: Solve the differential equation $\left(2xy + 3y^2\right)dx - \left(2xy + x^2\right)dy = 0$.

Ans.: $y^2 + xy = cx^3$.

Q.No.11.: Solve the differential equation $\left(x^3 + y^2\sqrt{x^2 + y^2}\right)dx - xy\sqrt{x^2 + y^2}\,dy = 0$.

Ans.: $\left(x^2 + y^2\right)^{3/2} = x^3 \log cx^3$.

Q.No.12.: Solve the differential equation $(2x - 5y)dx + (4x - y)dy = 0$, $y(1) = 4$.

Ans.: $(2x + y)^2 = 12(y - x)$.

Q.No.13.: Solve the differential equation $x.\sin\dfrac{y}{x}\dfrac{dy}{dx} = y\sin\dfrac{y}{x} + x$.

Ans.: $\cos\dfrac{y}{x} + \log cx = 0$.

Q.No.14.: Solve the differential equation $x^2 y' = 3\left(x^2 + y^2\right)\tan^{-1}\dfrac{y}{x} + xy$.

Ans.: $y = x\tan cx^3$.

Q.No.15.: Solve the differential equation $\left(x^2 + xy\right)dy = \left(x^2 + y^2\right)dx$.

Ans.: $(x - y)^2 = cx.e^{-y/x}$.

Q.No.16.: Solve the differential equation $x^2 ydy + \left(x^3 + x^2 y - 2xy^2 - y^3\right)dx = 0$

Ans.: $\log\left\{\dfrac{c(y - x)}{x^4(y + x)}\right\} = \dfrac{2x}{x + y}$.

Q.No.17.: Solve the differential equation $xy' = y + x.\cos^2\left(\dfrac{y}{x}\right)$, $y(1) = \dfrac{\pi}{4}$.

Ans.: $1 + \log x = \tan\dfrac{y}{x}$.

Q.No.19.: Solve the differential equation $y' = \dfrac{6x^2 - 5xy - 2y^2}{6x^2 - 8xy + y^2}$.

Ans.: $(y - x)(y - 3x)^9 = c(y - 2x)^{12}$.

Q.No.20.: Solve the differential equation

$$\left[2x\sin\frac{y}{x}+2x\tan\frac{y}{x}-y\cos\frac{y}{x}-y\sec^2\frac{y}{x}\right]dx+\left[x.\cos\frac{y}{x}+x\sec^2\frac{y}{x}\right]dy=0$$

Ans.: $x^2\left(\sin\frac{y}{x}+\tan\frac{y}{x}\right)=c$.

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3rd Topic

Ordinary Differential Equations of First order

Part-II:

"Equations reducible to the Homogeneous Form"

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Equations Reducible to Homogeneous Form:

The equation of the form

$$\boxed{\frac{dy}{dx} = \frac{ax + by + c}{a'x + b'y + c'}} \qquad \text{(i)}$$

is non-homogeneous form of differential equation due to the presence of c and c'.

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This equation can be reduced to homogeneous form as follows:

Case I.: When $\boxed{\frac{a}{a'} \neq \frac{b}{b'}}$.

Putting $x = X + h$, $y = Y + k$, (h, k being constants)

so that $dx = dX$, $dy = dY$, then (i) becomes

$$\frac{dY}{dX} = \frac{aX + bY + (ah + bk + c)}{a'X + b'Y + (a'h + b'k + c')}. \qquad \text{(ii)}$$

Choose h, k in such a way that (ii) becomes homogeneous.

So put $ah + bk + c = 0$ and $a'h + b'k + c' = 0$

$$\Rightarrow \frac{h}{bc' - b'c} = \frac{k}{ca' - c'a} = \frac{1}{ab' - ba'} \Rightarrow h = \frac{bc' - b'c}{ab' - b'a},\ k = \frac{ca' - c'a}{ab' - ba'}. \qquad \text{(iii)}$$

Thus, when $h=\dfrac{bc'-b'c}{ab'-b'a}$, $k=\dfrac{ca'-c'a}{ab'-ba'}$ and $ab'-ba'\neq 0$,

(ii) becomes $\dfrac{dY}{dX}=\dfrac{aX+bY}{a'X+b'Y}$,

which is homogeneous in X and Y.

Now above homogeneous form of differential equation can be solved by putting $Y=vX$.

Further process is same as we discuss in earlier.

Case II.: When $\boxed{\dfrac{a}{a'}=\dfrac{b}{b'}}$ i.e., $ab'-ba'=0$,

then the above method fails as h and k become infinite or indeterminate.

Now in that case put $\dfrac{a}{a'}=\dfrac{b}{b'}=\dfrac{1}{m}$ (say).

$\Rightarrow a'=am$, $b'=bm$, then (i) becomes

$$\frac{dy}{dx}=\frac{(ax+by)+c}{m(ax+by)+c'}. \tag{iv}$$

Put $ax+by=t$ so that $a+b\dfrac{dy}{dx}=\dfrac{dt}{dx}\Rightarrow\dfrac{dy}{dx}=\dfrac{1}{b}\left(\dfrac{dt}{dx}-a\right)$

$\therefore$ (iv) becomes $\dfrac{1}{b}\left(\dfrac{dt}{dx}-a\right)=\dfrac{t+c}{mt+c'}$

$\Rightarrow\dfrac{dt}{dx}=a+\dfrac{bt+bc}{mt+c'}=\dfrac{(am+b)t+ac'+bc}{mt+c'}$,

so that the variables are separable.

Integrating, we get the solution.

In that solution, put $t=ax+by$, we get the required solution of (i).

Now let us solve some differential equations, which can be reducible to homogeneous form:

- **Differential equations reducible to Homogeneous form of differential equations:**

  Case I.: When $\dfrac{a}{a'}\neq\dfrac{b}{b'}$.

  Case II.: When $\dfrac{a}{a'}=\dfrac{b}{b'}$ i.e., $ab'-ba'=0$.

Problems related with $\frac{a}{a'} \neq \frac{b}{b'}$:

Q.No.1.: Solve the differential equation $\frac{dy}{dx} = \frac{y+x-2}{y-x-4}$.

Sol.: Given equation is $\frac{dy}{dx} = \frac{y+x-2}{y-x-4}$. (i)

Here $\frac{a}{a'} \neq \frac{b}{b'}$.

Putting $x = X + h$, $y = Y + k$, (h, k being constants), so that $dx = dX$, $dy = dY$.

Then (i) becomes

$$\frac{dY}{dX} = \frac{Y + X + (k + h - 2)}{Y - X + (k - h - 4)}. \quad \text{(ii)}$$

Put $k + h - 2 = 0$ and $k - h - 4 = 0$ so that $h = -1$, $k = 3$.

$\therefore$ (ii) becomes $\frac{dY}{dX} = \frac{Y+X}{Y-X}$ which is homogeneous in X and Y. (iii)

$\therefore$ Put $Y = vX$, then $\frac{dY}{dX} = v + X\frac{dv}{dX}$.

Then (iii) becomes $v + X\frac{dv}{dX} = \frac{v+1}{v-1} \Rightarrow X\frac{dv}{dX} = \frac{v+1}{v-1} - v = \frac{1+2v-v^2}{v-1}$

$$\Rightarrow \frac{v-1}{1+2v-v^2}dv = \frac{dX}{X}.$$

Now this is a variable separable form.

Integrating both sides, we get

$$-\frac{1}{2}\int \frac{2-2v}{1+2v-v^2}dv = \int \frac{dX}{X} + c \Rightarrow -\frac{1}{2}\log\left(1+2v-v^2\right) = \log X + c$$

$$\Rightarrow \log\left(1 + \frac{2Y}{X} - \frac{Y^2}{X^2}\right) + \log X^2 = -2c \Rightarrow \log\left(X^2 + 2XY - Y^2\right) = -2c$$

$$\Rightarrow X^2 + 2XY - Y^2 = e^{-2c} = c'. \quad \text{(iv)}$$

Putting $X = x - h = x + 1$, $Y = y - k = y - 3$, then (iv) becomes

$$(x+1)^2 + 2(x+1)(y-3) - (y-3)^2 = c'$$

$\Rightarrow x^2 + 2xy - y^2 - 4x + 8y - 14 = c'$, which is the required solution.

Q.No.2.: Solve the differential equation $(3y-7x+7)dx+(7y-3x+3)dy=0$.

Sol.: Given equation is $\frac{dy}{dx}=\frac{3y-7x+7}{7y-3x+3}$. (i)

Here $\frac{a}{a'}\neq\frac{b}{b'}$.

Putting $x=X+h$, $y=Y+k$, (h, k being constants), so that dx = dX , dy = dY.

Then (i) becomes

$$\frac{dY}{dX}=-\frac{3(Y+k)-7(X+h)+7}{7(Y+k)-3(X+h)+3}=\frac{3Y-7X+(-7h+3k+7)}{7Y-3X+(-3h+7k+3)}. \quad \text{(ii)}$$

Now, choosing, h, k such that $-7h+3k+7=0$ and $-3h+7k+3=0$

Solving these equations, we get h = 1, k = 0.

With these values of h, k, equation (ii) reduces to $\frac{dY}{dX}=-\frac{3Y-7X}{7Y-3X}$. (iii)

Putting $Y=vX$ so that $\frac{dY}{dX}=v+X\frac{dv}{dX}$.

Equation (iii) becomes $v+X\frac{dv}{dX}=-\frac{3vX-7X}{7vX-3X}\Rightarrow X\frac{dv}{dX}=\frac{7-3v}{7v-3}-v=\frac{7-7v^2}{7v-3}$.

Now this is a variable separable form.

Separating the variables, we get

$$\frac{7v-3}{1-v^2}dv=7\frac{dX}{X}\quad\Rightarrow\left(\frac{2}{1-v}-\frac{5}{1+v}\right)dv=7\frac{dX}{X}$$

Integrating both sides, we get

$$-2\log(1-v)-5\log(1+v)=7\log X+c\Rightarrow 7\log X+2\log(1-v)+5\log(1+v)=-c$$

$$\Rightarrow\log\left[X^7\left(1-v^2\right)(1+v)^5\right]=-c\Rightarrow X^7\left(1-\frac{Y}{X}\right)^2\left(1+\frac{Y}{X}\right)^5=e^{-c}$$

$$\Rightarrow(X-Y)^2(X+Y)^5=C,\ \text{where } C=e^{-c}. \quad \text{(iv)}$$

Putting $X=x-h=x-1$, $Y=y-k=y$.

Equation (iv) becomes $(x-y-1)^2(x+y-1)^5=C$, which is the required solution.

Q.No.3.: Solve $(x-2y+1)dx+(4x-3y-6)dy=0$.

Sol.: Given equation is $(x-2y+1)dx+(4x-3y-6)dy=0$. (i)

This equation is non-homogeneous.

Here $\frac{a}{a'}\neq\frac{b}{b'}$.

Putting $x=x_1+h$, $y=y_1+k$, (h, k being constants), so that $dx = dx_1$, $dy = dy_1$.

Then (i) becomes

$$[x_1+h-2(y_1+k)+1]dx_1+[4(x_1+h)-3(y_1+k)-6]dy_1=0$$

$$\Rightarrow [(x_1-2y_1)+(h-2k+1)]dx_1+[(4x_1-3y_1)+(4h-3k-6)]dy_1=0. \quad \text{(ii)}$$

Now, choosing, h, k such that $h-2k+1=0$ and $4h-3k-6=0$.

Solving these equations, we get $h = 3$, $k = 2$.

With these values of h, k, equation (ii) reduces to

$$\frac{dy_1}{dx_1}=\frac{x_1-2y_1}{3y_1-4x_1}. \quad \text{(iii)}$$

Putting $u=\frac{y_1}{x_1}$, so that $x_1\frac{du}{dx_1}+u=\frac{dy_1}{dx_1}$.

Equation (iii) becomes $x_1\frac{du}{dx_1}+u=\frac{1-2u}{3u-4}$

$$x_1\frac{du}{dx_1}=\frac{1-2u}{3u-4}-u=\frac{1-2u-3u^2+4u}{3u-4}$$

$$x_1\frac{du}{dx_1}=\frac{1+2u-3u^2}{3u-4}.$$

Now this is a variable separable form.

Separating the variables, we get

$$\left(\frac{3u-4}{3u^2-2u-1}\right)du=\frac{-dx_1}{x_1}$$

Integrating both sides, we get

$$\frac{1}{2}\log\left|3u^2-2u-1\right|-\frac{3}{4}\log\left|\frac{3u-3}{3u+1}\right|=-\log|x_1|+\log|c_1|$$

$$\Rightarrow \log(3u^2-2u-1)^2 - \log\left|\frac{3u-3}{3u+1}\right|^3 = \log\left(\frac{c_1^4}{x_1^4}\right)$$

$$\Rightarrow \log\left|\frac{(3u+1)^5}{3(u+1)}\right| = \log\left(\frac{c_1^4}{x_1^4}\right)$$

$$\Rightarrow x_1^4(3u+1)^5 = c|u-1|, \quad \text{where } c = 3c_1^4.$$

Replacing $u = \frac{y_1}{x_1}$, we get $|3y_1 + x_1|^5 = c|y_1 - x_1|$.

Replacing $y_1 = y-2$, $x_1 = x-3$, we get

$|x+3y-9|^5 = c|y-x+1|$, which is the required solution.

Q.No.4.: Solve the differential equation $(2x^2+3y^2-7)xdx = (3x^2+2y^2-8)ydy$.

Sol.: Put $x^2 = X$, $y^2 = Y$, so that $2xdx = dX$, $2ydy = dY$.

$(2X+3Y-7)dX = (3X+2Y-8)dY$

$\frac{dY}{dX} = \frac{2X+3Y-7}{3X+2Y-8}$, which is not homogeneous equation. (i)

Here $\frac{a}{a'} \neq \frac{b}{b'}$.

Putting $X = X_1 + h$, $Y = Y_1 + k$, (h, k being constants), so that $dX = dX_1$, $dY = dY_1$.

Then (i) becomes

$$\frac{dY_1}{dX_1} = \frac{2X_1+3Y_1+(2h+3k-7)}{3X_1+2Y_1+(3h+2k-8)}. \qquad \text{(ii)}$$

To convert this into a homogeneous equation, put $2h+3k-7=0$, $3h+2k-8=0$

Solving these equations, we get $h = 2$, $k = 1$,

With these values of h, k, equation (ii) reduces to $\frac{dY_1}{dX_1} = \frac{2X_1+3Y_1}{3X_1+2Y_1}$. (iii)

Now this is a homogeneous equation.

To solve this, put $u = \frac{Y_1}{X_1}$, so that $\frac{dY_1}{dX_1} = u + X_1\frac{du}{dX_1}$.

Equation (iii) becomes $u + X_1\frac{du}{dX_1} = \frac{2X_1+3uX_1}{3X_1+2uX_1} = \frac{2+3u}{3+2u}$

$$X_1 \frac{du}{dX_1} = \frac{2+3u}{3+2u} - u = \frac{2\left(1-u^2\right)}{3+2u}$$

Separating the variables, we get

$$\frac{2dX_1}{X_1} = \left(\frac{3+2u}{1-u^2}\right)du = \frac{3du}{1-u^2} + \frac{2udu}{1-u^2}$$

But $\dfrac{1}{1-u^2} = \dfrac{1}{(1-u)(1+u)} = \dfrac{1}{2}\left[\dfrac{1}{(1-u)} + \dfrac{1}{1+u}\right]$

Integrating both sides, we get

$$2\int \frac{dX_1}{X_1} = 3.\frac{1}{2}\left[\int \frac{du}{1-u} + \frac{du}{1+u}\right] - 2\int \frac{udu}{u^2-1}$$

$$\Rightarrow 4\log X_1 + 2\log c = 3\log\left(\frac{u+1}{u-1}\right) - 2\log\left(u^2-1\right) \Rightarrow c^2 X_1^4 = \frac{(u+1)^3}{(u-1)^3}.\frac{1}{\left(u^2-1\right)^2}$$

$$\Rightarrow c^2 X_1^4 = \frac{u+1}{(u-1)^5}.$$

Replacing $u = \dfrac{Y_1}{X_1}$, we get $c^2 = \dfrac{(Y_1 + X_1)}{(Y_1 + X_1)^5}$.

Replacing $X_1 = X-2$, $Y_1 = Y-1$, we have $c^2 = \dfrac{(Y-1+X-2)}{[Y-1-(X-2)]^5}$

$\Rightarrow c^2\left(x^2 - y^2 + 1\right)^5 = \left(x^2 + y^2 - 3\right)$, which is the required solution.

Problems related with $\frac{a}{a'} = \frac{b}{b'}$:

Q.No.5.: Solve the differential equation $(3y+2x+4)dx - (4x+6y+5)dy = 0$.

Sol.: Given equation is $\dfrac{dy}{dx} = \dfrac{(2x+3y)+4}{2(2x+3y)+5}$. (i)

Here $\dfrac{a}{a'} = \dfrac{b}{b'}$.

Putting $2x+3y = t$ so that $2 + 3\dfrac{dy}{dx} = \dfrac{dt}{dx}$.

$\therefore$ (i) becomes $\dfrac{1}{3}\left(\dfrac{dt}{dx} - 2\right) = \dfrac{t+4}{2t+5}$

$$\Rightarrow \frac{dt}{dx} = 2 + \frac{3t+12}{2t+5} = \frac{7t+22}{2t+5} \Rightarrow \frac{2t+5}{7t+22}dt = dx$$

Now this is a variable separable form.

Integrating both sides, we get

$$\int \frac{2t+5}{7t+22}dt = \int dx + c \Rightarrow \int \left(\frac{2}{7} - \frac{9}{7}.\frac{1}{7t+22}\right)dt = x + c$$

$$\Rightarrow \frac{2}{7}t - \frac{9}{49}\log(7t+22) = x + c$$

Putting $t = 2x + 3y$, we have

$$14(2x+3y) - 9\log(14x+21y+22) = 49x + 49c$$

$\Rightarrow 21x - 42y + 9\log(14x+21y+22) = c'$, which is the required solution.

Q.No.6.: Solve the differential equation $\frac{dy}{dx} = \frac{y-x}{y-x+2}$.

Sol.: Given equation is $\frac{dy}{dx} = \frac{y-x}{y-x+2}$.

Here $\frac{a}{a'} = \frac{b}{b'}$.

Putting $z = y - x$, so that $\frac{dz}{dx} = \frac{dy}{dx} - 1$.

$\therefore$ (i) becomes $\frac{dy}{dx} = \frac{dz}{dx} + 1 = \frac{z}{z+2} \Rightarrow \frac{dz}{dx} = \frac{z}{z+2} - 1 = \frac{-2}{z+2} \Rightarrow (z+2)dz + 2dx = 0$

Now this is a variable separable form.

Integrating both sides, we get

$$\int (z+2)dz + 2\int dx = c \Rightarrow \frac{z^2}{2} + 2z + 2x = c.$$

Replacing z by $y - x$, we get

$(y-x)^2 + 4(y-x) + 4x = c \Rightarrow (y-x)^2 + 4y = c$, which is the required solution.

Home Assignments

Q.No.1.: Solve the differential equation $\frac{dy}{dx} = \frac{2y - x - 4}{y - 3x + 3}$.

Ans.: $X^2 - 5XY + Y^2 = c\left[\frac{2Y + \left(5 + \sqrt{21}\right)X}{2Y - \left(5 - \sqrt{21}\right)X}\right]^{1/\sqrt{21}}$, where $X = x - 2$, $Y = y - 3$.

Q.No.2.: Solve the differential equation $(2x + y - 3)dy = (x + 2y - 3)dx$.

Ans.: $(y - x)^3 = c(y + x - 2)$.

Q.No.3.: Solve the differential equation $(2x + 5y + 1)dx - (5x + 2y - 1)dy = 0$.

Ans.: $(x +)^7 = c\left(x - y - \frac{2}{3}\right)^3$.

Q.No.4.: Solve the differential equation $\frac{dy}{dx} + \frac{ax + hy + g}{hx + by + f} = 0$.

Ans.: $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$.

Q.No.5.: Solve the differential equation $\frac{dy}{dx} = \frac{x + y + 1}{2x + 2y + 3}$.

Ans.: $3(2y - x) + \log(3x + 3y + 4) = c$.

Q.No.6.: Solve the differential equation $(4x - 6y - 1)dx + (3y - 2x - 2)dy = 0$.

Ans.: $x - y + \frac{3}{4}\log(8x - 12y - 5) = c$.

Q.No.7.: Solve the differential equation $(x + 2y)(dx - dy) = dx + dy$.

Ans.: $\log\left(x + y + \frac{1}{3}\right) + \frac{3}{2}(y - x) = c$.

Q.No.8.: Solve the differential equation $\frac{dy}{dx} = \frac{x + 2y - 3}{2x + y - 3}$.

Ans.: $(x - y)^3 = c(x + y - 2)$.

Q.No.9.: Solve the differential equation $\frac{dy}{dx} + \frac{2x + 3y + 1}{3x + 4y - 1} = 0$.

Ans.: $x^2 + 3xy + 2y^2 + x - y = c$.

Q.No.10.: Solve the differential equation $(x+2y+3)dx-(2x-y+1)dy=0$..

Ans.: $\log\left[(x+1)^2+(y+1)^2\right]=4\tan^{-1}\dfrac{y+1}{x+1}+c$.

Q.No.11.: Solve the differential equation $(2x-2y+5)\dfrac{dy}{dx}=x-y+3$

Ans.: $x-2y+\log(x-y+2)=c$.

Q.No.12.: Solve the differential equation $\dfrac{dy}{dx}=\dfrac{6x-4y+3}{3x-2y+1}$.

Ans.: $2x-y=\log(3x-2y+3)+c$.

Q.No.13.: Solve the differential equation $(2x+y+1)dx+(4x+2y-1)dy=0$.

Ans.: $x+2y+\log(2x+y-1)=c$.

Q.No.14.: Solve the differential equation $(x+y)(dx-dy)=dx+dy$.

Ans.: $x-y=\log(x+y)+c$.

Q.No.15.: Solve the differential equation $(3x-y-9)y'=(10-2x+2y)$.

Ans.: $y-2x+7=c(x+y+1)^4$.

Q.No.16.: Solve the differential equation $y'=\dfrac{ax+by-a}{bx+ay-b}$.

Ans.: $(y-x+1)^{(a+b)/a}(y+x-1)^{(a-b)/a}$.

Q.No.17.: Solve the differential equation $(2x-5y+3)\,dx-(2x+4y-6)\,dy=0$.

Ans.: $(4y-x-3)(y+2x-x)^2=c$.

Q.No.18.: Solve the differential equation $(3y-7x+7)dx+(7y-3x+3)dy=0$.

Ans.: $(y-x+1)^2(y+x-1)^5=c$.

Q.No.19.: Solve the differential equation $y'=\dfrac{2x+3y+1}{3x-2y-5}$.

Ans.: $\log\left[(x-1)^2+(y+1)^2\right]-3\tan^{-1}\left(\dfrac{y+1}{x-1}\right)=c$.

Q.No.20.: Solve the differential equation $y'=\dfrac{2x+y-1}{4x+2y+5}$.

Ans.: $10y-5x+7\log(10x+5y+9)=c$.

Q.No.21.: Solve the differential equation $(2x+2y+1)dx+(x+y-1)dy=0$.

Ans.: $3\log(x+y+2)-2x-y=c$.

Q.No.22.: Solve the differential equation $y'=\dfrac{(x-2y+3)}{(2x-4y+5)}$.

Ans.: $x^2-4xy+4y^2+6x-10y=c$.

Q.No.23.: Solve the differential equation $y'+\dfrac{ax+hy+g}{hx+by+f}=0$.

Ans.: $ax^2+2hxy+by^2+2gx+2fy+c=0$.

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4th Topic

Differential Equations of First order

Part: I

"Linear Differential Equations"
(Leibnitz's Linear Equation)

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Linear differential equation:

An n^{th} order ordinary differential equation in the dependent variable y is said to be linear in y if

i. y and all its derivatives are of degree one.
ii. No product terms of y and/or any of it derivatives are present.
iii. No transcendental functions of y and/or its derivatives occur.

The general from of an n^{th} order linear O.D.E. in y with variables coefficient is

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \ldots.. + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = b(x),$$

where RHS b(x) and all the coefficients $a_0(x)$, $a_1(x), \ldots\ldots\ldots, a_n(x)$ are given functions of x and $a_0(x) \neq 0$.

If all the coefficients a_0, a_1, ……….a_n are constants then the above equation is known as n^{th} order linear O.D.E. with constant coefficients.

Non-linear differential equation:

An ordinary differential equation in the dependent variable y is said to be nonlinear in y if

(i) y and all its derivatives are of degree more than one.

(ii) Product terms of y and/or any of it derivatives are present.

(iii) Transcendental functions of y and/or its derivatives occur.

Note: A linear differential equation is of first degree differential equation, but a first degree differential equation need not to be linear, since it may contain nonlinear terms such y^2, $y^{\frac{-1}{2}}$, e^y, $\sin y$, etc.

S.No.	Diff. Equation	Ans. Kind	Order	Degree	linearity
1	$\frac{dy}{dx} = kx^2$	Ordinary	1	1	Yes
2	$\frac{dy}{dx} + P(x)y = y^n Q(x)$	Ordinary	1	1	No (Yes for n = 0, 1)
3	$e^x dx + e^y dy = 0$	Ordinary	1	1	Nonlinear (in x and y)
4	$\left(\frac{d^3y}{dx^2}\right)^4 - 6x^2\left(\frac{dy}{dx}\right)^8 + e^y = \sin xy$	Ordinary	3	4	No
5	$y\frac{d^2y}{dx^2} + \sin x = 0$	Ordinary	2	1	No
6	$x^2 dy + y^2 dx = 0$	Ordinary	1	1	No
7	$\frac{d^4y}{dx^4} + 3\left(\frac{d^2y}{dx^2}\right)^5 + 5y = 0$	Ordinary	4	1	No
8	$y^2 dx + (3xy - 1)dy = 0$	Ordinary	1	1	Nonlinear in y and linear in x
9	$k(y'')^2 = [1 + (y'')^2]^3$	Ordinary	2	2	No

10	$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$	Partial	2	1	Yes
11	$\frac{\partial^2 Y}{\partial t^2} = a^2 \frac{\partial^2 Y}{\partial x^2}$	Partial	2	1	Yes
12	$\left(\frac{dr}{ds}\right)^3 = \sqrt{\frac{d^2 r}{ds^2} + 1}$	Ordinary	2	1	No

Linear differential equations:

If the dependent variable and its differential coefficients occur only in the first degree and not multiplied together in a differential equation, then that differential equation is called a linear differential equation.

STANDARD FORM:

The standard form of a linear equation of the first order, commonly known as Leibnitz's linear equation, is

$$\boxed{\frac{dy}{dx} + Py = Q}, \qquad \text{(i)}$$

where P, Q are functions of x only or may be constant.

Method to solve a linear differential equation:

1. Multiplying both sides by $e^{\int Pdx}$ (known as integrating factor I.F.), we get

$$\frac{dy}{dx}.e^{\int Pdx} + y\left(e^{\int Pdx}P\right) = Qe^{\int Pdx} \Rightarrow \frac{d}{dx}\left(ye^{\int Pdx}\right) = Qe^{\int Pdx}.$$

2. Integrating on both sides, we get

$$y\left(e^{\int pdx}\right) = \int Q\left(e^{\int Pdx}\right)dx + c \quad \Rightarrow y(I.F.) = \int Q(I..F.)dx + c,$$

which is the required solution.

Remarks:

1. In the general form of a linear differential equation, the co-efficient of $\frac{dy}{dx}$ is unity.

2. **The equation $R\frac{dy}{dx}+Sy=T$, where R, S and T are functions of x only or constants, must be divided by R to bring it to the general linear form.**

3. **The factor $e^{\int P dx}$ on multiplying by which the left-hand side of (i) becomes the differential coefficient of a single function, is called the integrating factor (I.F.) of linear equation (i).**

4. **It is important to remember that $\boxed{\text{I. F.} = e^{\int P dx}}$**

   and the solution is $\boxed{y(\text{I.F.}) = \int Q(\text{I..F.})dx + c}$.

Now let us solve some linear differential equations:

Q.No.1.: Solve $(x+1)\frac{dy}{dx} - y = e^{3x}(x+1)^2$.

Sol.: Given differential equation is $(x+1)\frac{dy}{dx} - y = e^{3x}(x+1)^2$.

$$\Rightarrow \frac{dy}{dx} - \frac{y}{x+1} = e^{3x}(x+1), \qquad \text{(i)}$$

which is Leibnitz's linear equation in y.

Here $P = -\frac{1}{x+1}$ and $\int P dx = -\int \frac{dx}{x+1} = -\log(x+1) = \log(x+1)^{-1}$.

$\therefore$ I.F. $= e^{\int P dx} = e^{\log(x+1)^{-1}} = \frac{1}{x+1}$.

Thus, the solution of (i) is y (I. F.) $= \int \left[e^{3x}(x+1)\right](\text{I. F.})dx + c$

$$\Rightarrow \frac{y}{x+1} = \int e^{3x}dx + c = \frac{e^{3x}}{3} + c \Rightarrow y = \left(\frac{e^{3x}}{3} + c\right)(x+1),$$

which is the required solution of given differential equation.

Q.No.2.: Solve $\left(1+y^2\right)dx = \left(\tan^{-1}y - x\right)dy$.

Sol.: Given differential equation is $\left(1+y^2\right)dx = \left(\tan^{-1}y - x\right)dy$.

This equation contains y^2 and $\tan^{-1}y$ and is, therefore, not a linear in y; but since only x occurs, it can be written as

$$\left(1+y^2\right)\frac{dx}{dy} = \left(\tan^{-1}y - x\right) \Rightarrow \frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\left(\tan^{-1}y\right)}{1+y^2}, \qquad \text{(i)}$$

which is Leibnitz's linear equation in x.

$\therefore$ I. F. $= e^{\int P dy} = e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1} y}$.

Thus, the solution of equation (i) is $x(\text{I.F.}) = \int Q(\text{I..F.})dy + c$

$$\Rightarrow x\ (\text{I. F.}) = \int \frac{\tan^{-1}}{1+y^2}(\text{I. F.})dy + x$$

$$\Rightarrow xe^{\tan^{-1}y} = \int \frac{\tan^{-1}y}{1+y^2}.e^{\tan^{-1}y}dy + c$$

$$= \int te^t dt + c = t.e^t - \int 1.e^t dt + c \qquad \text{Here put } \tan^{-1}y = t \quad \therefore \frac{dy}{1+y^2} = dt$$

$$= t.e^t - e^t + c = \left(\tan^{-1}y - 1\right)e^{\tan^{-1}y} + c$$

$$\Rightarrow x = \tan^{-1}y - 1 + ce^{-\tan y},$$

which is the required solution of given differential equation.

Q.No.3.: Solve the differential equation $x^2\dfrac{dy}{dx} = 3x^2 - 2xy + 1$.

Sol.: Given differential equation is $x^2\dfrac{dy}{dx} + 2xy = 3x^2 + 1$

$$\Rightarrow \frac{dy}{dx} + \frac{2y}{x} = 3 + \frac{1}{x^2}, \qquad \text{(i)}$$

which is Leibnitz's linear equation in y.

$\therefore$ I.F. $= e^{\int \frac{2}{x}dx} = x^2$.

Thus, the solution of equation (i) is $y(\text{I.F.}) = \int Q(\text{I..F.})dx + c$

$$\Rightarrow yx^2 = \int \frac{3x^2+1}{x^2}.x^2 dx + c \Rightarrow yx^2 = \int \left(3x^2+1\right)dx + c \Rightarrow yx^2 = x^3 + x + c$$

$$\Rightarrow y = x + x^{-1} + cx^{-2},$$

which is the required solution of given differential equation.

Q.No.4.: Solve the differential equation $\cos^2 x\dfrac{dy}{dx}+y=\tan x$.

Sol.: Given differential equation is $\cos^2 x\dfrac{dy}{dx}+y=\tan x$

$$\Rightarrow \frac{dy}{dx}+y\sec^2 x=\frac{\tan x}{\cos^2 x}, \qquad \text{(i)}$$

which is Leibnitz's linear equation in y.

$\therefore$ I.F. $=e^{\int \sec^2 x dx}=e^{\tan x}$.

Thus, the solution of equation (i) is $y(\text{I.F.})=\int Q(\text{I..F.})dx+c$

$$\Rightarrow y.e^{\tan x}=\int \tan x\sec^2 x.e^{\tan x}dx+c.$$

Put $\tan x=t \therefore dt=\sec^2 xdx$.

Then $y.e^{\tan x}=t.e^t-\int e^t dt=e^t[t-1]+c \Rightarrow y.e^{\tan x}=e^{\tan x}[\tan x-1]+c$

$$\Rightarrow y=\tan x-1+ce^{-\tan x},$$

which is the required solution of given differential equation.

Q.No.5.: Solve the differential equation $x\log x\dfrac{dy}{dx}+y=\log x^2$.

Sol.: Given differential equation is $x\log x\dfrac{dy}{dx}+y=\log x^2 \Rightarrow x\log x\dfrac{dy}{dx}+y=2\log x$

$$\Rightarrow \frac{dy}{dx}+\frac{y}{x\log x}=\frac{2}{x}, \qquad \text{(i)}$$

which is Leibnitz's linear equation in y.

$\therefore$ I.F. $=e^{\int \frac{1}{x\log x}dx}=e^{\int \frac{1/x}{\log x}dx}=e^{\log(\log x)}=\log x$.

Thus, the solution of equation (i) is $y(\text{I.F.})=\int Q(\text{I..F.})dx+c$

$$\Rightarrow y(\log x)=\int \frac{2}{x}\log xdx+c \Rightarrow y\log x=(\log x)^2+c \qquad \left[\because \int f(x)f'(x)=\frac{[f(x)]^2}{2}+c\right]$$

$$\Rightarrow y=\log x+\frac{c}{\log x},$$

which is the required solution of given differential equation.

Q.No.6.: Solve the differential equation $\cosh x\dfrac{dy}{dx}+y\sinh x=2\cosh^2 x\sinh x$.

Sol.: Given differential equation is $\cosh x\dfrac{dy}{dx}+y\sinh x=2\cosh^2 x\sinh x$

$$\Rightarrow \frac{dy}{dx}+y\tanh x=2\cosh x\sinh x\,, \qquad \text{(i)}$$

which is Leibnitz's linear equation in y.

$$\therefore \text{I. F.}=e^{\int \tanh x dx}=e^{(\log\cosh x)}=\cosh x\,. \qquad \left[\because \int \tanh x dx=\log\cosh x\right]$$

Thus, the solution of equation (i) is $y\cosh x=2\int \cosh x\sin x.\cosh x dx+c$

$$\Rightarrow y\cosh x=2\int \cosh^2 x\sinh x dx+c$$

$$\Rightarrow y\cosh x=2\left(\frac{\cosh^3 x}{3}\right)+c \qquad \left[\because \int f^n(x).f'(x)dx=\frac{f^{n+1}(x)}{n+1}\right]$$

$$\therefore y\text{coxhx}=\frac{2}{3}\cosh^3 x+c\,,$$

which is the required solution of given differential equation.

Q.No.7.: Solve the differential equation $\dfrac{dy}{dx}-\dfrac{2y}{x+1}=(x+1)^3$.

Sol.: Given differential equation is $\dfrac{dy}{dx}-\dfrac{2y}{x+1}=(x+1)^3$, (i)

which is Leibnitz's linear equation in y.

$$\therefore \text{ I.F. }=e^{\int -\frac{2}{x+1}dx}=e^{-2\log(x+1)}=e^{\log(x+1)^{-2}}=\frac{1}{(x+1)^2}\,.$$

Thus, the solution of equation (i) is $y(\text{I.F.})=\int Q(\text{I..F.})dx+c$

$$\Rightarrow \frac{y}{(x+1)^2}=\int (x+1)^3.\frac{1}{(x+1)^2}dx+c=\int (x+1)dx+c$$

$$\Rightarrow \frac{y}{(x+1)^2}=\frac{x^2}{2}+x+c\,,$$

which is the required solution of given differential equation.

Q.No.8.: Solve the differential equation $\dfrac{dy}{dx}=-\dfrac{x+y\cos x}{1+\sin x}$.

Sol.: Given differential equation is $\frac{dy}{dx} = -\frac{x + y\cos x}{1+\sin x}$

$$\Rightarrow \frac{dy}{dx} + \frac{y\cos x}{1+\sin x} = \frac{-x}{1+\sin x}, \qquad \text{(i)}$$

which is Leibnitz's linear equation in y.

$\therefore$ I.F. $= e^{\int\left(\frac{\cos x}{1+\sin x}\right)dx} = e^{\log(1+\sin x)}$.

Thus, the solution of equation (i) is $y(\text{I.F.}) = \int Q(\text{I..F.})dx + c$

$\Rightarrow y(1+\sin x) = -\int -x dx + c' \Rightarrow 2y(1+\sin x) = -x^2 + c'$

$\Rightarrow y(1+\sin x) = c - \frac{x^2}{2}, \qquad \left(\text{where } c' = \frac{c}{2}\right)$

which is the required solution of given differential equation.

Q.No.9.: Solve the differential equation $\left[\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right]\frac{dx}{dy} = 1$

Sol.: Given differential equation is $\left[\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right]\frac{dx}{dy} = 1$

$$\Rightarrow \frac{dy}{dx} + \frac{y}{\sqrt{x}} = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}, \qquad \text{(i)}$$

which is Leibnitz's linear equation in y.

$\therefore$ I.F. $e^{\int\frac{1}{\sqrt{x}}dx} = e^{2\sqrt{x}}$.

Thus, the solution of equation (i) is $y(\text{I.F.}) = \int Q(\text{I..F.})dx + c$

$\Rightarrow ye^{2\sqrt{x}} = \int \frac{e^{-2\sqrt{x}}}{\sqrt{x}}.e^{2\sqrt{x}}dx + c \Rightarrow ye^{2x.\frac{1}{2}} = 2\sqrt{x} + c$

$ye^{2\sqrt{x}} = 2\sqrt{x} + c$,

which is the required solution of given differential equation.

Q.No.10.: Solve the differential equation $(x + 2y^3)\frac{dy}{dx} = y$.

Sol.: Given differential equation is $\left(x+2y^3\right)\frac{dy}{dx}=y$

$$\Rightarrow \frac{dx}{dy}=\frac{y}{x+2y^3}\Rightarrow \frac{dx}{dy}=\frac{x}{y}+2y^2\Rightarrow \frac{dx}{dy}-\frac{x}{y}=2y^2, \qquad \text{(i)}$$

which is Leibnitz's linear equation in x.

$$\therefore \text{ I.F. } = e^{\int -\frac{1}{y}dy}.e^{-\log y}=\frac{1}{y}.$$

Thus, the solution of equation (i) is $x(\text{I.F.})=\int Q(\text{I..F.})dy+c$

$$\Rightarrow x\frac{1}{y}=\int 2y^2.\frac{1}{y}dy+c\Rightarrow \frac{x}{y}=\int 2ydy+c\Rightarrow \frac{x}{y}=2\frac{y^2}{2}+c$$

$$\Rightarrow x=y^3+cy,$$

which is the required solution of given differential equation.

Q.No.11.: Solve the differential equation $\sqrt{1-y^2}dx=\left(\sin^{-1}y-x\right)dy$.

Sol.: Given differential equation is $\sqrt{1-y^2}dx=\left(\sin^{-1}y-x\right)dy$

$$\Rightarrow \sqrt{1-y^2}\frac{dx}{dy}+x=\sin^{-1}y\Rightarrow \frac{dx}{dy}+\frac{x}{\sqrt{1-y^2}}=\frac{\sin^{-1}y}{\sqrt{1-y^2}}, \qquad \text{(i)}$$

which is Leibnitz's linear equation in x.

$$\therefore \text{ I.F.} = e^{\int \frac{dy}{\sqrt{1-y^2}}}=e^{\sin^{-1}y}.$$

Thus, the solution of equation (i) is $x(\text{I.F.})=\int Q(\text{I..F.})dy+c$

$$\Rightarrow e^{\sin^{-1}y}.x=\int \frac{\sin^{-1}y}{\sqrt{1-y^2}}e^{\sin^{-1}y}dy+c$$

Put $\sin^{-1}y=t\Rightarrow \frac{dy}{\sqrt{1-y^2}}=dt$

$$\therefore e^t x=\int te^t dt+c\Rightarrow e^t x=te^t-e^t+c\Rightarrow e^t x=e^t(t-1)+c\Rightarrow x=(t-1)+ce^{-t}$$

$$\Rightarrow x=\sin^{-1}y+ce^{-\sin^{-1}y},$$

which is the required solution of given differential equation.

Q.No.12.: Solve the differential equation $y.e^y dx = \left(y^3 + 2xe^y\right)dy$

Sol.: Given differential equation is $y.e^y dx = \left(y^3 + 2xe^y\right)dy \Rightarrow \frac{dx}{dy} - \frac{2x}{y} = \frac{y^2}{e^y}$, (i)

which is Leibnitz's linear equation in x.

$\therefore$ I.F. $= e^{\int -\frac{2}{y}dy} = e^{-2\int \frac{1}{y}dy} = e^{-2\log y} = e^{\log y^{-2}} = \frac{1}{y^2}$.

Thus, the solution of equation (i) is $x(\text{I.F.}) = \int Q(\text{I..F.})dy + c$

$$\Rightarrow x.\frac{1}{y^2} = \int \frac{y^2}{e^y}.\frac{1}{y^2}dy + c \Rightarrow \frac{x}{y^2} = -e^{-y} + c$$

$$\Rightarrow xy^{-2} = c - e^{-y},$$

which is the required solution of given differential equation.

Q.No.13.: Solve the differential equation $y\log y dx + (x - \log y)dy = 0$.

Sol.: Given differential equation is $y\log y dx + (x - \log y)dy = 0$

$$\Rightarrow \frac{dx}{dy} = \frac{\log y}{y\log y} - \frac{x}{y\log y} \Rightarrow \frac{dx}{dy} + \frac{x}{y\log y} = \frac{1}{y}, \quad \text{(i)}$$

which is Leibnitz's linear equation in x.

$\therefore$ I.F. $= e^{\int \frac{dy}{y\log y}} = e^{\int \frac{1/y}{\log y}dy} = e^{\log(\log y)} = \log y$.

Thus, the solution of equation (i) is $x(\text{I.F.}) = \int Q(\text{I..F.})dy + c$

$$x\log y = \int \frac{1}{y} \times \log y dy + c. \quad \text{(ii)}$$

Put $\log y = t \Rightarrow \frac{i}{y}dy = dt$, then (ii) becomes

$$\Rightarrow x\log y = \int t dt + c \Rightarrow x\log y = \frac{t^2}{2} + c \Rightarrow x\log y = \frac{(\log y)^2}{2} + c$$

$$\Rightarrow x = \frac{\log y}{2} + c(\log y)^{-1},$$

which is the required solution of given differential equation.

Q.No.14.: Solve the differential equation $\left(1+y^2\right)dx+\left(x-e^{-\tan^{-1}y}\right)dy=0$.

Sol.: Given differential equation is $\left(1+y^2\right)dx+\left(x-e^{-\tan^{-1}y}\right)dy=0$

$$\Rightarrow \left(1+y^2\right)\frac{dx}{dy}+\left(x-e^{-\tan^{-1}y}\right)=0.$$

Dividing throughout by $\left(1+y^2\right)$, we get $\dfrac{dx}{dy}+\dfrac{x-e^{-\tan^{-1}y}}{1+y^2}=0$

$$\Rightarrow \frac{dx}{dy}+\frac{x}{1+y^2}-\frac{e^{-\tan^{-1}y}}{1+y^2}=0 \Rightarrow \frac{dx}{dy}+\frac{x}{1+y^2}=\frac{e^{-\tan^{-1}y}}{1+y^2}.$$

which is Leibnitz's linear equation in x.

$$\therefore \text{I.F.}=e^{\int Pdy}=e^{\int \frac{1}{1+y^2}dy}=e^{\tan^{-1}y}.$$

Hence, solution is $\quad x(\text{I.F.})=\int Q(\text{I..F.})dy+c$

$$\Rightarrow xe^{\tan^{-1}y}=\int \frac{1}{\left(1+y^2\right)}dy+c \Rightarrow xe^{\tan^{-1}y}=\tan^{-1}y+c,$$

which is the required solution of given differential equation.

Q.No.15.: Solve the differential equation $x\left(1-x^2\right)\dfrac{dy}{dx}+\left(2x^2-1\right)y=x^3$.

Sol.: Given equation is $x\left(1-x^2\right)\dfrac{dy}{dx}+\left(2x^2-1\right)y=x^3$.

Dividing by $x\left(1-x^2\right)$ to make the co-efficient of $\dfrac{dy}{dx}$ unity, the given equation becomes

$$\frac{dy}{dx}+\frac{2x^2-1}{x\left(1-x^2\right)}y=\frac{x^2}{1-x^2}, \qquad \text{(i)}$$

which is Leibnitz's linear equation in y.

Comparing it with $\dfrac{dy}{dx}+Py=Q$, we have $P=\dfrac{2x^2-1}{x\left(1-x^2\right)}$, $Q=\dfrac{x^2}{1-x^2}$.

Now $P=\dfrac{2x^2-1}{x(1-x)(1+x)}=-\dfrac{1}{x}+\dfrac{1}{2(1-x)}+\dfrac{1}{2(1+x)}$.

$$\int Pdx = -\log x - \frac{1}{2}\log(1-x) - \frac{1}{2}\log(1+x) = -\log\left[x(1-x)^{1/2}(1+x)^{1/2}\right]$$

$$= -\log\left[x\left(1-x^2\right)^{1/2}\right] = \log\frac{1}{x\sqrt{1-x^2}}.$$

$$\text{I.F.} = e^{\int Pdx} = e^{\log\frac{1}{x\sqrt{1-x^2}}} = \frac{1}{x\sqrt{1-x^2}}.$$

Thus, the solution of equation (i) is $y(\text{I.F.}) = \int Q(\text{I..F.})dx + c$

$$\Rightarrow y.\frac{1}{x\left(1-x^2\right)} = \int \frac{x^2}{1-x^2} \times \frac{1}{x\sqrt{1-x^2}}dx + c = -\frac{1}{2}\int \left(1-x^2\right)^{-3/2}(-2x)dx + c$$

$$= \left(1-x^2\right)^{-1/2} + c$$

$$\Rightarrow y = x + cx\sqrt{1-x^2},$$

which is the required solution of given differential equation.

Q.No.16.: Solve the differential equation $y' = 4y + 2x - 4x^2$.

Sol.: Given equation is $y' = 4y + 2x - 4x^2$.

$$\Rightarrow y' - 4y = 2x - 4x^2,$$

which is Leibnitz's linear equation in y.

Comparing it with $\frac{dy}{dx} + Py = Q$, we have $P(x) = -4$, $Q(x) = 2x - 4x^2$.

$$\text{I.F.} = e^{\int Pdx} = e^{\int -4dx} = e^{-4x}.$$

Thus, the solution of equation (i) is $y(\text{I.F.}) = \int Q(\text{I..F.})dx + c$

$$ye^{-4x} = \int \left(2x - 4x^2\right)e^{-4x}dx = 2\int xe^{-4x}dx + \int x^2 d\left(e^{-4x}\right)$$

$$\Rightarrow ye^{-4x} = \int 2xe^{-4x}dx + x^2e^{-4x} - \int e^{-4x}.2xdx + c$$

$$\Rightarrow ye^{-4x} = x^2e^{-4x} + c$$

$$\Rightarrow y = x^2 + ce^{4x},$$

which is the required solution of given differential equation.

Q.No.17.: Solve the differential equation $xy' + 2y - x\sin x = 0$.

Sol.: Given equation is $xy' + 2y - x\sin x = 0 \Rightarrow y' + \frac{2y}{x} = \sin x$,

which is Leibnitz's linear equation in y.

Comparing it with $\frac{dy}{dx} + Py = Q$, we have $Q = \sin x$, $P = \frac{2}{x}$

$$\text{I.F.} = e^{\int Pdx} = e^{\int \frac{2}{x}dx} = e^{\log x^2} = x^2.$$

Thus, the solution of equation (i) is $y(\text{I.F.}) = \int Q(\text{I..F.})dx + c$

$$yx^2 = \int x^2 \sin x.dx = \int x^2 d(\cos x)$$

$$\Rightarrow yx^2 = -x^2\cos x + \int 2x.\cos x dx$$

$$yx^2 = -x^2\cos x + 2x.\sin x + 2\cos x + c,$$

which is the required solution of given differential equation.

Q.No.18.: Solve the differential equation $\{y(1 - x\tan x) + x^2\cos x\}dx - xdy = 0$.

Sol.: Given equation is $\{y(1 - x\tan x) + x^2\cos x\}dx - xdy = 0$.

$$\Rightarrow y' + \frac{(x\tan x - 1)}{x}y = x\cos x,$$

which is Leibnitz's linear equation in y.

Comparing it with $\frac{dy}{dx} + Py = Q$, we have $P(x) = \frac{x\tan x - 1}{x}$, $Q(x) = x\cos x$.

$$\text{I.F.} = \exp\left\{\int \left(\frac{x\tan x - 1}{x}\right)dx\right\} = \exp\{-\log\cos x - \log x\} = \frac{1}{x\cos x}$$

Thus, the solution of equation (i) is $y(\text{I.F.}) = \int Q(\text{I..F.})dx + c$

$$\frac{y}{x\cos x} = \int dx + c = x + c$$

$$\Rightarrow y = x^2\cos x + cx\cos x,$$

which is the required solution of given differential equation.

Q.No.19.: Solve the differential equation $\frac{dy}{dx} = \frac{1}{(1 + x^2)}\left(e^{\tan^{-1}x} - y\right)$.

Sol.: Given equation is $\frac{dy}{dx} = \frac{1}{\left(1+x^2\right)}\left(e^{\tan^{-1}x} - y\right)$.

$$\Rightarrow \frac{dy}{dx} + \frac{1}{1+x^2}y = \frac{e^{\tan^{-1}x}}{1+x^2},$$

which is Leibnitz's linear equation in y.

Comparing it with $\frac{dy}{dx} + Py = Q$, we have $P(x) = \frac{1}{1+x^2}$, $Q(x) = \frac{e^{\tan^{-1}x}}{1+x^2}$.

I.F. $= e^{\int P(x)dx} = e^{\int \frac{dx}{1+x^2}} = e^{\tan^{-1}x}$.

Thus, the solution of equation (i) is $y(\text{I.F.}) = \int Q(\text{I..F.})dx + c$

$$y.e^{\tan^{-1}x} = \int \frac{\left(e^{\tan^{-1}x}\right)^2}{\left(1+x^2\right)}dx + c = \int \left(e^{\tan^{-1}x}\right)^2 .d\left(\tan^{-1}x\right) + c$$

$$y.e^{\tan^{-1}x} = \frac{\left(e^{\tan^{-1}x}\right)^2}{2} + c,$$

which is the required solution of given differential equation.

Q.No.20.: Solve the differential equation $y^2dx + (3xy-1)dy = 0$.

Sol.: Given equation is $y^2dx + (3xy-1)dy = 0$

$$\Rightarrow \frac{dy}{dx} = \frac{y^2}{1-3xy}$$

This is not linear in y (because of the presence of the term y^2). This is also not exact, not homogeneous, nor separable. Instead if we swap the roles of x and y by treating x as dependent variable and y as the independent variable, the given equation can be rearranged as

$$\frac{dx}{dy} + \frac{3}{y}x = \frac{1}{y^2},$$

which is Leibnitz's linear equation in x.

Comparing it with $\frac{dy}{dx} + Py = Q$, we have $P(y) = \frac{3}{y}$ and $Q(y) = \frac{1}{y^2}$.

Now we get

I.F. $= e^{\int P(y)dy} = e^{\int \frac{3}{y}dy} = e^{3\log y} = e^{\log y^3} = y^3$.

Thus, the solution of equation (i) is $y(\text{I.F.}) = \int Q(\text{I.F.})dx + c$

$$xy^3 = \int ydy + c = \frac{y^2}{2} + c$$

$$\Rightarrow 2xy^3 - y^2 = 2c,$$

which is the required solution of given differential equation.

Q.No.21.: Solve the differential equation $\frac{dy}{dx} + \frac{y\log y}{x - \log y} = 0$.

Sol.: Given equation is $\frac{dy}{dx} + \frac{y\log y}{x - \log y} = 0$.

This is not linear in y because of the presence of the term y. It is neither separable nor homogeneous nor exact. But with x taken as dependent variable the given equation can be rearranged as

$$\frac{dx}{dy} + \frac{1}{y\log y}.x = \frac{1}{y},$$

which is Leibnitz's linear equation in x.

Comparing it with $\frac{dy}{dx} + Py = Q$, we have $P(y) = \frac{1}{y\log y}$ and $Q(y) = \frac{1}{y}$ so that

I.F.$= e^{\int P(y)dy} = \exp\left\{\int \frac{dy}{y\log y}\right\} = e^{\log(\log y)} = \log y$.

Thus, the solution of equation (i) is $x(\text{I.F.}) = \int Q(\text{I.F.})dy + c$

$$x.\log y = \int \frac{\log y}{y}dy = \int \log y d(\log y) = \frac{(\log y)^2}{2} + c$$

$$\Rightarrow 2x\log y = (\log y)^2 + 2c,$$

which is the required solution of given differential equation.

Q.No.22.: Solve the differential equation $e^{-y}\sec^2 ydy = dx + xdy$

Sol.: Given differential equation is $e^{-y}\sec^2 ydy = dx + xdy$

$\Rightarrow e^{-y}\sec^2 y = \frac{dx}{dy} + x \Rightarrow \frac{dx}{dy} + x = e^{-y}\sec^2 y$, (i)

which is Leibnitz's linear equation in x.

$\therefore$ I.F. $= e^{\int P dy} = e^{\int dy} = e^y$. (Here P = 1)

Thus, the solution of equation (i) is $x(\text{I.F.}) = \int Q(\text{I..F.})dy + c$

$xe^y = \int \sec^2 y.dy + c \quad \Rightarrow e^y.x = \tan y + c$,

which is the required solution of given differential equation.

Q.No.23.: Solve the differential equation $\frac{dr}{d\theta} + 2r\cot\theta + \sin 2\theta = 0$

Sol.: Given differential equation is $\frac{dr}{d\theta} + 2r\cot\theta + \sin 2\theta = 0 \Rightarrow \frac{dr}{d\theta} + 2r\cot\theta = -\sin 2\theta$,(i)

which is Leibnitz's linear equation in r.

$\therefore$ I.F. $= e^{\int 2\cot\theta d\theta} = e^{\int \frac{2\cos\theta}{\sin\theta}d\theta} = e^{\int \frac{2\cos\theta\sin\theta}{\sin^2\theta}d\theta} = e^{\log\sin^2\theta} = \sin^2\theta$.

Thus, the solution of equation (i) is $r(\text{I.F.}) = \int Q(\text{I..F.})d\theta + c$

$\Rightarrow r\sin^2\theta = \int -\sin 2\theta \times \sin^2\theta d\theta + c = -\int 2\sin^3\theta.\cos\theta d\theta + c$.

By Putting $\sin\theta = t \Rightarrow \cos\theta d\theta = dt$, we get

$r\sin^2\theta = -2\int t^3 dt + c \quad \Rightarrow r\sin^2\theta = -\frac{2t^4}{4} + c \quad \Rightarrow 2r\sin^2\theta = -\sin^4\theta + c'$

$\Rightarrow 2r\sin^2\theta + \sin^4\theta = c'$,

which is the required solution of given differential equation.

Q.No.24.: Solve the differential equation $r\sin\theta d\theta + \left(r^3 - 2r^2\cos\theta + \cos\theta\right)dr = 0$.

Sol.: Given differential equation is $r\sin\theta d\theta + \left(r^3 - 2r^2\cos\theta + \cos\theta\right)dr = 0$. (i)

Putting $\cos\theta = z$ so that $-\sin\theta d\theta = dz$.

Then, equation (i) becomes $-rdz + \left(r^3 - 2r^2 z + z\right)dr = 0$

$\Rightarrow \frac{dz}{dr} - \left(r^2 - 2rz + \frac{z}{r}\right) = 0 \quad \Rightarrow \frac{dz}{dr} + \left(2r - \frac{1}{r}\right)z = r^2$,

which is Leibnitz's linear equation in z.

$\therefore$ I.F. $= e^{\int\left(2r-\frac{1}{r}\right)dy} = e^{r^2-\log r} = \dfrac{e^{r^2}}{r}$.

Thus, the solution is $z(\text{I.F.}) = \int Q(\text{I..F.})dr + c'$

$$\Rightarrow \cos\theta\frac{e^{r^2}}{r} = \int r^2\frac{e^{r^2}}{r}dr + c' \Rightarrow \cos\theta\frac{e^{r^2}}{r} = \frac{1}{2}\int 2re^{r^2}dr + c'. \quad \text{(ii)}$$

Putting $r^2 = t$ so that $2rdr = dt$.

$\therefore$ Equation (ii) becomes $\cos\theta\dfrac{e^{r^2}}{r} = \dfrac{1}{2}\int e^t dt + c'$

$$\Rightarrow \cos\theta\frac{e^{r^2}}{r} = \frac{1}{2}e^t + c' \Rightarrow \cos\theta\frac{e^{r^2}}{r} = \frac{1}{2}e^{r^2} + c'$$

$$\Rightarrow 2\cos\theta = r\left[1 + ce^{-r^2}\right],$$

which is the required solution of this differential equation.

INITIAL VALUE PROBLEM:

A differential equation together with an initial condition is called an initial value problem. In this type of problems, we can determine the value of the arbitrary constant in the general solution by using initial condition.

Q.No.1.: Solve the differential equation $\dfrac{dy}{dx} + y\cot x = 4x\,\mathrm{cosec}\,x$, if $y = 0$ when $x = \dfrac{\pi}{2}$.

Sol.: Given differential equation is $\dfrac{dy}{dx} + y\cot x = 4x\,\mathrm{cosec}\,x$, (i)

which is Leibnitz's equation in y.

$\therefore$ I.F. $= e^{\int \cot x dx} = e^{\log\sin x} = \sin x$.

Thus, the solution of equation (i) is $y(\text{I.F.}) = \int Q(\text{I..F.})dx + c$

$$\Rightarrow y\sin x = \int 4x\,\mathrm{cosec}\,x\sin x dx \Rightarrow y\sin x = \frac{4x^2}{2} + c \Rightarrow y\sin x = 2x^2 + c. \quad \text{(ii)}$$

Given when $x = \dfrac{\pi}{2} \Rightarrow y = 0$, we have

$$0 \times \sin\frac{\pi}{2} = \frac{2\pi^2}{4} - c \Rightarrow c = -\frac{\pi^2}{2}.$$

$\therefore$ (ii) becomes $\quad y\sin x = 2x^2 - \dfrac{\pi^2}{2}$,

which is the required solution of given differential equation.

Q.No.2.: Solve the differential equation $x(1-4y)dx - (x^2+1)dy = 0$ with $y(2) = 1$.

Sol.: Given equation is $x(1-4y)dx - (x^2+1)dy = 0$.

$$\Rightarrow y' + \frac{4x}{x^2+1}.y = \frac{x}{x^2+1},$$

which is Leibnitz's linear equation in y.

Comparing it with $\dfrac{dy}{dx} + Py = Q$, we have $P(x) = \dfrac{4x}{x^2+1}$ and $Q(x) = \dfrac{x}{x^2+1}$

$$\text{I.F.} = e^{\int Pdx} = e^{\int \frac{4x}{x^2+1}dx} = e^{2\log(x^2+1)} = (x^2+1)^2.$$

Thus, the solution of equation (i) is $\quad y(\text{I.F.}) = \int Q(\text{I..F.})dx + c$

$$y.(x^2+1)^2 = \int (x^3+x)dx + c = \frac{x^4}{4} + \frac{x^2}{2} + c$$

$$4y(x^2+1)^2 = x^4 + 2x^2 + 4c,$$

which is the required solution of given differential equation.

Since y(2) = 1, Put x = 2 and y = 1

$$4.25 = 16 + 8 + 4c \Rightarrow c = 19$$

The particular solution is $4y(x^2+1)^2 = x^4 + 2x^2 + 76$.

Home Assignments

Q.No.1.: Solve the differential equation $\dfrac{dy}{dx} + \dfrac{y}{x} = x^3 - 3$.

Ans.: $10xy = 2x^5 - 15x^2 + c$.

Q.No.2.: Solve the differential equation $(x+1)\dfrac{dy}{dx} - y = e^x(x+1)^2$.

Ans.: $y = (x+1)(e^x + c)$.

Q.No.3.: Solve the differential equation $(x^2+1)\dfrac{dy}{dx} + 2xy = x^2$.

Ans.: $y(x^2+1) = \dfrac{x^3}{3} + c$.

Q.No.4.: Solve the differential equation $(1+x^3)\dfrac{dy}{dx} + 6x^2y = 1 + x^2$.

Ans.: $y(1+x^3)^2 = x + \dfrac{x^3}{3} + \dfrac{x^4}{4} + \dfrac{x^6}{6} + c$.

Q.No.5.: Solve the differential equation $\dfrac{dy}{dx} + y\cot x = \cos x$.

Ans.: $y\sin x = \dfrac{1}{2}\sin^2 x + c$.

Q.No.6.: Solve the differential equation $(1+x^2)\dfrac{dy}{dx} + y = e^{\tan^{-1}x}$.

Ans.: $2ye^{\tan^{-1}x} = e^{2\tan^{-1}x} + c$.

Q.No.7.: Solve the differential equation $(1-x^2)\dfrac{dy}{dx} + 2xy = x\sqrt{1-x^2}$.

Ans.: $y = \sqrt{(1-x^2)} + c(1-x^2)$.

Q.No.8.: Solve the differential equation $x\dfrac{dy}{dx} + y = e^x - xy$.

Ans.: $xy = \dfrac{1}{2}e^x + ce^{-x}$.

Q.No.9.: Solve the differential equation $\dfrac{dy}{dx} = \dfrac{y}{2y\log y + y - x}$.

Ans.: $x = \dfrac{c}{y} + y\log y$.

Q.No.10.: Solve the differential equation $2(y-4x^2)dx + xdy = 0$.

Ans.: $x^2y = 2x^4 + c$.

Q.No.11.: Solve the differential equation $y' + y\cot x = 2x\,\mathrm{cosec}\,x$.

Ans.: $y = (x^2+c)\,\mathrm{cosec}\,x$.

Q.No.12.: Solve the differential equation $y' + y = \frac{1}{1+e^{2x}}$.

Ans.: $y = e^{-x}.\tan^{-1} e^{x} + ce^{-x}$.

Q.No.13.: Solve the differential equation $\left(1+x^2\right)dy + 2xydx = \cot xdx$.

Ans.: $y = \frac{\log(\sin x) + c}{\left(1+x^2\right)}$.

Q.No.14.: Solve the differential equation $y' + 2y = e^{x}(3\sin 2x + 2\cos 2x)$.

Ans.: $y = ce^{-2x} + e^{x}.\sin 2x$

Q.No.15.: Solve the differential equation $y' + y = e^{e^{x}}$.

Ans.: $ye^{x} = e^{e^{x}} + c$.

Q.No.16.: Solve the differential equation $\left[y(x+1)^2 e^{3x}\right]dx - (x+1)dy = 0$.

Ans.: $y = \left(\frac{1}{3}e^{3x} + c\right)(x+1)$.

Q.No.17.: Solve the differential equation $dx + (3y - x)dy = 0$.

Ans.: $x - 3y - 3 = ce^{y}$.

Q.No.18.: Solve the differential equation $ydx + (3x - xy + 2)dy = 0$.

Ans.: $xy^3 = 2y^2 + 4y + 4 + ce^{y}$.

Q.No.19.: Solve the differential equation $\left(1+y^2\right)dx + \left(x - \tan^{-1} y\right)dy = 0$.

Ans.: $x = \tan^{-1} y - 1 + ce^{-\tan^{-1} y}$.

Q.No.20.: Solve the differential equation $y^2dx + \left(xy - 2y^2 - 1\right)dy = 0$.

Ans.: $xy = y^2 + \log y + c$.

Q.No.21.: Solve the differential equation $dx - (x + y + 1)dy = 0$.

Ans.: $x = -(y+2) + ce^{y}$.

Q.No.22.: Solve the differential equation $ydx - \left(x + 2y^3\right)dy = 0$.

Ans.: $x = y^3 + cy$.

INITIAL VALUE PROBLEM:

Q.No.1.: Solve the differential equation

$$2y'\cos x + 4y\sin x = \sin 2x, \text{ given } y = 0 \text{ when } x = \frac{\pi}{3}.$$

Ans.: $y\sec^2 x = \sec x - 2$.

Q.No.2.: Solve the differential equation $\frac{dy}{dx} + y\cot x = 5e^{\cos x}$, if $y = -4$ when $x = \frac{\pi}{2}$.

Ans.: $y\sin x = 1 - 5e^{\cos x}$.

Q.No.3.: Solve the differential equation $\frac{dy}{dx} - y\tan x = 3e^{-\sin x}$, if $y = 4$ when $x = 0$.

Ans.: $y\cos x = 7 - 3e^{-\sin x}$

Q.No.4.: Solve the differential equation $y' + y\tan x = \sin 2x$, $y(0) = 1$.

Ans.: $y = 3\cos x - 2\cos^2 x$.

Q.No.5.: Solve the differential equation $\frac{dI}{dt} + 2I = 10e^{-2t}$, $I = 0$ when $t = 0$.

Ans.: $I = 10te^{2t}$.

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4^{th} Topic

Differential Equations of First order

Part: II

"Non-Linear Differential Equations"
(Bernoulli's Equation)
(Reducible to Linear Differential Equation)

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Equation reducible to the linear equation (Bernoulli's Equations)

Bernoulli's Equation:

The equation $\boxed{\dfrac{dy}{dx} + Py = Qy^n}$ (i)

where P, Q are functions of x only or constants, reducible to the Leibnitz's linear equation and is usually called the Bernoulli's equation.

To solve (i), dividing both sides by y^n, we have $y^{-n}\dfrac{dy}{dx} + Py^{1-n} = Q$. (ii)

Putting $y^{1-n} = z$ so that $(1-n)y^{-n}\dfrac{dy}{dx} = \dfrac{dz}{dx} \Rightarrow y^{-n}\dfrac{dy}{dx} = \dfrac{1}{1-n}\dfrac{dz}{dx}$

$\therefore$ Equation (ii) becomes $\dfrac{1}{1-n}\dfrac{dz}{dx} + Pz = Q \Rightarrow \dfrac{dz}{dx} + P(1-n)z = Q(1-n)$.

which is the Leibnitz's linear differential equation in z as the dependent variable.

Remarks:

General equation reducible to Leibnitz's linear form is

$$f'(y)\frac{dy}{dx}+Pf(y)=Q \qquad \text{(i)}$$

where P and Q are function of x only or constants.

Putting f(y) = z so that $f'(y)\frac{dy}{dx}=\frac{dz}{dx}$

Then equation (i) becomes $\frac{dz}{dx}+Pz=Q$, which is Leibnitz's linear equation form.

Now let us solve some differential equations which are reducible to the linear equation:

Q.No.1.: Solve $x\frac{dy}{dx}+y=x^3y^6$.

Sol.: The given equation is $x\frac{dy}{dx}+y=x^3y^6$

Dividing throughout by xy^6, we get $y^{-6}\frac{dy}{dx}+\frac{y^{-5}}{x}=x^2$. (i)

Putting $y^{-5}=z$, so that $-5y^{-6}\frac{dy}{dx}=\frac{dz}{dx}$

$\therefore$ (i) becomes $\frac{-1}{5}\frac{dz}{dx}+\frac{z}{x}=x^2 \Rightarrow \frac{dz}{dx}-\frac{5}{x}z=-5x^2$, (ii)

which is Leibnitz's linear equation in z.

$\therefore$ I. F. $=e^{-\int\left(\frac{5}{x}\right)dx}=e^{-5\log x}=e^{\log x^{-5}}=x^{-5}$

$\therefore$ The solution of (ii) is $z\,(\text{I. F.})=\int\left(-5x^2\right)(\text{I. F.})dx+c$

$\Rightarrow zx^{-5}=\int\left(-5x^2\right)x^{-5}dx+c \quad \Rightarrow y^{-5}x^{-5}=-5.\frac{x^{-2}}{-2}+c \quad \left[\because z=y^{-5}\right]$

Dividing throughout by $y^{-5}x^{-5}$, we get

$1=\left(2.5+cx^2\right)x^3y^5$, Ans. which is the required solution.

Q.No.2.: Solve $xy\left(1+xy^2\right)\frac{dy}{dx}=1$.

Sol.: The given equation is $xy\left(1+xy^2\right)\dfrac{dy}{dx}=1 \Rightarrow \dfrac{dx}{dy}-yx=y^3x^2$.

Dividing throughout by x^2, we get $x^{-2}\dfrac{dx}{dy}-yx^{-1}=y^3$. (i)

Putting $x^{-1}=z$ so that $x^{-2}\dfrac{dx}{dy}=-\dfrac{dz}{dy}$,

Then, equation (i) becomes $\dfrac{dz}{dy}+yz=-y^3$, (ii)

which is Leibnitz's linear equation in z.

$\therefore$ I. F. $=e^{\int y\,dy}=e^{y^2/2}$.

$\therefore$ The solution of (ii) is $z\,(\text{I. F.})=\int\left(-y^3\right)(\text{I. F.})dy+c$

$$\Rightarrow ze^{y^2/2}=-\int y^2.e^{\frac{1}{2}y^2}.ydy+c \qquad \left[\text{Put } \frac{1}{2}y^2=t, \text{so that } ydy=dt\right]$$

$$=-2\int t.e^t dt+c=-2\left[t.e^t-\int 1.e^t dt\right]+c$$

$$=-2\left[te^t-e^t\right]+c=\left(2-y^2\right)e^{y^2/2}+c$$

$$\Rightarrow z=\left(2-y^2\right)+ce^{-\frac{1}{2}y^2} \Rightarrow \frac{1}{x}=\left(2-y^2\right)+ce^{-\frac{1}{2}y^2}. \text{ Ans.}$$

which is the required solution.

Q.No.3.: Solve $\dfrac{dy}{dx}+x\sin 2y=x^3\cos^2 y$.

Sol.: The given equation is $\dfrac{dy}{dx}+x\sin 2y=x^3\cos^2 y$

Dividing throughout by $\cos^2 y$, we get $\sec^2 y\dfrac{dy}{dx}+2x\dfrac{\sin y\cos y}{\cos^2 y}=x^3$

$$\Rightarrow \sec^2 y\frac{dy}{dx}+2x\tan y=x^3 \qquad \text{(i)}$$

which is of the form $f'(y)\dfrac{dy}{dx}+Pf(y)=Q$.

$\therefore$ Putting $\tan y=z$, so that $\sec^2 y\dfrac{dy}{dx}=\dfrac{dz}{dx}$

Then equation (i) becomes $\frac{dz}{dx} + 2xz = x^3$ (ii)

which is Leibnitz's linear equation in z.

$\therefore \text{I. F.} = e^{\int 2x dx} = e^{x^2}$

$\therefore$ The solution of (ii) is $z(\text{I.F.}) = \int .Q(\text{I.F.})dx + c$

$$\Rightarrow ze^{x^2} = \int e^{x^2}.x^3 dx + c = \frac{1}{2}\left(x^2 - 1\right)e^{x^2} + c \quad \Rightarrow z = \frac{1}{2}\left(x^2 - 1\right) + ce^{-x^2}$$

$\tan y = \frac{1}{2}\left(x^2 - 1\right) + ce^{-x^2}$, Ans.

which is the required solution.

Q.No.:4.: Solve $\frac{dz}{dx} + \left(\frac{z}{x}\right)\log z = \frac{z}{x}(\log z)^2$.

Sol.: The given equation is $\frac{dz}{dx} + \left(\frac{z}{x}\right)\log z = \frac{z}{x}(\log z)^2$

Dividing by z, we get $\frac{1}{z}\frac{dz}{dx} + \frac{1}{x}\log z = \frac{1}{x}(\log z)^2$ (i)

which is of the form $f'(y)\frac{dy}{dx} + Pf(y) = Q$.

Putting $\log z = t$ so that $\frac{1}{z}\frac{dz}{dx} = \frac{dt}{dx}$.

$\therefore$ Equation (i) becomes $\frac{dt}{dx} + \frac{t}{x} = \frac{t^2}{x} \Rightarrow \frac{1}{t^2}\frac{dt}{dx} + \frac{1}{x}.\frac{1}{t} = \frac{1}{x}$ (ii)

which is Bernoulli's equation in t.

Putting $\frac{1}{t} = v \Rightarrow -\frac{1}{t^2}\frac{dt}{dx} = \frac{dv}{dx} \Rightarrow \frac{1}{t^2}\frac{dt}{dx} = -\frac{dv}{dx}$

$\therefore$ Equation (ii) reduces to $-\frac{dv}{dx} + \frac{v}{x} = \frac{1}{x} \Rightarrow \frac{dv}{dx} - \frac{1}{x}v = -\frac{1}{x}$ (ii)

which is Leibnitz's linear equation in v.

$\therefore \text{I. F.} = e^{-\int \frac{1}{x}dx} = \frac{1}{x}$

$\therefore$ The solution of (ii) is $v(\text{I.F.}) = \int .Q(\text{I.F.})dx + c$

$$v.\frac{1}{x} = -\int \frac{1}{x}.\frac{1}{x}dx + c = \frac{1}{x} + c$$

Replacing v by $\frac{1}{\log z}$, we get

$(x \log z)^{-1} = x^{-1} + c \Rightarrow (\log z)^{-1} = 1 + cx$, Ans. which is the required solution.

Q.No.5.: Solve the equation $\frac{dy}{dx} + y\tan x = y^3 \sec x$.

Sol.: The given equation is $\frac{dy}{dx} + y\tan x = y^3 \sec x$

which is Bernoulli's equation in y.

$\therefore$ Dividing throughout by y^3 on both sides, we get

$$y^{-3}\frac{dy}{dx} + y^{-2}\tan x = \sec x \qquad \text{(i)}$$

Putting $y^{-2} = u \quad \therefore -2y^{-3}dy = du \Rightarrow y^{-3}dy = -\frac{du}{2}$

Then equation (i) becomes $-\frac{1}{2}\frac{du}{dx} + 2\tan x = \sec x$

$$\Rightarrow \frac{du}{dx} - 2\tan x = -2\sec x, \qquad \text{(ii)}$$

which is Leibnitz's linear equation in u.

$\therefore$ I. F. $= e^{\int -2\tan x dx} = e^{2\log\cos x} = \cos^2 x$

$\therefore$ The solution of (ii) is $u(\text{I.F.}) = \int .Q(\text{I.F.})dx + c$

$u\cos^2 x = -2\int \cos x dx + c \quad \Rightarrow u\cos^2 x = 2\sin x + c$

$\Rightarrow u = 2\tan x \sec x + c\sec^2 x \quad \Rightarrow y^{-2} = 2\tan x\sec x + c\sec^2 x$

$\Rightarrow \cos^2 x = y^2(c + 2\sin x)$, Ans. which is the required solution.

Q.No.6.: Solve the equation $r\sin\theta - \cos\theta\frac{dr}{d\theta} = r^2$.

Sol.: The given equation is $r\sin\theta - \cos\theta\frac{dr}{d\theta} = r^2 \Rightarrow -\frac{dr}{d\theta} + r\tan\theta = r^2\sec\theta \qquad \text{(i)}$

which is Bernoulli's equation in r.

Dividing throughout by $r^2 \cos\theta$, we get

$$\Rightarrow -\frac{1}{r^2}\frac{dr}{d\theta} + \frac{1}{r}\tan\theta = \sec\theta \qquad \text{(ii)}$$

Putting $\frac{1}{r} = u \Rightarrow -\frac{1}{r^2}\frac{dr}{d\theta} = \frac{du}{d\theta}$

$\therefore$ Equation (ii) becomes $\frac{du}{d\theta} + u\tan\theta = \sec\theta$

which is Leibnitz's linear equation in u.

Hence $P = \tan\theta$ and $Q = \sec\theta$

$\therefore$ I. F. $= e^{\int \tan\theta d\theta} = e^{-\log(\cos\theta)} = e^{\log(\cos\theta)^{-1}} = \sec\theta$

$\therefore$ The solution of (ii) is $u(\text{I. F.}) = \int Q(\text{I. F.})d\theta + c$

$$\Rightarrow u\sec\theta = \int \sec\theta.\sec\theta d\theta + c \Rightarrow u\sec\theta = \tan\theta + c$$

$$\Rightarrow \frac{u}{\cos\theta} = \frac{\sin\theta}{\cos\theta} + c \qquad \Rightarrow \frac{1}{r} = \sin\theta + c\cos\theta,$$ Ans. which is the required solution.

Q.No.7.: Solve the equation $2xy' = 10x^3y^5 + y$.

Sol.: The given equation is $2x\frac{dy}{dx} = 10x^3y^5 + y$

$$\Rightarrow \frac{dy}{dx} = 5x^2y^5 + \frac{y}{2x} \qquad \Rightarrow \frac{dy}{dx} - \frac{y}{2x} = 5x^2y^5 \qquad \text{(i)}$$

which is Bernoulli's equation.

Dividing both sides by y^5, we get

$$y^{-5}\frac{dy}{dx} - \frac{y^{-4}}{2x} = 5x^2 \qquad \text{(ii)}$$

Putting $y^{-4} = t \Rightarrow -4y^{-5}\frac{dy}{dt} = \frac{dt}{dx} \Rightarrow y^{-5}\frac{dy}{dx} = -\frac{1}{4}\frac{dt}{dx}$

$\therefore$ Equation (ii) becomes $-\frac{1}{4}\frac{dt}{dx} - \frac{t}{2x} = 5x^2 \quad \Rightarrow \frac{dt}{dx} + \frac{2t}{x} = -20x^2$

which is Leibnitz's linear equation in t.

$\therefore$ I. F. $= e^{\int \frac{2}{x}dx} = x^2$

$\therefore$ The solution is $t(\text{I.F.}) = \int .Q(\text{I.F.})dx + c$

$x^2 t = \int -20x^4 dx + c \Rightarrow x^2 t = -4x^5 + c \qquad \Rightarrow x^2 y^{-4} = -4x^2 + c$

$\Rightarrow x^2 = -(4x^2 - c)y^4 \quad \Rightarrow x^2 + (4x^2 + c')y^4 = 0$. Ans. $\qquad$ [where $c' = c$]

which is the required solution.

Q.No.8.: Solve the equation $(x^3y^2 + xy)dx = dy$.

Sol.: The given equation is $(x^3y^2 + xy)dx = dy \Rightarrow \dfrac{dy}{dx} - xy = x^3y^2 \qquad$ (i)

which is Bernoulli's equation in y.

Dividing throughout by y^2, we get

$$y^{-2}\frac{dy}{dx} - y^{-1}x = x^3$$

Putting $y^{-1} = u \Rightarrow -y^{-2}dy = du \Rightarrow y^{-2}dy = -du \Rightarrow -\dfrac{du}{dx} - ux = x^3$

$$\frac{du}{dx} + xu = -x^3, \qquad \text{(ii)}$$

which is Leibnitz's linear equation in u.

$\therefore$ I. F. $= e^{\int x dx} = e^{x^2/2}$

Hence the solution is $\quad u(\text{I.F.}) = \int (-x^3)(\text{I.F.})dx + c$

$$\Rightarrow u\left(e^{x^2/2}\right) = \int (-x^3)\left(e^{x^2/2}\right)dx + c = -2\int x.\frac{x^2}{2}.e^{x^2/2}dx + c$$

$$= -2\int v.e^v du + c = -2\left[ve^v - \int e^v dv\right] + c$$

$$= -2[v-1]e^v + c. \quad \text{Here put } \frac{x^2}{2} = v \Rightarrow xdx = dv$$

By putting $u = \dfrac{1}{y}$, $\; v = \dfrac{x^2}{2}$, we get

$\dfrac{1}{y} = -2\left[\dfrac{x^2}{2} - 1\right] + ce^{-\frac{x^2}{2}} \Rightarrow \dfrac{1}{y} = 2 - x^2 + ce^{-x^2/2}$, Ans. which is the required solution.

Q.No.9.: Solve the equation $\frac{dy}{dx}=\frac{x^2+y^2+1}{2xy}$.

Sol.: The given equation is $\frac{dy}{dx}=\frac{x^2+y^2+1}{2xy} \Rightarrow \frac{dy}{dx}-\frac{1}{2x}y=\left(\frac{x^2+1}{2x}\right)y^{-1}$, (i)

which is Bernoulli's equation in y.

Dividing throughout by y^{-1}, we get

$$y\frac{dy}{dx}-\frac{1}{2x}y^2=\left(\frac{x^2+1}{2x}\right). \quad \text{(ii)}$$

Put $y^2=z \Rightarrow 2y\frac{dy}{dx}=\frac{dz}{dx} \Rightarrow y.\frac{dy}{dx}=\frac{1}{2}\frac{dz}{dx}$

$\therefore$ Equation (ii) becomes $\frac{1}{2}.\frac{dz}{dx}-\frac{1}{2x}.z=\frac{x^2+1}{2x} \Rightarrow \frac{dz}{dx}-\frac{1}{x}z=\frac{x^2+1}{x}$, (iii)

which is Leibnitz's linear equation in z.

$$\therefore \text{I. F.}=e^{\int -\frac{1}{x}dx}=e^{-\log x}=e^{\log\left(\frac{1}{x}\right)}=\frac{1}{x}$$

$\therefore$ The solution is $z(\text{I. F.})=\int\left(\frac{x^2+1}{x}\right)(\text{I. F.})dx+c$

$$\Rightarrow \frac{z}{x}=\int\frac{x^2+1}{x}.\frac{1}{x}dx+c \Rightarrow \frac{y^2}{x}=\int\left(1+\frac{1}{x^2}\right)dx+c \Rightarrow \frac{y^2}{x}=x-\frac{1}{x}+c$$

$$\Rightarrow y^2=x^2+cx-1,$$

which is the required solution.

Q.No.10.: Solve the equation $x(x-y)dy+y^2dx=0$.

Sol.: The given equation is $x(x-y)dy+y^2dx=0 \Rightarrow \frac{dx}{dy}=-\frac{x^2}{y^2}+xy$

$$\Rightarrow \frac{dx}{dy}=-x^{-2}y^{-2}+xy^{-1}, \quad \text{(i)}$$

which is Bernoulli's equation in y.

Dividing throughout by x^2, we get

$$x^{-2}\frac{dx}{dy} - x^{-1}y^{-1} = -y^{-2}.$$

Putting $x^{-1} = u \Rightarrow x^{-2}dx = -du$, then

$$-\frac{du}{dy} - y^{-1}u = -y^{-2} \Rightarrow \frac{du}{dy} + uy^{-1} = y^{-2}, \qquad \text{(ii)}$$

which is Leibnitz's linear equation in u.

$\therefore$ I.F. $= e^{\int y^{-1}dy} = e^{\log y} = y$.

Then the solution is $u(\text{I.F.}) = \int y^{-2}(\text{I.F})dy + c$

$u(y) = \int y^{-2}y dy + c \Rightarrow \frac{y}{x} = \log y + c$, which is the required solution.

Q.No.11.: Solve the equation $\frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x).e^x.\sec y$.

Sol.: The given equation is $\frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x).e^x.\sec y$

which is Bernoulli's equation in y.

Dividing both sides by $\sec y$, we get

$$\cos y\frac{dy}{dx} - \frac{\sin y}{1+x} = e^x(1+x) \qquad \text{(i)}$$

Putting $\sin y = t$, $\cos x\frac{dy}{dx} = \frac{dt}{dx}$.

Then (i) becomes $\frac{dt}{dx} - \frac{1}{1+x}t = e^x(1+x)$,

which is Leibnitz's linear equation in t.

$$\therefore \text{I.F.} = e^{-\int \frac{1}{1+x}dx} = e^{-\log(1+x)} = \frac{1}{(1+x)}.$$

$\therefore$ The solution is $t(\text{I.F.}) = \int e^x(1+x)(\text{I.F.})dx + c$

$\Rightarrow \frac{\sin x}{1+x} = e^x + c \Rightarrow \sin y = e^x(1+x) + c(1+x) = (1+x)(e^x + c)$. Ans.

which is the required solution.

Q.No.12.: Solve the equation $e^y\left(\frac{dy}{dx} + 1\right) = e^x$.

Sol.: The given equation is $e^y\left(\frac{dy}{dx}+1\right)=e^x \Rightarrow e^y\frac{dy}{dx}+e^y=e^x$ (i)

which is Bernoulli's equation in y.

Putting $e^y=z \Rightarrow e^y\frac{dy}{dx}=\frac{dz}{dx}$.

$\therefore$ Equation (i) becomes $\frac{dz}{dx}+z=e^x$, (ii)

which is Leibnitz's linear equation in z.

Hence I. F. $=e^{\int 1dx}=e^x$.

$\therefore$ The solution of the equation is $z(\text{I. F.})=\int e^x(\text{I. F})dx+c'$

$\Rightarrow z.e^x=\int e^{2x}dx+c' \Rightarrow e^y.e^x=\frac{e^{2x}}{2}+c' \Rightarrow 2.e^{x+y}=e^{2x}+c$, (where $c=2c'$)

$\Rightarrow e^{x+y}=\frac{1}{2}e^{2x}+c$, Ans. which is the required solution.

Q.No.13.: Solve the equation $\sec^2 y\frac{dy}{dx}+\tan y=x^3$.

Sol.: The given equation is $\sec^2 y\frac{dy}{dx}+\tan y=x^3$

which is Bernoulli's equation in y.

Putting $\tan y=z$ so that $\sec^2 y\frac{dy}{dx}=\frac{dz}{dx}$

$\therefore$ Equation (i) becomes $\frac{dz}{dx}+z=x^3$, (ii)

which is Leibnitz's linear equation in z.

Hence I. F. $=e^{\int 1dx}=e^x$.

$\therefore$ The solution of the equation is $z(\text{I. F.})=\int e^x(\text{I. F})dx+c$

$\Rightarrow z.e^x=\int x^3e^xdx+c \Rightarrow \tan y.e^x=x^3e^x-\int 3x^2e^xdx+c$

$\Rightarrow \tan y.e^x=x^3e^x-3x^2e^x+\int 6xe^xdx+c$

$\Rightarrow \tan y.e^x=x^3e^x-3x^2e^x+6xe^x-\int 6e^xdx+c$

$\Rightarrow \tan y.e^x = x^3e^x - 3x^2e^x + 6xe^x - 6e^x + c$

$\Rightarrow \tan y = x^3 - 3x^2 + 6x - 6 + c\,e^{-x}$, Ans. which is the required solution.

Q.No.14.: Solve the equation $\tan y\dfrac{dy}{dx} + \tan x = \cos y.\cos^2 x$.

Sol.: The given equation is $\tan y\dfrac{dy}{dx} + \tan x = \cos y.\cos^2 x$

Dividing both sides by $\cos y$, we get

$$\frac{\tan y}{\cos y}\frac{dy}{dx} + \frac{\tan x}{\cos y} = \cos^2 x \Rightarrow \sec y.\tan y.\frac{dy}{dx} = \cos^2 x \qquad \text{(i)}$$

Putting $\sec y = t$ $\quad \therefore\ \sec y \tan y\dfrac{dy}{dx} = \dfrac{dt}{dx}$

$\therefore$ Equation (i) becomes $\dfrac{dt}{dx} + \tan x.t = \cos^2 x$

which is Leibnitz's linear equation in t.

$\therefore$ I. F. $= e^{\int \tan x dx} = e^{\log|\sec x|} = \sec x$

Hence the solution is $t(\text{I. F.}) = \int \cos^2 x(\text{I. F.})dx + c$

$\Rightarrow t.\sec x = \int \cos^2 x.\sec x dx + c \Rightarrow t.\sec x = \int \cos x dx + c \Rightarrow t.\sec x = \sin x + c$

$\Rightarrow t = \cos x(\sin x + c) \Rightarrow \sec y = \cos x(\sin x + c)$. Ans.

which is the required solution.

Q.No.15.: Solve the equation $\dfrac{dy}{dx} = \dfrac{y}{x + \sqrt{xy}}$.

Sol.: The given equation is $\dfrac{dy}{dx} = \dfrac{y}{x + \sqrt{xy}}$.

$$\Rightarrow \frac{dx}{dy} = \frac{x + \sqrt{xy}}{y} \Rightarrow \frac{dx}{dy} - \frac{x}{y} = \sqrt{\frac{x}{y}} \Rightarrow \frac{dx}{dy} + \left(-\frac{1}{y}\right)x = \sqrt{\frac{x}{y}}, \qquad \text{(i)}$$

which is Bernoulli's equation in x.

Dividing both sides by $\sqrt{x}$, we get $\dfrac{1}{\sqrt{x}}\dfrac{dx}{dy} + \left(-\dfrac{1}{y}\right)\sqrt{x} = \dfrac{1}{\sqrt{y}}$. (ii)

Putting $\sqrt{x} = z$, so that $\frac{1}{2\sqrt{x}}dx = dz \Rightarrow \frac{1}{\sqrt{x}}dx = 2dz$

$\therefore$ Equation (ii) becomes $2\frac{dz}{dy} + \left(-\frac{z}{y}\right) = \frac{1}{\sqrt{y}} \Rightarrow \frac{dz}{dy} + \left(-\frac{1}{2y}\right)z = \frac{1}{2\sqrt{y}}$,

which is Leibnitz's linear equation in z.

$$\therefore \text{I. F.} = e^{\int -\frac{1}{2y}dy} = e^{-\frac{1}{2}\log y} = y^{-1/2} = \frac{1}{\sqrt{y}}.$$

Hence, the solution is $z(\text{I. F.}) = \int Q(\text{I. F.})dx + c$

$$\Rightarrow z.\frac{1}{\sqrt{y}} = \int \frac{1}{2\sqrt{y}}\frac{1}{\sqrt{y}}dy + c \Rightarrow z.\frac{1}{\sqrt{y}} = \frac{1}{2}\int \frac{1}{y}dy + c \Rightarrow z.\frac{1}{\sqrt{y}} = \frac{1}{2}\log y + c$$

$$\Rightarrow \sqrt{x}.\frac{1}{\sqrt{y}} = \log\sqrt{y} + c \Rightarrow \sqrt{x} = \sqrt{y}\left[\log\sqrt{y} + c\right],$$

which is the required solution.

Q.No.16.: Solve $\frac{dy}{dx} + \frac{x}{1-x^2}y = x\sqrt{y}$.

Sol.: The given equation is $\frac{dy}{dx} + \frac{x}{1-x^2}y = x\sqrt{y}$, (i)

which is Bernoulli's equation in y.

Dividing by $\sqrt{y}$, then given equation becomes

$$y^{-1/2}\frac{dy}{dx} + \frac{x}{(1-x^2)}y^{1/2} = x. \quad \text{(ii)}$$

Putting $y^{1/2} = z$ so that $\frac{1}{2}y^{-1/2}\frac{dy}{dx} = \frac{dz}{dx} \Rightarrow y^{-1/2}\frac{dy}{dx} = 2\frac{dz}{dx}$.

$\therefore$ Equation (ii) becomes

$$2\frac{dz}{dx} + \frac{x}{1-x^2}z = x \Rightarrow \frac{dz}{dx} + \frac{x}{2(1-x^2)}z = \frac{x}{2},$$

which is Leibnitz's linear equation in z.

$$\therefore \text{I.F.} = e^{\int \frac{x}{2(1-x^2)}dx} = e^{-\frac{1}{4}\int \frac{-2x}{1-x^2}dx} = e^{-\frac{1}{4}\log(1-x^2)} = (1-x^2)^{-1/4}.$$

Hence, the solution is $z(\text{I. F.}) = \int Q(\text{I. F.})dx + c$

$$z(\text{I.F.}) = \int \frac{x}{2}(\text{I.F.})dx + c$$

$$\Rightarrow z\left(1-x^2\right)^{-1/4} = \int \frac{x}{2}\left(1-x^2\right)^{-1/4} dx + c$$

$$= -\frac{1}{4}\int \left(1-x^2\right)^{-1/4}(-2x)dx + c = -\frac{1}{4}.\frac{\left(1-x^2\right)^{3/4}}{3/4} + c$$

$$\Rightarrow z = -\frac{1}{3}\left(1-x^2\right) + c\left(1-x^2\right)^{1/4}$$

$$\Rightarrow \sqrt{z} = -\frac{1}{3}\left(1-x^2\right) + c\left(1-x^2\right)^{1/4}, \qquad \left[\because z = \sqrt{y}\right]$$

which is the required solution.

Q.No.17.: Solve $\dfrac{dy}{dx} + x\sin 2y = x^3\cos^2 y$.

Sol.: The given equation is $\dfrac{dy}{dx} + x\sin 2y = x^3\cos^2 y$, (i)

which is Bernoulli's equation in y.

Dividing by $\cos^2 y$, we have $\sec^2 y\dfrac{dy}{dx} + \dfrac{2\sin y\cos y}{\cos^2 y} = x^3$

$$\Rightarrow \sec^2 y\frac{dy}{dx} + 2x\tan y = x^3. \qquad \text{(ii)}$$

Putting $\tan y = z$ so that $\sec^2 y\dfrac{dy}{dx} = \dfrac{dz}{dx}$

$\therefore$ Equation (ii) becomes $\dfrac{dz}{dx} + 2xz = x^3$,

which is Leibnitz's linear equation in z.

$\therefore$ I.F. $= e^{\int 2xdx} = e^{x^2}$.

Hence, the solution is $z(\text{I.F.}) = \int Q(\text{I.F.})dx + c$

$$z.e^{x^2} = \int x^3.e^{x^2}dx + c = \int x^2e^{x^2}.xdx + c$$

$$= \frac{1}{2}\int te^t dt + c \quad \text{where } t = x^2 = \frac{1}{2}(t-1)e^t + c = \frac{1}{2}\left(x^2-1\right)e^{x^2} + c$$

$$\Rightarrow z = \frac{1}{2}\left(x^2-1\right) + ce^{-x^2}$$

$\Rightarrow \tan y = \frac{1}{2}\left(x^2 - 1\right) + ce^{-x^2}$, $\qquad [\because z = \tan y]$

which is the required solution.

Q.No.18.: Solve $3y' + xy = xy^{-2}$.

Sol.: The given equation is $3y' + xy = xy^{-2} \Rightarrow y' + \frac{x}{3}y = \frac{x}{3}y^{-2}$, (i)

which is Bernoulli's equation in y.

Introducing $z = y^{1-a} = y^{1-(-2)} = y^3$, so that $\frac{dz}{dx} = 3y^2\frac{dy}{dx}$.

$\therefore$ Equation (i) becomes $\frac{1}{3}\frac{dz}{dx} + \frac{1}{3}xz = \frac{1}{3}x$,

which is Leibnitz's linear equation in z but is also a separable equation

$\frac{dz}{dx} = x(1 - z) \quad \Rightarrow \frac{dz}{z-1} = -xdx$

Integrating both sides, we get $\log(z-1) = -\frac{x^2}{2} + c_0$

$\Rightarrow (z-1) = e^{-x^2/2}.c \quad$ where $c = e^{c_0}$

Replacing z, we get

$y^3 = 1 + ce^{-x^2/2}$,

which is the required solution.

Q.No.19.: Solve $\cos x dy = y(\sin x - y)dx$.

Sol.: The given equation is $\cos x dy = y(\sin x - y)dx$

$\Rightarrow y' - y.\tan x = -\sec x.y^2$. (i)

which is Bernoulli's equation in y.

Put $z = y^{1-a} = y^{1-2} = y^{-1}$, so that $\frac{dz}{dx} = -\frac{1}{y^2}\frac{dy}{dx}$.

$\therefore$ Equation (i) becomes $\frac{dz}{dx} + z.\tan x = \sec x$,

which is Leibnitz's linear equation in z.

$\therefore$ I.F. $= e^{\int \tan x d} = \sec x$.

Hence, the solution is $z(\text{I.F.}) = \int Q(\text{I.F.})dx + c$

$$z\sec x = \int \sec^2 x dx + c$$

$$z\sec x = \tan x + c$$

Replacing z, we get

$$\frac{1}{y}\sec x = \tan x + c \Rightarrow \sec x = y(\tan x + c),$$

which is the required solution.

Q.No.20.: Solve $2xyy' = y^2 - 2x^3$, $y(1) = 2$.

Sol.: The given equation is $2xyy' = y^2 - 2x^3$.

$$\Rightarrow y' - \frac{1}{2x}y = -x^2y^{-1}, \tag{i}$$

which is Bernoulli's equation in y.

Put $z = y^{1-a} = y^{1-(-1)} = y^2$, and $\frac{dz}{dx} = 2y\frac{dy}{dx}$.

$\therefore$ Equation (i) becomes $\frac{dz}{dx} - \frac{1}{x}z = -2x^2$, which is Leibnitz's linear equation in z.

$$\therefore \text{I.F.} = e^{\int -\frac{1}{x}dx} = e^{-\log x} = e^{\log 1/x} = \frac{1}{x}.$$

Hence, the solution is $z(\text{I.F.}) = \int Q(\text{I.F.})dx + c$

$$z.\frac{1}{x} = \int -2xdx + c = -x^2 + c$$

$$z = -x^3 + cx$$

Replacing z, we get

$$y^2 = cx - x^3$$

Since $y(1) = 2$, $2^2 = c.1 - 1^3 \Rightarrow c = 5$.

Thus the required solution is

$$y^2 = x(5 - x^2).$$

Q.No.21.: Solve $(xy^5 + y)dx - dy = 0$.

Sol.: The given equation is $\left(xy^5+y\right)dx-dy=0 \Rightarrow y'-y=xy^5$, (i)

which is Bernoulli's equation in y.

Put $z=y^{1-a}=y^{1-5}=y^{-4}$, and $\frac{dz}{dx}=-4.y^{-5}\frac{dy}{dx}$.

$\therefore$ Equation (i) becomes $\frac{dz}{dx}+4z=-4x$,

which is Leibnitz's linear equation in z.

$\therefore$ I.F.$=e^{\int 4dx}=e^{4x}$.

Hence, the solution is $z(\text{I. F.})=\int Q(\text{I. F.})dx+c$

$$z.e^{4x}=-4\int xe^{4x}dx+c \Rightarrow ze^{4x}=-xe^{4x}+\frac{1}{4}e^{4x}+c.$$

Replacing z by y^{-4}, we get

$$y^{-4}e^{4x}=-xe^{4x}+\frac{1}{4}e^{4x}+c,$$

which is the required solution.

Q.No.22.: Solve $y'-2\cos x.\cot y+\sin^2 x.\operatorname{cosec} y.\cos x=0$.

Sol.: The given equation is $y'-2\cos x.\cot y+\sin^2 x.\operatorname{cosec} y.\cos x=0$

$$\Rightarrow \sin y.y'=2\cos x.\cos y-\cos x.\sin^2 x \Rightarrow -\sin y\frac{dy}{dx}+(\cos y)(2\cos x)=\sin^2 x.\cos x, \quad \text{(i)}$$

which is Bernoulli's equation in y.

Put $v=\cos y$, so that $\frac{dv}{dy}=-\sin y$.

$\therefore$ Equation (i) becomes $\frac{dv}{dx}+2v\cos x=\sin^2 x.\cos x$,

which is Leibnitz's linear equation in v.

$\therefore$ I.F. $=e^{\int 2\cos x dx}=e^{2\sin x}$.

Hence, the solution is $v(\text{I. F.})=\int Q(\text{I. F.})dx+c$

$$ve^{2\sin x}=\int e^{2\sin x}.\sin^2 x.\cos x dx=\frac{1}{2}e^{2\sin x}.\sin^2 x-\frac{1}{2}e^{2\sin x}.\sin x+\frac{1}{4}e^{2\sin x}+c.$$

Replacing v by $\cos y$, we get the required solution as

$$\cos y = \frac{1}{2}\sin^2 x - \frac{1}{2}\sin x + \frac{1}{4}ce^{-2\sin x},$$

which is the required solution.

Home Assignments:

Q.No.1.: Solve the differential equation $2\frac{dy}{dx} = \frac{y}{x} + \frac{y^2}{x^2}$.

Ans.: $\frac{x}{y} = 1 + c\sqrt{x}$.

Q.No.2.: Solve the differential equation $\frac{dy}{dx} - x^2 y = y^2 e^{-\frac{1}{3}x^3}$.

Ans.: $y(c - x) = e^{\frac{1}{3}x^3}$.

Q.No.3.: Solve the differential equation $(x+1)\frac{dy}{dx} + 1 = 2e^{-y}$.

Ans.: $(x+1)e^y = 2x + c$.

Q.No.4.: Solve the differential equation $\left(xy^2 - e^{x^{1/3}}\right)dx - x^2 y dy = 0$.

Ans.: $3y^2 = x^2\left(2e^{x^{1/3}} + c\right)$.

Q.No.5.: Solve the differential equation $\frac{dy}{dx} + y\tan x = y^3 \cos x$.

Ans.: $\cos^2 x = y^2\left(c - 2\sin x + \frac{2}{3}\sin^3 x\right)$.

Q.No.6.: Solve the differential equation $\frac{dy}{dx} + \frac{y\log y}{x} = \frac{y(\log y)^2}{x^2}$.

Ans.: $\frac{1}{x\log y} = \frac{1}{2x^2} + c$.

Q.No.7.: Solve the differential equation $y - \cos x\frac{dy}{dx} = y^2(1 - \sin x)\cos x$, given that $y = 2$ when $x = 0$.

Ans.: $2(\tan x + \sec x) = y(2\sin x + 1)$.

Q.No.8.: Solve the differential equation $y\left(2xy + e^x\right)dx = e^x dy$.

Ans.: $e^x = y\left(c - x^2\right)$.

Q.No.9.: Solve the differential equation $\left(y - y^2 x^2 \sin x\right)dx + xdy = 0$.

Ans.: $yx(c + \cos x) = 1$.

Q.No.10.: Solve the differential equation $y' + y + y^2(\sin x - \cos x) = 0$.

Ans.: $y\left(ce^x - \sin x\right) = 1$.

Q.No.11.: Solve the differential equation $\dfrac{dz}{dx} + \left(\dfrac{z}{x}\right)\log z = \dfrac{z}{x}(\log z)^2$.

Ans.: $(1 + cx)\log z = 1$.

Q.No.12.: Solve the differential equation $y' + \dfrac{y}{2x} = \dfrac{x}{y^3}$, $y(1) = 2$.

Ans.: $x^2 y^4 = x^4 + 15$.

Q.No.13.: Solve the differential equation $dx - \left(x^2 y^3 + xy\right)dy = 0$.

Ans.: $x\left(2 - y^2\right) + cxe^{\frac{-x^2}{2}} = 1$.

Q.No.14.: Solve the differential equation $y^2 dx + \left(xy - x^3\right)dy = 0$.

Ans.: $2x^2 - 3y = cx^2 y^3$.

Q.No.15.: Solve the differential equation $xy'' - 3y' = 4x^2$.

Ans.: $y = c_1 x^4 - \dfrac{4}{3}x^3 + c_2$.

Q.No.16.: Solve the differential equation $y' = Ay - By^n$ with A, B, n ; constants and $n \neq 0,1$.

Ans.: $y^{1-n} = \left[\dfrac{B}{A} + ce^{(1-n)Ax}\right]$.

Q.No.17.: Solve the differential equation $y^2.y' - y^3 \tan x - \sin x.\cos^2 x = 0$.

Ans.: $2y^3 + \cos^3 x = 2c\sec^3 x$.

Q.No.18.: Solve the differential equation $y' + x \sin 2y = x^3 \cos^2 y$.

Ans.: $2y = \left(x^2 - 1\right) + 2ce^{-x^2}$.

Q.No.19.: Solve the differential equation $\left[(x+1)^4 + 2\sin y^2\right]dx - 2y(x+1)\cos y^2 dy = 0$.

Ans.: $2\sin y^2 = (x+1)^4 + 2c(x+1)^2$.

Q.No.20.: Solve the differential equation $\left(4e^{-y}\sin x - 1\right)dx - dy = 0$.

Ans.: $e^y = 2(\sin x - \cos x) + ce^{-x}$.

Q.No.21.: Solve the differential equation $y' - \cot y + x\cot y = 0$.

Ans.: $\sec y = x + 1 + ce^x$.

Q.No.22.: Solve the differential equation $x\left(4y - 8y^{-3}\right)dx + dy = 0$.

Ans.: $y^4 = \left(2 + ce^{-8x^2}\right)$.

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5^{th} Topic

Differential Equations of First order

Part: I

"Exact Differential Equations"

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Exact Differential equations:

A differential equation of the form

$$\boxed{M(x, y)dx + N(x, y)\,dy = 0}$$

is said to be exact if its left hand member is the exact differential of some function u(x, y), i.e., $Mdx + Ndy \equiv du$.

Hence, its solution is u(x, y) = c.

Necessary and sufficient condition for the differential equation to be exact:

Theorem:

The necessary and sufficient condition for the differential equation $Mdx + Ndy = 0$ to be exact is

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}.$$

or

Discuss the necessary and sufficient condition for the differential equation $Mdx + Ndy = 0$ to be exact. Also derive the method of solution.

Proof: Condition is necessary: **Given** the equation $Mdx + Ndy = 0$ is exact.

To show: $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$.

Since the equation $Mdx + Ndy = 0$ is exact $\Rightarrow Mdx + Ndy \equiv du$. (i)

But also $du = \dfrac{\partial u}{\partial x}dx + \dfrac{\partial u}{\partial y}dy$. (total differential) (ii)

$\therefore$ Equating coefficients of dx and dy in (i) and (ii), we get

$$M = \frac{\partial u}{\partial x} \quad \text{and} \quad N = \frac{\partial u}{\partial y}$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \partial x} \quad \text{and} \quad \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial y \partial x},$$

But $\dfrac{\partial^2 u}{\partial y \partial x} = \dfrac{\partial^2 u}{\partial x \partial y}$.

$\therefore \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, which is the necessary condition for exactness.

This completes the first part.

Condition is sufficient: **Given** $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$.

To show: $Mdx + Ndy = 0$ is exact.

Let $\int Mdx = u$, where y is supposed constant while performing integration.

Then $\dfrac{\partial}{\partial x}\left(\int Mdx\right) = \dfrac{\partial u}{\partial x} \Rightarrow M = \dfrac{\partial u}{\partial x}$. (iii)

Differential partially w.r.t. y, we get

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \partial x}$$

$$\Rightarrow \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$$

$$\left[\because \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \ (\text{given}) \ \text{ and } \ \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}\right]$$

Integrating both sides w.r.t. x, taking y as constant.

$N=\frac{\partial u}{\partial y}+f(y)$, where f(y) is a function y alone. (iv)

Now start from the LHS of the given equation.

$\therefore Mdx+Ndy=\frac{\partial u}{\partial x}dx+\left\{\frac{\partial u}{\partial y}+f(y)\right\}dy$ {by (iii) and (iv)}

$$=\left\{\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy\right\}+f(y)dy=du+f(y)dy=d\left[u+\int f(y)dy\right]. \quad \text{(v)}$$

This shows that the equation $Mdx+Ndy=0$ is exact.

Method of solution:

By (v), the equation $Mdx+Ndy=0$

$$\Rightarrow d\left[u+\int f(y)dy\right]=0.$$

Integrating, we get $u+\int f(y)dy=c$.

But $u=\int_{y\text{ const.}} Mdx$ and $f(y)=$ terms of N not containing x.

$\therefore$ The solution of $Mdx+Ndy=0$ is

$$\int_{y\text{ const.}} Mdx+\int (\text{terms of N not containing x})dy=c, \quad \text{provided } \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}.$$

Now let us solve some differential equations, which are exact:

Q.No.1.: Solve $\left(y^2e^{xy^2}+4x^3\right)dx+\left(2xy\,e^{xy^2}-3y^2\right)dy=0$.

Sol.: The given equation is $\left(y^2e^{xy^2}+4x^3\right)dx+\left(2xy\,e^{xy^2}-3y^2\right)dy=0$

Here $M=y^2e^{xy^2}+4x^3$ and $N=2xy\,e^{xy^2}-3y^2$

$\therefore \frac{\partial M}{\partial y}=2ye^{xy^2}+y^2e^{xy^2}.2xy=\frac{\partial N}{\partial x}$.

Thus the equation is exact and its solution is

$$\int_{y\text{ const.}} Mdx+\int (\text{terms of N not containing x})dy=c$$

$$\Rightarrow \int\left(y^2e^{xy^2}+4x^3\right)dx+\int\left(-3y^2\right)dy=c \quad \Rightarrow y^2\frac{e^{xy^2}}{y^2}+x^4-y^3=c$$

$\Rightarrow e^{xy^2} + x^4 - y^3 = c$, Ans.

which is the required solution.

Q.No.2.: Solve $\dfrac{dy}{dx} + \dfrac{y\cos x + \sin y + y}{\sin x + x\cos y + x} = 0$.

Sol.: The given equation is $\dfrac{dy}{dx} + \dfrac{y\cos x + \sin y + y}{\sin x + x\cos y + x} = 0$

$\Rightarrow (y\cos x + \sin y + y)dx + (\sin x + x\cos y + x)dy = 0$

Here $M = y\cos x + \sin y + y$ and $N = \sin x + x\cos y + x$.

$\therefore \dfrac{\partial M}{\partial y} = \cos x + \cos y + 1 = \dfrac{\partial N}{\partial x}$.

Thus the equation is exact and its solution is

$$\int_{y\ \text{const.}} M dx + \int (\text{terms of N not containing x})dy = c$$

$\Rightarrow \int (y\cos x + \sin y + y)\,dx + \int (0)\,dy = c \quad \Rightarrow y\sin x + (\sin y + y)x = c$, Ans.

which is the required solution.

Q.No.3.: Solve the equation $\left(2x^3 - xy^2 - 2y + 3\right)dx - \left(x^2y + 2x\right)dy = 0$

Sol.: The given equation is $\left(2x^3 - xy^2 - 2y + 3\right)dx - \left(x^2y + 2x\right)dy = 0$

Here $M = 2x^3 - xy^2 - 2y + 3$ and $N = \left(x^2y + 2x\right)$.

$\therefore \dfrac{\partial M}{\partial y} = -2xy - 2 = \dfrac{\partial N}{\partial x}$.

Thus the equation is exact and its solution is

$$\int_{y\ \text{const.}} M dx + \int (\text{terms of N not containing x})dy = c$$

$\Rightarrow \int \left(2x^3 - xy^2 - 2y + 3\right)dx + \int (0) = c' \quad \Rightarrow \dfrac{2x^4}{4} - \dfrac{y^2x^2}{2} - 2yx + 3x = c'$

$\Rightarrow \dfrac{x^4}{2} - \dfrac{y^2x^2}{2} - 2yx + 3x = c' \quad \Rightarrow x^4 - x^2y^2 - 4xy + 6x = c$, Ans. $\quad$ (where $c = 2c'$)

which is the required solution.

Q.No.4.: Solve the equation $\left(2x^2 + 6xy - y^2\right)dx + \left(3x^2 - 2xy + y^2\right)dy = 0$.

Sol.: The given equation is $\left(2x^2 + 6xy - y^2\right)dx + \left(3x^2 - 2xy + y^2\right)dy = 0$

Here $M = 2x^2 + 6xy - y^2$ and $N = 3x^2 - 2xy + y^2$

$\therefore \dfrac{\partial M}{\partial y} = 6x - 2y = \dfrac{\partial N}{\partial x}$.

Thus the equation is exact and its solution is

$$\int_{y\ \text{const.}} Mdx + \int (\text{terms of N not containing x})dy = c$$

$$\Rightarrow \int \left(2x^2 + 6xy - y^2\right)dx + \int y^2 dy = c \Rightarrow \frac{2x^3}{3} + \frac{6yx^2}{2} - y^2x + \frac{y^3}{3} = c$$

$\Rightarrow 2x^3 + 9x^2y - 3y^2x + y^3 = c$, Ans. $\quad$ (where $c = 3c'$)

which is the required solution.

Q.No.5.: Solve the equation $\left(x^2 - ay\right)dx = \left(ax - y^2\right)dy$

Sol.: The given equation is $\left(x^2 - ay\right)dx = \left(ax - y^2\right)dy \Rightarrow \left(x^2 - ay\right)dx + \left(y^2 - ax\right)dy = 0$

Here $M = x^2 - ay$ and $N = y^2 - ax$

$\therefore \dfrac{\partial M}{\partial y} = -a = \dfrac{\partial N}{\partial x}$.

Thus the equation is exact and its solution is

$$\int_{y\ \text{const.}} Mdx + \int (\text{terms of N not containing x})dy = c'$$

$$\Rightarrow \int \left(x^2 - ay\right)dx + \int y^2 dy = c' \Rightarrow \frac{x^3}{3} - (ay)x + \frac{y^3}{3} = c'$$

$\Rightarrow x^3 - 3axy + y^3 = c$, Ans. $\quad$ (where $c = 3c'$)

which is the required solution.

Q.No.6.: Solve the equation $\left(x^2 + y^2 - a^2\right)xdx + \left(x^2 - y^2 - b^2\right)ydy = 0$

Sol.: The given equation is $\left(x^2 + y^2 - a^2\right)xdx + \left(x^2 - y^2 - b^2\right)ydy = 0$

Here $M = \left(x^2 + y^2 - a^2\right)x$ and $N = \left(x^2 + y^2 - b^2\right)y$

$\therefore \dfrac{\partial M}{\partial y} = 2xy = \dfrac{\partial N}{\partial x}$.

Thus the equation is exact and its solution is

$$\int_{\text{y const.}} M dx + \int (\text{terms of N not containing x}) dy = c'$$

$$\Rightarrow \int \left(x^2 + y^2 - a^2\right) dx + \int \left(-y^2 - b^2\right) y dy = c'$$

$$\Rightarrow \left[\frac{x^4}{4} + y^2.\frac{x^2}{2} - \frac{a^2x^2}{2}\right] + \left[\frac{y^4}{4} + \frac{b^2y^2}{2}\right] = c'$$

$$\Rightarrow x^4 + 2x^2y^2 - 2a^2x^2 - y^4 - 2b^2y^2 = c \ , \qquad (\text{where } c = 4c')$$

$$\Rightarrow x^4 - y^4 + 2x^2y^2 - 2\left(a^2x^2 + b^2y^2\right) = c \text{ , Ans.}$$

which is the required solution.

Q.No.7.: Solve the equation $\left(x^2 - 4xy - 2y^2\right)dx + \left(y^2 - 4xy - 2x^2\right)dy = 0$

Sol.: The given equation is $\left(x^2 - 4xy - 2y^2\right)dx + \left(y^2 - 4xy - 2x^2\right)dy = 0$

Here $M = \left(x^2 - 4xy - 2y^2\right)$ and $N = \left(y^2 - 4xy - 2x^2\right)$

$$\therefore \frac{\partial M}{\partial y} = -4x - 4y = \frac{\partial N}{\partial x}.$$

Thus the equation is exact and its solution is

$$\int_{\text{y const.}} M dx + \int (\text{terms of N not containing x}) dy = c'$$

$$\Rightarrow \int \left(x^2 - 4y - 2y^2\right) dx + \int y^2 dy = c' \quad \Rightarrow \left[\frac{x^3}{3} - 4y.\frac{x^2}{2} - 2y^2x\right] + \frac{y^3}{3} = c'$$

$$\Rightarrow x^3 - 6x^2y - 6xy^2 + y^3 = c \qquad (\text{where } c = 3c')$$

$$\Rightarrow x^3 + y^3 - 6\left(x^2y + xy^2\right) = c \text{ , Ans.}$$

which is the required solution.

Q.No.8.: Solve the equation $\left(x^4 - 2xy^2 + y^4\right)dx - \left(2x^2y - 4xy^3 + \sin y\right)dy = 0$.

Sol.: The given equation is $\left(x^4 - 2xy^2 + y^4\right)dx - \left(2x^2y - 4xy^3 + \sin y\right)dy = 0$

Here $M = x^4 - 2xy^2 + y^4$ and $N = -\left(2x^2y - 4xy^3 + \sin y\right)$.

$$\therefore \frac{\partial M}{\partial y} = -4xy + 4y^3 = \frac{\partial N}{\partial x}.$$

Thus the equation is exact and its solution is

$$\int_{y\text{ const.}} Mdx + \int (\text{terms of N not containing x})dy = c$$

$$\Rightarrow \int \left(x^4 - 2xy^2 + y^4\right)dx + \int (-\sin y)dy = c \qquad \Rightarrow \left(\frac{x^5}{5} - 2y^2\frac{x^2}{2} + y^4x\right) + \cos y = c$$

$$\Rightarrow \frac{x^5}{5} - x^2y^2 + xy^4 + \cos y = c \text{ , Ans.}$$

which is the required solution.

Q.No.9.: Solve the equation $ye^{xy}dx + \left(xe^{xy} + 2y\right)dy = 0$.

Sol.: The given equation is $ye^{xy}dx + \left(xe^{xy} + 2y\right)dy = 0$.

Here $M = ye^{xy}$ and $N = xe^{xy} + 2y$.

$$\therefore \frac{\partial M}{\partial y} = ye^{xy}.x + e^{xy} = \frac{\partial N}{\partial x}.$$

Thus the equation is exact and its solution is

$$\int_{y\text{ const.}} Mdx + \int (\text{terms of N not containing x})dy = c$$

$$\Rightarrow \int ye^{xy}dx + \int 2ydy = c \Rightarrow y\frac{e^{xy}}{y} + \frac{2y^2}{2} = c \Rightarrow e^{xy} + y^2 = c\text{, Ans.}$$

which is the required solution.

Q.No.10.: Solve the equation $\left(5x^4 + 3x^2y^3 - 2xy^3\right)dx + \left(2x^3y - 3x^2y^2 - 5y^4\right)dy = 0$

Sol.: The given equation is $\left(5x^4 + 3x^2y^3 - 2xy^3\right)dx + \left(2x^3y - 3x^2y^2 - 5y^4\right)dy = 0$.

Here $M = 5x^4 + 3x^2y^2 - 2xy^3$ and $N = 2x^3y - 3x^2y^2 - 5y^4$.

$$\therefore \frac{\partial M}{\partial y} = 6x^2y - 6xy^2 = \frac{\partial N}{\partial x}.$$

Thus the equation is exact and its solution is

$$\int_{y\text{ const.}} Mdx + \int (\text{terms of N not containing x})dy = c$$

$$\Rightarrow \int \left(5x^4 + 3x^2y^2 - 2xy^3\right)dx + \int \left(-5y^4\right)dy = c \quad \Rightarrow \frac{5x^5}{5} + 3y^2\frac{x^3}{3} - 2y^3\frac{x^2}{2} - \frac{5y^5}{5} = c$$

$$\Rightarrow x^5 + y^2x^3 - y^3x^2 - y^5 = c\text{, Ans.}$$

which is the required solution.

Q.No.11.: Solve the equation $(3x^2+6xy^2)dx+(6x^2y+4y^3)dy=0$.

Sol.: The given equation is $(3x^2+6xy^2)dx+(6x^2y+4y^3)dy=0$.

Here $M=3x^2+6xy^2$ and $N=3x^2y+6y^3$.

$\therefore \dfrac{\partial M}{\partial y}=12y=\dfrac{\partial N}{\partial x}$.

Thus the equation is exact and its solution is

$$\int_{y\text{ const.}} Mdx+\int(\text{terms of N not containing x})dy=c$$

$$\Rightarrow \int(3x^2+6xy^2)dx+\int 4y^3dy=c \Rightarrow \frac{3x^3}{3}+\frac{6x^2}{2}y^2+\frac{4y^4}{4}=c$$

$\Rightarrow x^3+3x^2y^2+y^4=c$, Ans.

which is the required solution.

Q.No.12.: Solve the equation $\dfrac{2x}{y^3}dx+\dfrac{y^2-3x^2}{y^4}dy=0$.

Sol.: The given equation is $\dfrac{2x}{y^3}dx+\dfrac{y^2-3x^2}{y^4}dy=0$.

Here $M=2xy^{-3}$ and $N=\dfrac{y^2-3x^2}{y^4}=y^{-2}-3x^2y^{-4}$.

$\therefore \dfrac{\partial M}{\partial y}=-6xy^{-4}=\dfrac{\partial N}{\partial x}$.

Thus the equation is exact and its solution is

$$\int_{y\text{ const.}} Mdx+\int(\text{terms of N not containing x})dy=c$$

$$\Rightarrow \int 2xy^{-3}dx+\int y^{-2}dy=c \quad \Rightarrow 2y^{-3}\frac{x^2}{2}+\frac{y^{-1}}{-1}=c$$

$\Rightarrow \dfrac{x^2}{y^3}-\dfrac{1}{y}=c \Rightarrow x^2-y^2=cy^3$, Ans.

which is the required solution.

Q.No.13.: Solve the equation $y\sin 2x dx - \left(1 + y^2 + \cos^2 x\right)dy = 0$

Sol.: The given equation is $y\sin 2x dx - \left(1 + y^2 + \cos^2 x\right)dy = 0$.

Here $M = y\sin 2x$ and $N = -\left(1 + y^2 + \cos^2 x\right)$.

$\therefore \dfrac{\partial M}{\partial y} = \sin 2x = \dfrac{\partial N}{\partial x}$.

Thus the equation is exact and its solution is

$$\int_{y\ \text{const.}} M dx + \int (\text{terms of N not containing x}) dy = c'$$

$$\Rightarrow \int (y\sin 2x) dx + \int \left(1 + y^2\right)dy = c' \Rightarrow \left(y\frac{\cos 2x}{2}\right) + \left(y + \frac{y^3}{3}\right) = c'$$

$\Rightarrow 3y\cos 2x + 6y + 2y^3 = c$, Ans. (where $c = 6c'$)

which is the required solution.

Q.No.14.: Solve the equation $\left(\sec x \tan x \tan y - e^x\right)dx + \sec x \sec^2 y dy = 0$

Sol.: The given equation is $\left(\sec x \tan x \tan y - e^x\right)dx + \sec x \sec^2 y dy = 0$.

Here $M = \sec x \tan x \tan y - e^x$ and $N = \sec x \sec^2 y$.

$\therefore \dfrac{\partial M}{\partial y} = \sec x \tan x \sec^2 y = \dfrac{\partial N}{\partial x}$.

Thus the equation is exact and its solution is

$$\int_{y\ \text{const.}} M dx + \int (\text{terms of N not containing x}) dy = c'$$

$$\Rightarrow \int \left(\sec x \tan x \tan y - e^x\right)dx + \int (0) dy = c'$$

$\Rightarrow \left(\sec x \tan y - e^x\right) = c' \Rightarrow e^x = \sec x \tan y + c$, Ans. (where $c = -c'$)

which is the required solution.

Q.No.15.: Solve the equation $(2xy + y - \tan y)dx + \left(x^2 - x\tan y + \sec^2 y\right)dy$.

Sol.: The given equation is $(2xy + y - \tan y)dx + \left(x^2 - x\tan y + \sec^2 y\right)dy$.

Here $M = (2xy + y - \tan y)$ and $N = x^2 - x\tan y + \sec^2 y$.

$$\therefore \frac{\partial M}{\partial y} = 2x + 1 - \sec^2 y = \frac{\partial N}{\partial x}.$$

Thus the equation is exact and its solution is

$$\int_{y\ \text{const.}} Mdx + \int (\text{terms of N not containing x})dy = c'$$

$$\Rightarrow \int (2xy + y - \tan y)dx + \int \sec^2 y dy = c \quad \Rightarrow 2y\frac{x^2}{2} + xy - x\tan y + \tan y = c$$

$$\Rightarrow x^2 y + xy + \tan y(1 - x) = c \qquad\qquad \Rightarrow xy(1 + x) + \tan y(1 - x) = c\text{, Ans.}$$

which is the required solution.

Q.No.16.: Solve the equation $\left[y\left(1 + \frac{1}{x}\right) + \cos y\right]dx + (x + \log x - x\sin y)dy = 0$.

Sol.: The given equation is $\left[y\left(1 + \frac{1}{x}\right) + \cos y\right]dx + (x + \log x - x\sin y)dy = 0$.

Here $M = \left[y\left(1 + \frac{1}{x}\right) + \cos y\right]$ and $N = (x + \log x - x\sin y)$.

$$\therefore \frac{\partial M}{\partial y} = 1 + x^{-1} - \sin y = \frac{\partial N}{\partial x}.$$

Thus, the equation is exact and its solution is

$$\int_{y\ \text{const.}} Mdx + \int (\text{terms of N not containing x})dy = c'$$

$$\Rightarrow \int (y + yx^{-1} + \cos y)dx + \int (0)dy = c \qquad \Rightarrow xy + y\log x + x\cos y = c$$

$$\Rightarrow y(x + \log x) + x\cos y = c\text{, Ans.}$$

which is the required solution.

Q.No.17.: Solve $[\cos x \tan y + \cos(x + y)]dx + [\sin x \sec^2 y + \cos(x + y)]dy = 0$.

Sol.: The given equation is $[\cos x \tan y + \cos(x + y)]dx + [\sin x \sec^2 y + \cos(x + y)]dy = 0$.

Here $M = \cos x \tan y + \cos(x + y)$

and $N = \sin x \sec^2 y + \cos(x + y)$

$$\therefore \frac{\partial M}{\partial y} = \cos x \sec^2 y - \sin(x + y) = \frac{\partial N}{\partial x}$$

Thus, the equation is exact and its solution is

$$\int_{y\ \text{const.}} M dx + \int (\text{terms of N not containing x}) dy = c'$$

$$\Rightarrow \int_{y\ \text{constant}} [\cos x \tan y + \cos(x+y)] dx = c$$

$\Rightarrow \sin x \tan y + \sin(x+y) = c$, which is the required solution.

Home assignments

Q.No.1.: Solve the differential equation $\left(1+e^{x/y}\right)dx + \left(1-\dfrac{x}{y}\right)e^{x/y}dy = 0$.

Ans.: $x + ye^{x/y} = c$.

Q.No.2.: Solve the differential equation $(y\cos x + 1)dx + \sin x dy = 0$.

Ans.: $y\sin x + x = c$.

Q.No.3.: Solve the differential equation $\left(\sec x \tan x \tan y - e^x\right)dx + \sec x \sec^2 y dy = 0$.

Ans.: $\sec x \tan y - e^x = c$.

Q.No.4.: Solve the differential equation $\left(2xy\cos x^2 - 2xy + 1\right)dx + \left(\sin x^2 - x^2\right)dy = 0$.

Ans.: $y\sin x^2 - x^2 y + x = c$.

Q.No.5.: Solve the differential equation $\left(\sin x \cos y + e^{2x}\right)dx + (\cos x \sin y + \tan y)dy = 0$.

Ans.: $-\cos x \cos y + \dfrac{1}{2}e^{2x} + \log \sec y = c$.

Q.No.6.: Solve the differential equation $xdy + ydx + \dfrac{xdy - ydx}{x^2 + y^2} = 0$.

Ans.: $xy + \tan^{-1}\dfrac{y}{x} = c$.

Q.No.7.: Solve the differential equation $\left(2x^3 - xy^2 - 2y + 3\right)dx - \left(x^2 y + 2x\right)dy = 0$.

Ans.: $x^4 - x^2 y^2 - 4xy + 6x = c$.

Q.No.8.: Solve the differential equation $(\cos x.\cos y - \cot x)dx - (\sin x \sin y)dy = 0$.

Ans.: $\sin x.\cos y = \log(c\sin x)$.

Q.No.9.: Solve the differential equation $\left(y - x^3\right)dx + \left(x + y^3\right)dy = 0$.

Ans.: $4xy - x^4 + y^4 = c$.

Q.No.12.: Solve the differential equation $\left(\sin x.\sin y - xe^y\right)dy = \left(e^y + \cos x.\cos y\right)dx$.

Ans.: $xe^y + \sin x.\cos y = c$.

Q.No.13.: Solve the differential equation $\left(x^2 + y^2 - a^2\right)xdx + \left(x^2 + y^2 - b^2\right)ydy = 0$.

Ans.: $x^4 + 2x^2y^2 - y^4 - 2a^2x^2 - 2b^2y^2 = c$.

Q.No.14.: Solve the differential equation

$$(\sin x.\cosh y)dx - (\cos x \sinh y)dy = 0, \ y(0) = 3.$$

Ans.: $\cos x.\cosh y = 10.07$.

Q.No.15.: Solve the differential equation $\left[\frac{y}{(x+y)^2} - 1\right]dx + \left[1 - \frac{x}{(x+y)^2}\right]dy = 0$.

Ans.: $x + y^2 - x^2 = c(x + y)$.

Q.No.16.: Solve the differential equation $y' = \frac{y - 2x}{2y - x}$; $y(1) = 2$.

Ans.: $x^2 - xy + y^2 = 3$.

Q.No.17.: Solve $(\cos x - x\cos y)dy - (\sin y + y\sin x)dx = 0$.

Ans.: $y\cos x - x\sin y = c$.

Q.No.18.: Solve $\left(3x^2y + \frac{y}{x}\right)dx + \left(x^3 + \log x\right)dy = 0$.

Ans.: $x^3y + y\log x = c$.

Q.No.19.: Solve $e^y dx + \left(xe^y + 2y\right)dy = 0$.

Ans.: $xe^y + y^2 = c$.

Determine which of the following are exact and solve the ones that are exact:

Q.No.20.: $\left(y + xy^2 + x^2y^3\right)dx + \left(x - x^2y + x^3y^2\right)dy = 0$.

Ans.: Not exact.

Q.No.21.: $(\sin x.\tan y + 1)dx + \cos x.\sec^2 ydy = 0$.

Ans.: Not exact.

Q.No.22.: $xdy + 2y^2dx = 0$.

Ans.: Not exact.

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5^{th} Topic

Differential Equations of First order

Part: II

"Equations Reducible to Exact Differential Equations"

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Equation reducible to exact equations:

Sometimes a differential equation, which is not exact, can be reduced to an exact differential equation after multiplication by a suitable factor (a function of x and/or y) called an integrating factor (I.F.).

Now let us discuss few methods for evaluating integrating factor of differential equations $Mdx + Ndy = 0$, which are not exact:

CATEGORIES:

(1) Differential equations, where Integrating factor (I.F.) can be found by inspection:

(2) I.F. of a differential equation, which is given in homogeneous form:

(3) I.F. for a differential of the type $f_1(xy)ydx + f_2(xy)xdy = 0$:

(4) I.F. of a differential equation $Mdx + Ndy = 0$,

(a) where $\dfrac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N}$ is a function of x only, say f(x).

(b) where $\dfrac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M}$ **is a function of y only, say F(y).**

(5) I.F. for the equation of the type: $x^a y^b(mydx+nxdy)+x^{a'}y^{b'}(m'ydx+n'xdy)=0.$

(1) Integrating factor (I.F.) found by inspection:

In a number of problems, the integrating factor can be found after regrouping the terms of the equation recognizing each group as being a part of exact differential.

The following differentials are useful in selecting a suitable integrating factor.

(i) $xdy+ydx=d(xy)$; **(ii)** $\dfrac{xdy-ydx}{x^2}=d\left(\dfrac{y}{x}\right)$;

(iii) $\dfrac{xdy-ydx}{y^2}=-d\left(\dfrac{x}{y}\right)$; **(iv)** $\dfrac{xdx-ydx}{xy}=d\left[\log\left(\dfrac{y}{x}\right)\right]$;

(v) $\dfrac{ydx+xdy}{xy}=d[\log(xy)]$; **(vi)** $\dfrac{xdy-ydx}{x^2+y^2}=d\left(\tan^{-1}\dfrac{y}{x}\right)$;

(vii) $\dfrac{xdy-ydx}{x^2-y^2}=d\left(\dfrac{1}{2}\log\dfrac{x+y}{x-y}\right)$; **(viii)** $\dfrac{xdx+ydy}{x^2+y^2}=d\left(\dfrac{1}{2}\log\left(x^2+y^2\right)\right)$.

Q.No.1.: Solve $y\left(2xy+e^x\right)dx=e^x dy$.

Sol.: The given equation is $y\left(2xy+e^x\right)dx=e^x dy$

$\Rightarrow \left(ye^x dx-e^x dy\right)+2xy^2 dx=0$.

Now we observe that the term $2xy^2dx$ should not involve y^2. This suggest that $\dfrac{1}{y^2}$ may be I.F. Multiplying throughout by $\dfrac{1}{y^2}$, the equation becomes

$$\frac{ye^x dx-e^x dy}{y^2}+2xdx=0 \quad\Rightarrow d\left(\frac{e^x}{y}\right)+2xdx=0.$$

Integrating, we get $\dfrac{e^x}{y}+x^2=c$, which is the required solution.

(2) I.F. of a differential equation, which is given in homogeneous form:

If $Mdx + Ndy = 0$ **be a differential equation in homogeneous form in x and y, then** $\dfrac{1}{(Mx+Ny)}$ **is an integrating factor, provided** $Mx + Ny \neq 0$**.**

Q.No.2.: Solve $\left(x^2y - 2xy^2\right)dx - \left(x^3 - 3x^2y\right)dy = 0$.

Sol.: The given equation is $\left(x^2y - 2xy^2\right)dx - \left(x^3 - 3x^2y\right)dy = 0$

Here $M = x^2y - 2xy^2$ and $N = -x^3 + 3x^2y$ $\therefore \dfrac{\partial M}{\partial y} = x^2 - 4xy$, $\dfrac{\partial N}{\partial x} = -3x^2 + 6xy$

$\Rightarrow \dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$. Thus the equation is not exact.

Now this differential equation is in homogeneous form in x and y.

$$\therefore \text{I. F.} = \frac{1}{Mx+Ny} = \frac{1}{\left(x^2y - 2xy^2\right)x - \left(x^3 - 3x^2y\right)y} = \frac{1}{x^2y^2}.$$

Multiplying throughout by $\dfrac{1}{x^2y^2}$, the equation becomes

$\left(\dfrac{1}{y} - \dfrac{2}{x}\right)dx - \left(\dfrac{x}{y^2} - \dfrac{3}{y}\right)dy = 0$, which is an exact equation, $\because \dfrac{\partial M}{\partial y} = -\dfrac{1}{y^2} = \dfrac{\partial N}{\partial x}$.

$$\therefore \text{The solution is } \int\limits_{y\text{ const.}} Mdx + \int (\text{terms of } N \text{ not containing } x)dy = c$$

$\Rightarrow \dfrac{x}{y} - 2\log x + 3\log y = c$, which is the required solution.

(3) I.F. for an equation of the type $f_1(xy)ydx + f_2(xy)xdy = 0$:

If the equation $Mdx + Ndy = 0$ **is of the form** $f_1(xy)ydx + f_2(xy)xdy = 0$**, then** $\dfrac{1}{(Mx-Ny)}$ **is an integrating factor, provided** $Mx - Ny \neq 0$**.**

Q.No.3.: Solve $(1+xy)ydx + (1-xy)xdy = 0$.

Sol.: The given equation is $(1+xy)ydx + (1-xy)xdy = 0$

Here $M = y + xy^2$ and $N = x - x^2y$ $\therefore \dfrac{\partial M}{\partial y} = 1 + 2xy$, $\dfrac{\partial N}{\partial x} = 1 - 2xy$

$\Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$. Thus the equation is not exact.

Now the given equation is of the form $f_1(xy)ydx + f_2(xy)xdy = 0$

Here $M = (1+xy)y$, $N = (1-xy)x$.

$\therefore$ I. F. $= \frac{1}{Mx - Ny} = \frac{1}{(1+xy)yx - (1-xy)xy} = \frac{1}{2x^2y^2}$.

Multiplying throughout by $\frac{1}{2x^2y^2}$, the equation becomes

$$\left(\frac{1}{2x^2y} + \frac{1}{2x}\right)dx + \left(\frac{1}{2xy^2} - \frac{1}{2y}\right)dy = 0,$$

which is an exact equation, $\because \frac{\partial M}{\partial y} = -\frac{1}{2x^2y^2} = \frac{\partial N}{\partial x}$.

$\therefore$ The solution is $\int\limits_{y\text{ const.}} Mdx + \int (\text{terms of } N \text{ not containing } x)dy = c$

$$\Rightarrow \frac{1}{2y}\left(-\frac{1}{x}\right) + \frac{1}{2}\log x - \frac{1}{2}\log y = c \quad \Rightarrow \log\frac{x}{y} - \frac{1}{xy} = c',$$

which is the required solution.

(4) In the equation $Mdx + Ndy = 0$,

(a) If $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$ is a function of x only, say f(x),

then $e^{\int f(x)dx}$ is an integrating factor.

(b) If $\dfrac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}$ is a function of y only, say F(y),

then $e^{\int F(y)dy}$ is an integrating factor.

Theorem: If in an equation $Mdx + Ndy = 0$ which is not exact, $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$ depends only on x, say R(x), then show that $e^{\int R(x)dx}$ is an integrating factor.

Proof: The given equation is $Mdx + Ndy = 0$. (i)

Since this equation is not exact $\Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$.

Let F be an integrating factor.

To show: $F = e^{\int R(x)dx}$.

Since F be an integrating factor. Then (i) becomes

$FMdx + FNdy = 0$. Now here $\frac{\partial (FM)}{\partial y} = \frac{\partial (FN)}{\partial x} \Rightarrow F_y M + FM_y = F_x N + FN_x$. (ii)

Note: In the general case, this would be complicated and useless. So we follow the Golden Rule: If you cannot solve your problem, try to solve a simpler one--the result may be useful (and may also help you later on). Hence, we look for an integrating factor depending only on one variable; fortunately, in many practical cases, there are such factors, as we shall see.

Thus let $F = F(x),\ F_y = 0$, then (ii) becomes $\Rightarrow FM_y = F_x N + FN_x$.

$$\Rightarrow F\frac{\partial M}{\partial y} = \frac{dF}{dx}N + F\frac{\partial N}{\partial x} \Rightarrow \frac{1}{F}\frac{dF}{dx} = \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = R(x)$$

$$\Rightarrow \frac{dF}{F} = R(x)dx.$$

Integrating, we get

$$\log F = \int R(x)dx \Rightarrow F = \exp\int R(x)dx \Rightarrow F = e^{\int R(x)dx}.$$

This proves the theorem.

Theorem: If in an equation $Mdx + Ndy = 0$ which is not exact, $\dfrac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}$ depends only on y, say $R(y)$, then show that $e^{\int R(y)dy}$ is an integrating factor.

Proof: The given equation is $Mdx + Ndy = 0$. (i)

Since this equation is not exact $\Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$.

Let F be an integrating factor.

To show: $F = e^{\int R(y)dy}$.

Since F be an integrating factor. Then (i) becomes

$FMdx + FNdy = 0$. Now here $\frac{\partial(FM)}{\partial y} = \frac{\partial(FN)}{\partial x} \Rightarrow F_y M + FM_y = F_x N + FN_x$. (ii)

Note: In the general case, this would be complicated and useless. So we follow the Golden Rule: If you cannot solve your problem, try to solve a simpler one--the result may be useful (and may also help you later on). Hence, we look for an integrating factor depending only on one variable; fortunately, in many practical cases, there are such factors, as we shall see.

Thus let $F = F(y)$, $F_x = 0$, then (ii) becomes $\Rightarrow F_y M + FM_y = FN_x$.

$$\Rightarrow M\frac{\partial F}{\partial y} + F\frac{\partial M}{\partial y} = F\frac{\partial N}{\partial x} \Rightarrow \frac{1}{F}\frac{dF}{dy} = \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = R(y)$$

$$\Rightarrow \frac{dF}{F} = R(y)dy .$$

Integrating, we get

$$\log F = \int R(y)dy \Rightarrow F = \exp\int R(y)dy \Rightarrow F = e^{\int R(y)dy} .$$

This proves the theorem.

Q.No.4.: Solve $\left(xy^2 - e^{1/x^3}\right)dx - x^2 y dy = 0$.

Sol.: The given equation is $\left(xy^2 - e^{1/x^3}\right)dx - x^2 y dy = 0$

Here $M = xy^2 - e^{1/x^3}$ and $N = -x^2 y$ $\therefore \frac{\partial M}{\partial y} = 2xy$, $\frac{\partial N}{\partial x} = -2xy$

$\Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$. Thus the equation is not exact.

Now since $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \dfrac{2xy - (2xy)}{-x^2 y} = -\dfrac{4}{x}$, which is a function of x only.

$$\therefore \text{I.F.} = e^{\int\left(-\frac{4}{x}\right)dx} = e^{-4\log x} = x^{-4}$$

Multiplying throughout by x^{-4}, the equation becomes

$$\left(\frac{y^2}{x^3}-\frac{1}{x^4}e^{1/x^3}\right)dx-\frac{y}{x^2}dy=0,$$

which is an exact equation, $\because \frac{\partial M}{\partial y}=\frac{2y}{x^3}=\frac{\partial N}{\partial x}$.

$\therefore$ The solution is $\int\limits_{y\text{ const.}} Mdx+\int(\text{terms of } N \text{ not containing } x)dy=c$

$$\Rightarrow \int\left(\frac{y^2}{x^3}-\frac{1}{x^4}e^{1/x^3}\right)dx+\int(0)dy=c \Rightarrow -\frac{y^2x^{-2}}{2}+\frac{1}{3}\int e^{x^{-3}}\left(-3x^{-4}\right)dx=c$$

$\Rightarrow \frac{1}{3}e^{x^{-3}}-\frac{1}{2}\frac{y^2}{x^2}=c$, which is the required solution.

Q.No.5.: Solve $\left(xy^3+y\right)dx+2\left(x^2y^2+x+y^4\right)dy=0$.

Sol.: The given equation is $\left(xy^3+y\right)dx+2\left(x^2y^2+x+y^4\right)dy=0$.

Here $M=xy^3+y$, $N=2\left(x^2y^2+x+y^4\right)$ $\therefore \frac{\partial M}{\partial y}=3xy^2+1$, $\frac{\partial N}{\partial x}=2\left(2xy^2+1\right)$.

$\Rightarrow \frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$. Thus the equation is not exact.

Now since $\frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)=\frac{1}{y\left(xy^2+1\right)}\left(4xy^2+2-3xy^2-1\right)=\frac{1}{y}$, which is a function of y alone.

$\therefore$ I.F. $=e^{\int\left(\frac{1}{y}\right)dy}=e^{\log y}=y$.

Multiplying throughout by y, the equation becomes

$$\left(xy^4+y^2\right)dx+\left(2x^2y^3+2xy+2y^5\right)dy=0,$$

which is an exact equation, $\because \frac{\partial M}{\partial y}=4xy^3+2y=\frac{\partial N}{\partial x}$.

$\therefore$ The solution is $\int\limits_{y\text{ const.}} Mdx+\int(\text{terms of } N \text{ not containing } x)dy=c$

$\Rightarrow \frac{1}{2}x^2y^4+xy^2+\frac{1}{3}y^6=c$, which is the required solution.

(5) I.F. for the equation of the type: $x^ay^b\left(mydx+nxdy\right)+x^{a'}y^{b'}\left(m'ydx+n'xdy\right)=0$

If the equation $Mdx + Ndy = 0$ **is of the form**

$$x^a y^b\left(mydx + nxdy\right) + x^{a'} y^{b'}\left(m'ydx + n'xdy\right) = 0,$$

where a, b, a', b', m, n, m', n' **are all constants, then an integrating factor is** $x^h y^k$,

where $\dfrac{a+h+1}{m} = \dfrac{b+k+1}{n}$, $\dfrac{a'+h+1}{m'} = \dfrac{b'+k+1}{n'}$.

Q.No.6.: Solve $y\left(xy + 2x^2y^2\right)dx + x\left(xy - x^2y^2\right)dy = 0$.

Sol.: The given equation is $y\left(xy + 2x^2y^2\right)dx + x\left(xy - x^2y^2\right)dy = 0$

Here $M = xy^2 + 2x^2y^3$, $N = x^2y - x^3y^2$.

$\therefore \dfrac{\partial M}{\partial y} = 2xy + 6x^2y^2$, $\dfrac{\partial N}{\partial x} = 2xy - 3x^2y^2$.

$\Rightarrow \dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$. Thus the equation is not exact.

Now the given equation can be written as

$xy(ydx + xdy) + x^2y^2(2ydx - xdy) = 0$ and comparing with

$x^a y^b\left(mydx + nxdy\right) + x^{a'} y^{b'}\left(m'ydx + n'xdy\right) = 0$.

We have $a = b = 1$, $m = n = 1$; $a' = b' = 2$, $m' = 2$, $n' = -1$. $\therefore$ I.F. $= x^h y^k$.

where $\dfrac{a+h+1}{m} = \dfrac{b+k+1}{n}$, $\dfrac{a'+h+1}{m'} = \dfrac{b'+k+1}{n'}$

i. e. $\dfrac{1+h+1}{1} = \dfrac{1+k+1}{1}$, $\dfrac{2+h+1}{2} = \dfrac{2+k+1}{-1}$ $\Rightarrow h - k = 0$, $h + 2k + 9 = 0$

Solving these, we get $h = k = -3$. $\therefore$ I.F. $= \dfrac{1}{x^3y^3}$.

Multiplying throughout by $\dfrac{1}{x^3y^3}$, the equation becomes

$\left(\dfrac{1}{x^2y} + \dfrac{2}{x}\right)dx + \left(\dfrac{1}{xy^2} - \dfrac{1}{y}\right)dy = 0$, which is an exact equation, $\because \dfrac{\partial M}{\partial y} = -\dfrac{1}{x^2y^2} = \dfrac{\partial N}{\partial x}$.

$\therefore$ The solution is $\displaystyle\int\limits_{y\text{ const.}} Mdx + \int (\text{terms of } N \text{ not containing } x)dy = c$

$$\Rightarrow \frac{1}{y}\left(-\frac{1}{x}\right) + 2\log x - \log y = c \Rightarrow 2\log x - \log y - \frac{1}{xy} = c,$$

which is the required solution.

Let us understand more about the above methods by solving the following problems:

Q.No.7.: Solve the equation $xdy - ydx + a\left(x^2 + y^2\right)dx = 0$.

Sol.: The given equation is $xdy - ydx + a\left(x^2 + y^2\right)dx = 0$.

Dividing throughout by $\left(x^2 + y^2\right)$, we get

$$\frac{xdy - ydx}{x^2 + y^2} + adx = 0 \Rightarrow d\left(\tan^{-1}\frac{y}{x}\right) + adx = 0.$$

Integrating both sides, we get $\tan^{-1}\frac{y}{x} + ax = c$,

which is the required solution.

Q.No.8.: Solve the equation $xdx + ydy = \dfrac{a^2(xdy - ydx)}{x^2 + y^2}$.

Sol.: The given equation is $xdx + ydy = \dfrac{a^2(xdy - ydx)}{x^2 + y^2}$

$$\Rightarrow xdx + ydy = a^2 d\left(\tan^{-1}\frac{y}{x}\right)$$

Integrating both sides, we get

$$\frac{x^2}{2} + \frac{y^2}{2} = a^2\tan^{-1}\frac{y}{x} + c' \Rightarrow x^2 + y^2 - 2a^2\tan^{-1}\frac{y}{x} = c, \qquad \text{where } (c = 2c')$$

which is the required solution.

Q.No.9.: Solve the equation $ydx - xdy + \log xdx = 0$.

Sol.: The given equation is $ydx - xdy + \log xdx = 0$

$$\Rightarrow (y + \log x)dx - xdy = 0$$

Here $M = y + \log x$ and $N = -x$

$\therefore \dfrac{\partial M}{\partial y} = 1$, $\dfrac{\partial N}{\partial x} = -1$ $\Rightarrow \dfrac{\partial M}{\partial y} \ne \dfrac{\partial N}{\partial x}$. Thus the equation is not exact.

Now $\frac{1}{N}\left[\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right]=\frac{1}{-x}\left[1-(-1)\right]=\frac{-2}{x}$, which is a function a of x alone.

$\therefore$ I. F. $= e^{\int\left(-\frac{2}{x}\right)dx} = e^{-2\log x} = e^{\log x^{-2}} = x^{-2} = \frac{1}{x^2}$.

Multiplying throughout by x^{-2}, the equation becomes $\left(\frac{y+\log x}{x^2}\right)dx+\left(\frac{-x}{x^2}\right)dy=0$.

Now here $M=\frac{y+\log x}{x^2}$ and $N=-\frac{1}{x}$.

$\therefore \frac{\partial M}{\partial y}=\frac{1}{x^2}=\frac{\partial N}{\partial x}$, which is an exact equation.

$\therefore$ The solution of equation is $\int\limits_{y\ \text{const.}} M dx+\int(\text{terms of } N \text{ not containing } x)dy=c$

$\Rightarrow \int\left(\frac{y+\log x}{x^2}\right)dx+\int(0)dy=c \qquad \Rightarrow \int\left(\frac{y}{x^2}+\frac{\log x}{x^2}\right)dx=c$

$\Rightarrow y\frac{x^{-1}}{-1}+\left[-\log x.\frac{1}{x}+\int\frac{1}{x}.\frac{1}{x}dx\right]=c \qquad \Rightarrow -\frac{y}{x}-\frac{\log x}{x}-\frac{1}{x}=c$

$\Rightarrow y+cx+\log x+1=0$,

which is the required solution.

Q.No.10.: Solve the equation $\frac{dy}{dx}=\frac{x^3+y^3}{xy^2}$.

Sol.: The given equation is $\frac{dy}{dx}=\frac{x^3+y^3}{xy^2}$

$\Rightarrow xy^2dy=\left(x^3+y^3\right)dx \quad \Rightarrow \left(x^3+y^3\right)dx-xy^2dy=0$ (i)

Here $M=\left(x^3+y^3\right)$ and $N=xy^2$ $\therefore \frac{\partial M}{\partial y}=3y^2$, $\frac{\partial N}{\partial x}=-y^2$.

$\Rightarrow \frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$. Thus the equation is not exact.

Now since the given equation is homogeneous in x and y.

$\therefore$ I. F. $=\frac{1}{Mx+Ny}=\frac{1}{x^4+y^3x-xy^3}=\frac{1}{x^4}$.

Multiplying throughout by $\frac{1}{x^4}$, the equation becomes

$\left(\frac{1}{x}+\frac{y^3}{x^4}\right)dx-\frac{y^2}{x^3}dy=0$, which is an exact equation, $\because \frac{\partial M}{\partial y}=\frac{3y^2}{x^4}=\frac{\partial N}{\partial x}$.

$\therefore$ The solution of equation is $\int\limits_{y\ \text{const.}} Mdx+\int(\text{terms of } N \text{ not containing } x)dy=c$

$$\Rightarrow \int\left(\frac{1}{x}+\frac{y^3}{x^4}\right)dx+\int(0)dy=c \Rightarrow \log x+\frac{y^3}{x^3(-3)}=c'$$

$$\Rightarrow 3\log x-\left(\frac{y}{x}\right)^3=c',\quad [\text{where } c'=3c]$$

which is the required solution.

Q.No.11.: Solve the equation $\left(x^3y^2+x\right)dy+\left(x^2y^3-y\right)dx=0$ **.**

Sol.: The given equation is $\left(x^3y^2+x\right)dy+\left(x^2y^3-y\right)dx=0$

Here $N=x^3y^2+x$ and $M=x^2y^3-y$ $\therefore \frac{\partial M}{\partial y}=3x^2y^2-1$, $\frac{\partial N}{\partial x}=3x^2y^3+1$.

$\Rightarrow \frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$. Thus the equation is not exact.

Now since the given equation is of the form $f_1(xy)ydx+f_2(xy)xdy=0$.

$$\therefore \text{I. F.}=\frac{1}{Mx-Ny}=\frac{1}{x^3y^3-xy-x^3y^3-xy}=-\frac{1}{2xy}.$$

Multiplying throughout by $-\frac{1}{2xy}$, the equation becomes.

$\left(\frac{-xy^2}{2}+\frac{1}{2x}\right)dx+\left(\frac{-x^2y}{2}-\frac{1}{2y}\right)dy=0$, which is an exact equation, $\because \frac{\partial M}{\partial y}=-xy=\frac{\partial N}{\partial x}$.

$\therefore$ The solution of equation is $\int\limits_{y\ \text{const.}} Mdx+\int(\text{terms of } N \text{ not containing } x)dy=c$

$$\Rightarrow \int\left(\frac{-xy^2}{2}+\frac{1}{2x}\right)dx+\int\left(-\frac{1}{2y}\right)dy=c \Rightarrow -\frac{x^2y^2}{4}+\frac{1}{2}\log x-\frac{1}{2}\log y=c$$

$$\Rightarrow \log\frac{y}{x}+\frac{1}{2}x^2y^2=c',\qquad [\text{where } c'=2c]$$

which is the required solution.

Q.No.12.: Solve the equation $\left(x^2y^2 + xy + 1\right)ydx + \left(x^2y^2 - xy + 1\right)xdy = 0$.

Sol.: The given equation is $\left(x^2y^2 + xy + 1\right)ydx + \left(x^2y^2 - xy + 1\right)xdy = 0$

Here $M = \left(x^2y^3 + xy^2 + y\right)$, $N = \left(x^3y^2 - x^2y + x\right)$.

$\Rightarrow \dfrac{\partial M}{\partial y} = 3x^2y^2 + 2xy + 1$, $\dfrac{\partial N}{\partial x} = 3x^2y^2 - 2xy + 1$.

$\Rightarrow \dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$. Thus the equation is not exact.

Now since the given equation is of the form $f_1(xy)ydx + f_2(xy)xdy = 0$.

$$\therefore \text{I. F.} = \frac{1}{Mx - Ny} = \frac{1}{\left(x^2y^2 + xy^2 + y\right)x - \left(x^3y^2 - x^2y + x\right)y} = \frac{1}{2x^2y^2}$$

Multiplying throughout by $\dfrac{1}{2x^2y^2}$, the equation becomes.

$$\left(\frac{y}{2} + \frac{1}{2x} + \frac{1}{2x^2y}\right)dx + \left(\frac{x}{2} - \frac{1}{2y} + \frac{1}{2xy^2}\right)dy = 0,$$ which is an exact equation.

$\because$ Here $M = \left(\dfrac{y}{2} + \dfrac{1}{2x} + \dfrac{1}{2x^2y}\right)$ and $N = \left(\dfrac{x}{2} - \dfrac{1}{2y} + \dfrac{1}{2xy^2}\right)$ $\therefore \dfrac{\partial M}{\partial y} = \dfrac{1}{2} + \dfrac{\log y}{2x^2} = \dfrac{\partial N}{\partial x}$.

$\therefore$ The solution is $\displaystyle\int_{y\text{ const.}} Mdx + \int (\text{terms of } N \text{ not containing } x)dy = c$.

$$\Rightarrow \int \left(\frac{y}{2} + \frac{1}{2x} + \frac{1}{2x^2y}\right)dx + \int \left(-\frac{1}{2y}\right)dy = c.$$

$$\Rightarrow \frac{xy}{2} + \frac{1}{2}\log x - \frac{1}{2xy} - \frac{1}{2}\log y = c \quad \Rightarrow xy + \log\frac{x}{y} - \frac{1}{xy} = c', \quad [\text{where } c' = 2c]$$

which is the required solution.

Q.No.13.: Solve the equation $\left(y^4 + 2y\right)dx + \left(xy^3 + 2y^4 - 4x\right)dy = 0$.

Sol.: The given equation is $\left(y^4 + 2y\right)dx + \left(xy^3 + 2y^4 - 4x\right)dy = 0$

Here $M = y^4 + 2y$ and $N = xy^3 + 2y^4 - 4x$.

$\therefore \dfrac{\partial M}{\partial y} = 4y^3 + 2$ and $\dfrac{\partial N}{\partial x} = y^3 - 4 \Rightarrow \dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$. Thus the equation is not exact.

Now $\dfrac{\dfrac{\partial N}{\partial x}-\dfrac{\partial M}{\partial y}}{M}=\dfrac{\left(y^3-4\right)-\left(4x^3+2\right)}{y^4+2y}=\dfrac{-3y^3-6}{y^4+2y}=-\dfrac{3}{y}$, which is a function of y alone.

$\therefore$ I. F. $=e^{\int\left(-\frac{3}{y}\right)dy}=e^{-3\log y}=\dfrac{1}{y^3}$.

Multiplying throughout by $\dfrac{1}{y^3}$, the equation becomes.

$\left(y+\dfrac{2}{y^2}\right)dx+\left(x+2y-\dfrac{4x}{y^3}\right)dy=0$, which is an exact equation, $\because \dfrac{\partial M}{\partial y}=1-\dfrac{4}{y^3}=\dfrac{\partial N}{\partial x}$

$\therefore$ The solution of equation is $\displaystyle\int\limits_{y\text{ const.}} M dx+\int(\text{terms of } N \text{ not containing } x)dy=c$

$\Rightarrow \displaystyle\int\left(y+\frac{2}{y^2}\right)dx+\int 2y dy=c \Rightarrow yx+\frac{2x}{y^2}+\frac{2y^2}{2}=c \Rightarrow x\left(y+\frac{2}{y^2}\right)+y^2=c$,

which is the required solution.

Q.No.14.: Solve the equation $\left(3xy-2ay^2\right)dx+\left(x^2-2axy\right)dy=0$.

Sol.: The given equation is $\left(3xy-2ay^2\right)dx+\left(x^2-2axy\right)dy=0$

Here $M=3xy-2ay^2$ and $N=x^2-2axy$ $\therefore \dfrac{\partial M}{\partial y}=3x-4ay$, $\dfrac{\partial N}{\partial x}=2x-2ay$

$\Rightarrow \dfrac{\partial M}{\partial y}\neq\dfrac{\partial N}{\partial x}$. Thus the equation is not exact.

Now $\dfrac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{N}=\dfrac{(3x-4ay)-(2x-2ay)}{x^2-2axy}=\dfrac{x-2ay}{x(x-2ay)}=\dfrac{1}{x}$,

which is a function of x alone.

$\therefore$ I. F. $=e^{\int\left(\frac{1}{x}\right)dx}=e^{\log x}=x$.

Multiplying throughout by x, the equation becomes.

$\left(3x^2y-2axy^2\right)dx+\left(x^3-2ax^2y\right)dy=0$,

which is an exact equation, $\because \dfrac{\partial M}{\partial y}=3x^2-4axy=\dfrac{\partial N}{\partial x}$

$\therefore$ The solution of the equation is $\int\limits_{y\text{ const.}} Mdx + \int (\text{terms of } N \text{ not containing } x)dy = c'$

$\int (3x^2y - 2axy^2)dx + \int (0)dy = c'$

$\Rightarrow 3y\dfrac{x^3}{3} - 2ay^2\dfrac{x^2}{2} = c' \quad \Rightarrow yx^3 - ax^2y^2 = c'$

$\Rightarrow x^2(ay^2 - xy) = c,$ [where $c = -c'$]

which is the required solution.

Q.No.15.: Solve the equation $x^4\dfrac{dy}{dx} + x^3y + \operatorname{cosec}(xy) = 0$.

Sol.: The given equation is $x^4\dfrac{dy}{dx} + x^3y + \operatorname{cosec}(xy) = 0$

$\Rightarrow [x^3y + \operatorname{cosec}(xy)]dx + x^4dy = 0$ (i)

Here $M = [x^3y + \operatorname{cosec}(xy)]$ and $N = x^4$

$\therefore \dfrac{\partial M}{\partial y} = x^3 - \operatorname{cosec}(xy)\cot(xy)x, \quad \dfrac{\partial N}{\partial x} = 4x^3$

$\Rightarrow \dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$. Thus the equation is not exact.

Now equation (i) can also be written as $\left(1 + \dfrac{\operatorname{cosec} xy}{xy}\right)ydx + (1)xdy = 0$.

Now comparing with $f_1(xy)ydx + f_2(xy)xdy = 0$

$\therefore$ I. F. $= \dfrac{1}{Mx - Ny} = \dfrac{1}{x^4y + x^3\operatorname{cosec} xy - x^4y} = \dfrac{1}{x^3\operatorname{cosec} xy}$

Multiplying throughout by $\dfrac{1}{x^3\operatorname{cosec} xy}$, the equation becomes

$\left(y\sin xy + \dfrac{1}{x}\right)dx + x\sin xy\,dy = 0$, which is an exact equation.

$\because$ Here $M = y\sin xy + \dfrac{1}{x}$ and $N = x\sin xy$ $\therefore \dfrac{\partial M}{\partial y} = \sin xy + y\cos xy.x = \dfrac{\partial N}{\partial x}$.

$\therefore$ The solution of the equation is $\int\limits_{y\text{ const.}} Mdx + \int (\text{terms of } N \text{ not containing } x)dy = c'$

$$\int\left(y\sin xy+\frac{1}{x}\right)dx+\int(0)dy=c'$$

$$\Rightarrow \frac{-y\cos xy}{y}-\frac{1}{2x^2}=c' \Rightarrow 2\cos xy+x^{-2}=c,\ [\text{where } c=-2c']$$

which is the required solution.

Or (Another simple method)

The given equation is $x^4\frac{dy}{dx}+x^3y+\operatorname{cosec}(xy)=0$

$$\Rightarrow x^3\left[x\frac{dy}{dx}+y\right]+\operatorname{cosec}xy=0 \qquad \text{(i)}$$

Put $xy=z$ and therefore $y+x\frac{dy}{dx}=\frac{dz}{dx}$.

Thus (i) $\Rightarrow x^3\left[\frac{dz}{dx}\right]+\operatorname{cosec}z=0 \Rightarrow \frac{dx}{x^3}=-\sin z dz$

$\Rightarrow -\frac{1}{2x^2}=\cos z+c \Rightarrow 2\cos xy+x^{-2}=c$, which is the required solution.

Q.No.16.: Solve the equation $\left(x^4e^x-2mxy^2\right)dx+2mx^2ydy=0$**.**

Sol.: The given equation is $\left(x^4e^x-2mxy\right)dx+2mx^2ydy=0$**,**

Here $M=x^4e^x-2mxy^2$ and $N=2mx^2y$

$\therefore \frac{\partial M}{\partial y}=-4mxy$, $\frac{\partial N}{\partial x}=4mxy \Rightarrow \frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$. Thus the equation is not exact.

Now $\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=\frac{1}{2mx^2y}[-4mxy-4mxy]=\frac{-8mxy}{2mx^2y}=-\frac{4}{x}$, which is a function of x alone.

$\therefore$ I. F. $=e^{\left(\int -\frac{4}{x}\right)dx}=e^{-4\log x}=e^{\log x^{-4}}=x^{-4}$.

Multiplying throughout by x^{-4}, the equation becomes.

$\left(e^x-\frac{2my^2}{x^3}\right)dx+\left(\frac{2my}{x^2}\right)dy$, which is an exact equation, $\because \frac{\partial M}{\partial y}=-\frac{4xy}{x^3}=\frac{\partial N}{\partial x}$.

$\therefore$ The solution of the equation is $\displaystyle\int_{y\text{ const.}} Mdx + \int (\text{terms of N not containing x})dy = c$

$$\Rightarrow \int \left(e^x - \frac{2my^2}{x^3}\right)dx + \int (0)dy = c \Rightarrow e^x + \frac{my^2}{x^2} = c \Rightarrow e^x + m\left(\frac{y}{x}\right)^2 = c,$$

which is the required solution.

Q.No.17.: Solve the equation $ydx - xdy + 3x^2y^2e^{x^3}dx = 0$.

Sol.: The given equation is $ydx - xdy + 3x^2y^2e^{x^3}dx = 0$

$$\Rightarrow \frac{ydx - xdy}{y^2} = -3x^2e^{x^3} \quad \Rightarrow d\left(\frac{x}{y}\right) = -3x^2e^{x^3}$$

Integrating both sides, we get

$$\int d\left(\frac{x}{y}\right)dx = -\int 3x^2e^{x^3}dx + c', \qquad \text{Putting } x^3 = t \text{, so that } 3x^2dx = dt$$

$$\therefore \int d\left(\frac{x}{y}\right)dx = -\int e^t dt + c' \Rightarrow \frac{x}{y} = -e^t + c' \quad \Rightarrow \frac{x}{y} + e^{x^3} + c = 0, \quad [\text{where } c = -c']$$

which is the required solution.

Q.No.18.: Solve the equation $\left(y^2 + 2x^2y\right)dx + \left(2x^3 - xy\right)dy = 0$.

Sol.: The given equation is $\left(y^2 + 2x^2y\right)dx + \left(2x^3 - xy\right)dy = 0$ (i)

Here $M = y^2 + 2x^2y$ and $N = 2x^3 - xy$

$\therefore \dfrac{\partial M}{\partial y} = 2y + 2x^2, \quad \dfrac{\partial N}{\partial x} = 6x^2 - y \Rightarrow \dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$. Thus the equation is not exact.

Now the equation (i) can be written as

$$x^2y^0(2ydx + 2xdy) + x^0y^1(ydx - xdy) = 0$$

Comparing with $x^ay^b(mydx + nxdy) + x^{a'}y^{b'}(m'ydx + n'xdy) = 0$, we get

$a = 2,\ b = 0,\ m = 2,\ n = 2,\ a' = 0,\ b' = 1,\ m' = 1,\ n' = -1,$

$\therefore$ I. F. $= x^hy^k$

where $\dfrac{a+h+1}{m} = \dfrac{b+k+1}{n}, \quad \dfrac{a'+h+1}{m'} = \dfrac{b'+k+1}{n'}$

$$\Rightarrow \frac{2+h+1}{2}=\frac{0+k+1}{2}, \quad \frac{0+h+1}{1}=\frac{1+k+1}{-1}$$

$$\Rightarrow \frac{h+3}{2}=\frac{k+1}{2}, \quad \frac{h+1}{1}=\frac{k+2}{-1} \quad \Rightarrow h+3-k-1=0, \quad h+1+k+2=0$$

Solving we get $h=-\frac{5}{2},\ k=-\frac{1}{2}$. $\quad \therefore$ I. F. $= x^{-5/2}y^{-1/2}$

Multiplying throughout by $x^{-5/2}y^{-1/2}$, the equation becomes.

$$\left(y^2.x^{-5/2}y^{-1/2}+2x^2yx^{-5/2}y^{-1/2}\right)dx+\left(2x^3x^{-5/2}y^{-1/2}-xyx^{-5/2}y^{-1/2}\right)dy=0$$

$$\Rightarrow \left(x^{-5/2}y^{3/2}+2x^{-1/2}y^{1/2}\right)dx+\left(2x^{1/2}y^{-1/2}-x^{-3/2}y^{1/2}\right)dy=0$$

$\therefore$ The solution of the equation is $\int\limits_{y\ \text{const.}} Mdx+\int(\text{terms of } N \text{ not containing } x)dy=c$

$$\Rightarrow \int\left(x^{-5/2}y^{3/2}+2x^{-1/2}y^{1/2}\right)dx+\int(0)dy=c$$

$$\Rightarrow y^{3/2}.\frac{x^{-\frac{5}{2}+1}}{-\frac{5}{2}+1}+2y^{1/2}\frac{x^{1/2}}{\frac{1}{2}}=c \quad \Rightarrow -\frac{2}{3}\left(\frac{y}{x}\right)^{3/2}+4(xy)^{1/2}=c,$$

which is the required solution.

Q.No.19.: Solve the differential equation $xdy-ydx=x\sqrt{x^2-y^2}dx$.

Sol.: The given equation is $xdy-ydx=x^2\sqrt{1-\left(\frac{y}{x}\right)^2}dx \Rightarrow \dfrac{\frac{xdy-ydx}{x^2}}{\sqrt{1-\left(\frac{y}{x}\right)^2}}dx$

$\Rightarrow d\left(\sin^{-1}\frac{y}{x}\right)=dx$, which is exact differential equation.

Integrating, we get

$\sin^{-1}\frac{y}{x}=x+c \Rightarrow y=x\sin(x+c)$, which is the required solution.

Q.No.20.: Solve the differential equation $y\left(xy+2x^2y^2\right)dx+x\left(xy-x^2y^2\right)dy=0$.

Sol.: The given equation is of the form $f_1(xy)ydx+f_2(xy)xdy=0$.

Here $M=xy^2+2x^2y^3$ and $N=x^2y-x^3y^2$

Now $Mx - Ny = x^2y^2 + 2x^3y^3 - x^2y^2 + x^3y^3 = 3x^3y^3 \neq 0$

$\therefore$ I.F. $\dfrac{1}{Mx - Ny} = \dfrac{1}{3x^3y^3}$.

Multiplying throughout by $\dfrac{1}{3x^3y^3}$, the given equation becomes

$\left(\dfrac{1}{3x^2y} + \dfrac{2}{3x}\right)dx + \left(\dfrac{1}{3xy^2} - \dfrac{1}{3y}\right)dy = 0$, which is exact.

The solution is

$$\int_{y\ \text{constant}} \left(\frac{1}{3x^2y} + \frac{2}{3x}\right)dx + \int -\frac{1}{3y}dy = c \quad \Rightarrow -\frac{1}{3y} + \frac{2}{3}\log x - \frac{1}{3}\log y = c$$

$\Rightarrow -\dfrac{1}{xy} + 2\log x - \log y = C$, where C = 3c, which is the required solution.

Q.No. 21.: Solve the differential equation $(2x^2y^2 + y)dx + (3x - x^3y)dy = 0$.

Sol.: The equation can be written as $2(x^2y^2dx - x^3ydy) + (ydx + 3xdy) = 0$

$\Rightarrow x^2y(2ydx - xdy) + x^0y^0(ydx + 3xdy) = 0$.

Therefore, it has an I.F. of the form x^hy^k.

Multiplying the given equation by x^hy^k, we get

$(2x^{h+2}y^{k+2} + x^hy^{k+1})dx + (3x^{h+1}y^k - x^{h+3}y^{k+1})dy = 0$.

For this equation to be exact, we must have

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

i.e., $2(k+2)x^{h+2}y^{k+1} + (k+1)x^hy^k = 3(h+1)x^hy^k - (h+3)x^{h+2}y^{k+1}$.

which holds when $2(k+2) = -(h+3)$ and $k+1 = 3(h+1)$

i.e. when $h + 2k + 7 = 0$ and $3h - k + 2 = 0$.

Solving these equation, we have $h = -\dfrac{11}{7}$, $k = -\dfrac{19}{7}$

$\therefore$ I.F. $= x^{-\frac{11}{7}}y^{-\frac{19}{7}}$.

Multiplying the given equation by $x^{-\frac{11}{7}}y^{-\frac{19}{7}}$, we have

$$\left(2x^{\frac{3}{7}}y^{-\frac{5}{7}}+x^{-\frac{11}{7}}y^{-\frac{12}{7}}\right)dx+\left(3x^{-\frac{4}{7}}y^{-\frac{19}{7}}-x^{\frac{10}{7}}y^{-\frac{12}{7}}\right)dy=0,$$ which is an exact equation.

The solution is $$\int_{y\text{ constant}}\left(2x^{\frac{3}{7}}y^{-\frac{5}{7}}+x^{-\frac{11}{7}}y^{-\frac{12}{7}}\right)dx=c$$

$$\Rightarrow \frac{7}{5}x^{\frac{10}{7}}y^{-\frac{5}{7}}-\frac{7}{4}x^{-\frac{4}{7}}y^{-\frac{12}{7}}=c \quad \Rightarrow 4x^{\frac{10}{7}}y^{-\frac{5}{7}}-5x^{-\frac{4}{7}}y^{-\frac{12}{7}}=C,$$ where $C=\frac{20}{7}=c$,

which is the required solution.

Q.No. 22.: Solve the differential equation $y\left(y^3-x\right)dx+x\left(y^3+x\right)dy=0$.

Sol.: The equation can be written as $y^4dx-xydx+xy^3dy+x^2dy=0$**.**

Regrouping the terms, we obtain

$$y^3(ydx+xdy)+x(xdy-ydx)=0\Rightarrow y^3d(xy)+x.x^2d\left(\frac{y}{x}\right)=0$$

$$\Rightarrow d(xy)+\left(\frac{x}{y}\right)^3d\left(\frac{y}{x}\right)=0\Rightarrow d(xy)+\left(\frac{y}{x}\right)^{-3}d\left(\frac{x}{y}\right)=0$$

Integrating both sides, we get

$$xy-\frac{1}{2}\left(\frac{y}{x}\right)^{-2}=c_1$$

Rearranging, we get $2xy^3-x^2=cy^2$ where $c=2c_1$, which is the required solution.

Q.No. 23.: Solve the differential equation $\left(x^3y^3+1\right)dx+x^4y^2dy=0$.

Sol.: The equation can be written as $x^3y^3dx+dx+x^4y^2dy=0$.

Dividing by x throughout, we get

$$x^2y^3dx+\frac{dx}{x}+x^3y^2dy=0$$

Regrouping, we get

$$x^2y^2(ydx+xdy)+\frac{dx}{x}=0\Rightarrow (xy)^2d(xy)+\frac{dx}{x}=0$$

$\Rightarrow \dfrac{(xy)^3}{3} + \log x = c$, which is the required solution.

Q.No. 24.: Solve the differential equation $y\left(x^3e^{xy} - y\right)dx + x\left(xy + x^3e^{xy}\right)dy = 0$.

Sol.: The equation can be written as $yx^3e^{xy}dx - y^2dx + xydy + x^4e^{xy}dy = 0$.

Regrouping, we get

$$x^3e^{xy}(ydx + xdy) + y(xdy - ydx) = 0$$

$$x^3d\left(e^{xy}\right) + y.x^2.\left(\frac{xdy - ydx}{x^2}\right) = 0.$$

Dividing throughout by x^3, we get

$$d\left(e^{xy}\right) + \frac{y}{x}.d\left(\frac{y}{x}\right) = 0.$$

Integrating both sides, we get

$e^{xy} + \left(\dfrac{y}{x}\right)^2 \dfrac{1}{2} = c$, which is the required solution.

Q.No. 25.: Solve the differential equation $\left(x^{n+1}.y^n + ay\right)dx + \left(x^n y^{n+1} + ax\right)dy = 0$.

Sol.: Regrouping the terms, we get

$$x^n y^n(xdx + ydy) + a(ydx + xdy) = 0.$$

Dividing throughout by $x^n y^n$, we get

$$(xdx + ydy) + \frac{ad(xy)}{(xy)^n} = 0.$$

If $n \neq 1$

Integrating, we get $\dfrac{x^2 + y^2}{2} + a.\dfrac{1}{(-n+1)(xy)^{n-1}} = c_0$

$(n-1)\left(x^2 + y^2 - c\right)(xy)^{n-1} = 2a$, where $c = 2c_0$

If $n = 1$, $\dfrac{x^2 + y^2}{2} + a\log xy = c$, which is the required solution.

Q.No. 26.: Solve the differential equation $y\left(x^2y^2 - 1\right)dx + x\left(x^2y^2 + 1\right)dy = 0$.

Sol.: The equation can be written as $x^2y^3dx - ydx + x^3y^2dy + xdy = 0$

Regrouping, we get

$$x^2y^2(ydx + xdy) + (xdy - ydx) = 0 \Rightarrow x^2y^2d(xy) + x^2\frac{(xdy - ydx)}{x^2} = 0$$

$$\Rightarrow x^2y^2d(xy) + x^2d\left(\frac{y}{x}\right) = 0.$$

The second term in LHS becomes an exact differential if k = n + 2, while the second term in LHS becomes an exact differential if $n = -k$. Solving these two equations,

$k = n + 2$ and $n = -k$, we get $n = -1$ and $k = 1$.

Substituting these values, the DE reduces to

$$x'.y^{-1+2}d(xy) + x'y^{-1}d\left(\frac{y}{k}\right) = 0 \Rightarrow xyd(xy) + \frac{x}{y}d\left(\frac{y}{x}\right) = 0 \Rightarrow xyd(xy) + \frac{d\left(\frac{y}{x}\right)}{\left(\frac{y}{x}\right)} = 0$$

Integrating both sides, we get

$\frac{(xy)^2}{2} + \log\left(\frac{y}{x}\right) = c$, which is the required solution.

Q.No. 27.: Solve the differential equation $y(2x^2 - xy + 1)dx + (x - y)dy = 0$.

Sol.: Here $M = 2yx^2 - xy^2 + y$, $N = x - y$

$\frac{\partial M}{\partial y} = 2x^2 - 2xy + 1 \neq 1 = \frac{\partial N}{\partial x}$, which is not exact.

Since $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{2x^2 - 2xy + 1 - 1}{x - y} = 2x = f(x)$ is a function of x only.

We get an I.F. as

$IF = e^{\int 2xdx} = e^{x^2}$.

Multiplying the given DE by e^{x^2}, we get

$$ye^{x^2}(2x^2 - xy + 1)dx + e^{x^2}(x - y)dy = 0$$

which is of the form

$M^*dx + N^*dy = 0$.

This is exact, since $\frac{\partial M^*}{\partial N} = \left(2x^2 - 2xy + 1\right)e^{x^2} = \frac{\partial N^*}{\partial x}$

Integrating, we get

$$\frac{\partial f}{\partial y} = e^{x^2}(x - y) = N^*$$

Integrating partially w.r.t. y, we have

$$f(x, y) = e^{x^2}.\left(xy - \frac{y^2}{2}\right) + h(x).$$

Differentiating partially w.r.t. and equating it to M*, we get

$$e^{x^2}\left[xy - \frac{y^2}{2}\right].2x + e^{x^2}[y] + \frac{dh}{dx}$$

$$= M^* = ye^{x^2}\left(2x^2 - xy + 1\right)$$

Simplifying, $\frac{dh}{dx} = 0$, so h is constant.

Thus, the general solution is

$$e^{x^2}\left(2xy - y^2\right) = c.$$

Q.No. 28.: Solve the differential equation $(x - y)dx - dy = 0$, $y(0) = 2$.

Sol.: $M = x - y$**,** $N = -1$**,** $M_y = -1$**,** $N_x = 0$**,** not exact.

$\frac{1}{N}\left(M_y - N_x\right) = \frac{1}{-1}[-1 - 0] = 1$ is a function of x.

$IF = e^{\int 1dx} = e^x$.

Multiplying DE by IF, we get $(x - y)e^x dx - e^x dy = 0$.

Rewriting, we get

$$xe^x dx - ye^x dx - e^x dy = 0 \Rightarrow xe^x dx - \left[d\left(ye^x\right)\right] = 0$$

$$\Rightarrow xe^x dx + e^x dx - e^x dx - d\left(ye^x\right) = 0 \Rightarrow (x - 1)e^x dx + e^x dx - d\left(ye^x\right) = 0$$

$$\Rightarrow d\left[(x - 1)e^x\right] - d\left(ye^x\right) = 0$$

The solution is

$$(x - 1)e^x - ye^{-x} = c.$$

Put $x = 0$, $y = 2$, so that $c = -3$

$\therefore (x-1)e^{x} - ye^{-x} = -3$ is the particular solution.

Q.No. 29.: Solve the differential equation $\left(3x^2y^4 + 2xy\right)dx + \left(2x^3y^3 - x^2\right)dy = 0$.

Sol.: Here $M = 3x^2y^4 + 2xy$, $N = 2x^3y^3 - x^2$.

$M_y = 12x^2y^3 + 2x \neq 6x^2y^3 - 2x = N_x$, Not exact

Since

$$\frac{1}{M}\left(N_x - M_y\right) = \frac{6x^2y^3 - 2x - \left(12x^2y^3 + 2x\right)}{3x^2y^4 + 2xy} = -\frac{2}{y} = g(x) = \text{function of y alone.}$$

We get an IF $= e^{\int g(y)dy} = e^{-\int \frac{2}{y}dy} = e^{-2\log y}e = \dfrac{1}{y^2}$.

Multiplying the given DE throughout by $\dfrac{1}{y^2}$, we get

$$\left(3x^2y^2 + \frac{2x}{y}\right)dx + \left(2x^3y + \frac{x^2}{y^2}\right)dy = 0.$$

Since $M_y^{*} = 6x^2y - \dfrac{2x}{y^2} = N_x^{*}$, this DE is exact.

Rearranging the terms, we get

$$\left(3x^2y^2dx + 2x^3ydy\right) + \left(\frac{2x}{y}dx - \frac{x^2}{y^2}dy\right) = 0$$

$$\Rightarrow y^2d\left(x^3\right) + x^3d\left(y^2\right) + \frac{1}{y}d\left(x^2\right) + x^2d\left(\frac{1}{y}\right) = 0$$

Regrouping, we get $d\left(x^3y^2\right) + d\left(\dfrac{x^2}{y}\right) = 0$.

Integrating, we get

$x^3y^2 + \dfrac{x^2}{y} = c$, which is the required solution.

Q.No. 30.: Solve $y\left(y^2 - 2x^2\right)dx + x\left(2y^2 - x^2\right)dy = 0$.

Sol.: Here $M = y(y^2 - 2x^2)$ and $N = x\left(2y^2 - x^2\right)$ are both homogeneous functions of degree 3.

Since $M + yN = xy\left(y^2 - 2x^2\right) + yx\left(2y^2 - x^2\right) = 3(xy)\left(y^2 - x^2\right) \neq 0$ unless y = x, DE has an IF

IF$= \dfrac{1}{xM + yN} = \dfrac{1}{3xy\left(y^2 - x^2\right)}$.

Multiplying the DE by IF, we get $\dfrac{y\left(y^2 - 2x^2\right)}{3xy\left(y^2 - x^2\right)}dx + \dfrac{x\left(2y^2 - x^2\right)}{3xy\left(y^2 - x^2\right)} = 0$

Rewriting, we get $\dfrac{\left(y^2 - x^2\right) - x^2}{x\left(y^2 - x^2\right)}dx + \dfrac{y^2 + \left(y^2 - x^2\right)}{y\left(y^2 - x^2\right)}dy = 0$

$\Rightarrow \dfrac{dx}{x} - \dfrac{xdx}{y^2 - x^2} + \dfrac{ydy}{y^2 - x^2} + \dfrac{dy}{y} = 0$.

Regrouping, we get $d(\log xy) + \dfrac{1}{2}\dfrac{d\left(y^2 - x^2\right)}{\left(y^2 - x^2\right)} = 0$

Integrating $d\left(\log\left\{x^2y^2\left(y^2 - x^2\right)\right\}\right) = 0$, we get

$\log x^2y^2\left(y^2 - x^2\right) = c$

$\Rightarrow x^2y^2\left(y^2 - x^2\right) = c_1$ where $c_1 = e^c$, which is the required solution.

Q.No. 31.: Solve the differential equation $\left(x^2y^2 + xy + 1\right)ydx + \left(x^2y^2 - xy + 1\right)xdy = 0$.

Sol.: $M = \left(x^2y^2 + xy + 1\right)y$, $N = \left(x^2y^2 - xy + 1\right)x$ so

$M_y = 3x^2y^2 + 2xy + 1 \neq 3x^2y^2 - 2xy + 1 = N_x$.

DE is not exact. But $M = yg(xy)$ and $N = xh(xy)$, so the given DE is of the form

$yg(xy)dx + xh(xy)dy = 0$,

which has an integrating factor given by

$$\frac{1}{xM - yN} = \frac{1}{xy\left(x^2y^2 + xy + 1\right) - xy\left(x^2y^2 - xy + 1\right)} = \frac{1}{2x^2y^2}.$$

Multiplying DE with IF $\dfrac{1}{2x^2y^2}$, we get

$$\frac{\left(x^2y^2+xy+1\right)ydx}{x^2y^2}+\frac{\left(x^2y^2-xy+1\right)xdy}{x^2y^2}=0.$$

Rearranging, we get

$$\left(ydx+\frac{dx}{x}+\frac{dx}{x^2y}\right)+\left(xdy-\frac{dy}{y}+\frac{dy}{xy^2}\right)=0.$$

Regrouping the terms, we get

$$(ydx+xdy)+\left(\frac{ydx}{x^2y^2}+\frac{xdy}{x^2y^2}\right)+\left(\frac{dx}{x}-\frac{dy}{y}\right)=0 \Rightarrow d(xy)+\frac{d(xy)}{(xy)^2}+d\left(\log\frac{x}{y}\right)=0.$$

Integrating both sides, we get

$xy-\dfrac{1}{xy}+\log\dfrac{x}{y}=c$, which is the required solution.

Q.No. 32.: Solve the differential equation $\left(2y^2+4x^2y\right)dx+\left(4xy+3x^3\right)dy=0$.

Sol.: Here $M=2y^2+4x^2y$, $N=4xy+3x^3$,

$M_y=4y+4x^2\neq 4y+9x^2=N_x$, not exact.

It is also not homogeneous.

It is not $M=yh(xy)$ and $N=xg(xy)$ form.

So let us try to find an integrating factor of the form x^hy^k.

Consider the DE

$$2y^2dx+4x^2ydx+4xydy+3x^3dy=0$$

Rearranging the terms

$$x^2(4ydx+3xdy)+y(2ydx+4xdy)=0$$

Comparing this with

$$x^ay^b(mydx+nxdy)+x^cy^d(pydx+qxdy)=c$$

Here $a=2,\ \ b=0,\ \ m=4,\ \ n=3,\ \ c=0,\ \ d=1,\ \ p=2,\ \ q=4.$

Also $mp-nq=8-12=-4\neq 0$

The unknown constants in the integrating factor are determined from the following

$$\frac{a+h+1}{m}=\frac{b+k+1}{n}\Rightarrow\frac{2+h+1}{4}=\frac{0+k+1}{3}\Rightarrow 4k-3h=5.$$

$$\frac{c+h+1}{p}=\frac{d+k+1}{q}\Rightarrow\frac{0+h+1}{2}=\frac{1+k+1}{4}\Rightarrow k-2=0.$$

Solving for h, k, we get $h=1$, $k=2$.

Thus, the required integrating factor is $x^1.y^2$.

Multiplying the given DE by $x.y^2$, we get

$$xy^2\left(2y^2+4x^2y\right)dx+xy^2\left(4xy+3x^3\right)dy=0$$

$$\Rightarrow 2xy^4+4x^3y^3dx+4x^2y^3dy+3x^4y^2dy=0$$

Regrouping the terms (1^{st} and 3^{rd}) and (2^{nd} and 4^{th})

$$d\left(x^2y^4\right)+d\left(x^4y^3\right)=0.$$

Integrating, we get $x^2y^4+x^4y^3=c$, which is the required solution.

Home assignments

Total =50 Problems

Note: Solve the following differential equations (by regrouping the terms):

Q.No.1.: Solve the differential equation $\left(4x^3y^3-2xy\right)dx+\left(3x^4y^2-x^2\right)dy=0$.

Ans.: $x^4y^3-x^2y=c.$

Q.No.2.: Solve the differential equation $3x^2ydx+\left(y^4-x^3\right)dy=0$.

Ans.: $3x^3+y^4=cy.$

Q.No.3.: Solve the differential equation $\left(x^3+xy^2+y\right)dx+\left(y^3+x^2y+x\right)dy=0$.

Ans.: $\left(x^2+y^2\right)=c-4xy$.

Q.No.4.: Solve the differential equation $ydx+\left(x+x^3y^2\right)dy=0$.

Ans.: $2x^2y^2.\log(cy)=1.$

Q.No.5.: Solve the differential equation $y\left(x^3-y\right)dx-x\left(x^3+y\right)dy=0$.

Ans.: $x^2+2y=cxy^2$.

Q.No.6.: Solve the differential equation $ydx-xdy=xy^3dy$.

Ans.: $\log\frac{x}{y}=\frac{1}{3}y^3+c.$

Q.No.7.: Solve the differential equation $xdy=\left(x^5+x^3y^2+y\right)dx$.

Ans.: $\tan^{-1}\frac{x}{y}=-\frac{1}{4}x^4+c.$

Q.No.8.: Solve the differential equation $xdy=\left(y+x^2+9y^2\right)dx$.

Ans.: $\tan^{-1}\frac{3y}{x}=3x+c.$

Q.No.9.: Solve the differential equation $y\left(2xy+e^x\right)dx=e^xdy$.

Ans.: $\frac{e^x}{y}+x^2=c.$

Q.No.10.: Solve the differential equation $\left(y^2e^{xy^2}+4x^3\right)dx+\left(2xye^{xy^2}-3y^2\right)dy=0$.

Ans.: $e^{xy^2}+x^4-y^3=c.$

(2) I.F. of a homogeneous differential equation:

If $Mdx+Ndy=0$ **be a homogeneous differential equation in x and y, then** $\frac{1}{(Mx+Ny)}$ **is an integrating factor, provided** $Mx+Ny\neq 0$**.**

Q.No.1.: Solve the differential equation $x^2ydx-\left(x^3+y^3\right)dy=0$.

Ans.: $y=ce^{-x^3/\left(3y^2\right)}$**.**

Q.No.2.: Solve the differential equation $\left(x^2y-2xy^2\right)dx-\left(x^3-3x^2y\right)dy=0$.

Ans.: $\frac{x}{y}-2\log x+3\log y=c$**.**

Q.No.3.: Solve the differential equation $\left(x^4+y^4\right)dx-xy^3dy=0$.

Ans.: $y^4=4x^4\log x+cx^4$**.**

Q.No.4.: Solve the differential equation $y^2dx+\left(x^2-xy-y^2\right)dy=0$.

Ans.: $(x-y)y^2=c(x+y)$**.**

Q.No.5.: Solve the differential equation $(y-x)dx+(y+x)dy=0$.

Ans.: $\log\left(x^2+y^2\right)^{1/2}-\tan^{-1}\left(\frac{x}{y}\right)=c$.

(3) I.F. for an equation of the type $f_1(xy)ydx + f_2(xy)xdy = 0$:

If the equation $Mdx + Ndy = 0$ **is of the form** $f_1(xy)ydx + f_2(xy)xdy = 0$**, then** $\frac{1}{(Mx-Ny)}$ **is an integrating factor, provided** $Mx - Ny \neq 0$.

Q.No.1.: Solve the differential equation $y\left(x^2y^2+2\right)dx + x\left(2-2x^2y^2\right)dy = 0$.

Ans.: $x = cy^2 e^{\frac{1}{\left(x^2y^2\right)}}$.

Q.No.2.: Solve the differential equation $y\left(xy+2x^2y^2\right)dx + x\left(xy - x^2y^2\right)dy = 0$.

Ans.: $\frac{2}{3}\log x - \frac{1}{3}\log y - \frac{1}{3xy} = c$.

Q.No.3.: Solve the differential equation

$$(xy\sin xy + \cos xy)ydx + (xy\sin xy - \cos xy)xdy = 0.$$

Ans.: $x\sec xy = cy$.

Q.No.4.: Solve the differential equation $y(1+xy)dx + (1-xy)xdy = 0$.

Ans.: $\log\frac{x}{y} - \frac{1}{xy} = c$.

Q.No.5.: Solve the differential equation

$$\left(x^3y^3 + x^2y^2 + xy + 1\right)ydx + \left(x^3y^3 - x^2y^2 + xy + 1\right)xdy = 0.$$

Ans.: $x^2y^2 - 2xy\log cy = 1$.

Q.No.6.: Solve the differential equation $\left(2xy^2 + y\right)dx = \left(x + 2x^2y - x^4y^3\right)dy = 0$.

Ans.: $y = ce^{\frac{(-3xy+1)}{\left(3x^3y^3\right)}}$.

(4) In the equation $Mdx + Ndy = 0$,

(a) **If** $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$ **is a function of x only, say f(x),**

then $e^{\int f(x)dx}$ **is an integrating factor.**

Q.No.1.: Solve the differential equation $\left(4xy+3y^2-x\right)dx+x(x+2y)dy=0$.

Ans.: $x^3\left(4xy+4y^2-x\right)=c$.

Q.No.2.: Solve the differential equation $y(x+y)dx+(x+2y-1)dy=0$.

Ans.: $y(x-1+y)=ce^{-x}$.

Q.No.3.: Solve the differential equation $2xydy-\left(x^2+y^2+1\right)dx=0$.

Ans.: $y^2-x^2+1=cx$.

Q.No.4.: Solve the differential equation $\left(3xy-2ay^2\right)dx+\left(x^2-2axy\right)dy=0$.

Ans.: $x^3y-ax^2y^2=c$.

Q.No.5.: Solve the differential equation $2\sin\left(y^2\right)dx+xy\cos\left(y^2\right)dy=0$, $y(2)=\sqrt{\dfrac{\pi}{2}}$.

Ans.: $x^4\sin\left(y^2\right)=16$.

Q.No.6.: Solve the differential equation

$$\left(2x^3y^2+4x^2y+2xy^2+xy^4+2y\right)dx+2\left(y^3+x^2y+x\right)dy=0.$$

Ans.: $\left(2x^2y^2+4xy+y^4\right)e^{x^2}=c$.

(4) In the equation $Mdx+Ndy=0$**,**

(b) **If** $\dfrac{\dfrac{\partial N}{\partial x}-\dfrac{\partial M}{\partial y}}{M}$ **is a function of y only, say F(y),**

then $e^{\int F(y)dy}$ **is an integrating factor.**

Q.No.1.: Solve the differential equation $\left(y+xy^2\right)dx-xdy=0$.

Ans.: $\dfrac{x}{y}+\dfrac{x^2}{2}=c$.

Q.No.2.: Solve the differential equation $y(x+y+1)dx+x(x+3y+2)dy=0$.

Ans.: $xy^2(x+2y+2)=c$.

Q.No.3.: Solve the differential equation $3\left(x^2+y^2\right)dx+x\left(x^2+3y^2+6y\right)dy=0$.

Ans.: $xe^{y}\left(x^{2}+3y^{2}\right)=c$.

Q.No.4.: Solve the differential equation $\left(xy^{3}+y\right)dx+2\left(x^{2}y^{2}+x+y^{4}\right)dy=0$.

Ans.: $3x^{2}y^{4}+6xy^{2}+2y^{6}=c$.

Q.No.5.: Solve the differential equation $\left(y^{4}+2y\right)dx+\left(xy^{3}+2y^{4}-4x\right)dy=0$.

Ans.: $\left(y^{3}+2\right)x+y^{4}=cy^{2}$.

Q.No.6.: Solve the differential equation $2xydx+\left(y^{2}-x^{2}\right)dy=0$, $y(2)=1$.

Ans.: $x^{2}+y^{2}=5y$.

(5) I.F. for the equation of the type: $x^{a}y^{b}(mydx+nxdy)+x^{a'}y^{b'}(m'ydx+n'xdy)=0$

If the equation $Mdx+Ndy$ **is of the form**

$$x^{a}y^{b}(mydx+nxdy)+x^{a'}y^{b'}(m'ydx+n'xdy)=0,$$

where $a, b, a', b', n, n, m', n'$ **are all constant, then an integrating factor is** $x^{h}y^{k}$,

where $\dfrac{a+h+1}{m}=\dfrac{b+k+1}{n}$, $\dfrac{a'+h+1}{m'}=\dfrac{b'+k+1}{n'}$.

Q.No.1.: Solve the differential equation $x(4ydx+2xdy)+y^{3}(3ydx+5xdy)=0$.

Ans.: $x^{4}y^{2}+x^{3}y^{3}=c$.

Q.No.2.: Solve the differential equation $(8ydx+8xdy)+x^{2}y^{3}(4ydx+5xdy)=0$.

Ans.: $4x^{2}y^{2}+x^{4}y^{5}=c$.

Q.No.3.: Solve the differential equation $x^{3}y^{3}(2ydx+xdy)-(5ydx+7xdy)=0$.

Ans.: $x^{3}y^{3}+2=cx^{5/3}y^{7/3}$.

Q.No.4.: Solve the differential equation $y\left(xy+2x^{2}y^{2}\right)dx+x\left(xy-x^{2}y^{2}\right)dy=0$.

Ans.: $2\log x-\log y-\dfrac{1}{xy}=c$.

Now let us solve some more assignments for experience:

Q.No.1.: Solve the differential equation $xdy-ydx=\left(x^{2}+y^{2}\right)dx$.

Ans.: $y=x\tan(x+c)$.

Q.No.2.: Solve the differential equation $xdy - ydx = xy^2 dx$.

Ans.: $\frac{x^2}{2} + \frac{x}{y} = c$.

Q.No.3.: Solve the differential equation $\left(xy^{x/y} + y^2\right)dx - x^2 e^{x/y} dy = 0$.

Ans.: $e^{x/y} + \log x = c$.

Q.No.4.: Solve the differential equation $x^2 ydx - \left(x^3 + y^3\right)dy = 0$.

Ans.: $\log y - \frac{x^3}{3y^3} = c$.

Q.No.5.: Solve the differential equation $\left(3xy^2 - y^3\right)dx - \left(2x^2 y - xy^2\right)dy = 0$.

Ans.: $3\log x - 2\log y + \frac{y}{x} = c$.

Q.No.6.: Solve the differential equation $y(2xy + 1)dx + x\left(1 + 2xy - x^3 y^3\right)dy = 0$.

Ans.: $\frac{1}{x^2 y^2} + \frac{1}{3x^3 y^3} + \log y = c$.

Q.No.7.: Solve the differential equation $\left(x^2 + y^2 + 2x\right)dx + 2ydy = 0$.

Ans.: $e^x\left(x^2 + y^2\right) = c$.

Q.No.8.: Solve the differential equation $\left(x^2 + y^2 + 1\right)dx - 2xydy = 0$.

Ans.: $x - \frac{y^2}{x} - \frac{1}{x} = c$.

Q.No.9.: Solve the differential equation $\left(y + \frac{y^3}{3} + \frac{x^2}{2}\right)dx + \frac{1}{4}\left(x + xy^2\right)dy = 0$.

Ans.: $3x^4 y + x^4 y^3 + x^6 = c$

Q.No.10.: Solve the differential equation $\left(y^4 + 2y\right)dx + \left(xy^3 + 2y^4 - 4x\right)dy = 0$.

Ans.: $\left(y + \frac{2}{y^2}\right)x + y^2 = c$.

Q.No.11.: Solve the differential equation $\left(3xy - 2ay^2\right)dx + \left(x^2 - 2axy\right)dy = 0$.

Ans.: $x^2\left(ay^2 - xy\right) = c$.

Q.No.12.: Solve the differential equation $\left(xy^2 + 2x^2y^3\right)dx + \left(x^2y - x^3y^2\right)dy = 0$.

Ans.: $-\dfrac{1}{xy} + 2\log x - \log y = c$.

Q.No.13.: Solve the differential equation $\left(2x^2y - 3y^4\right)dx + \left(3x^3 + 2xy^3\right)dy = 0$.

Ans.: $5x^{-\frac{36}{13}}y^{\frac{24}{13}} - 12x^{-\frac{10}{13}}y^{-\frac{15}{13}} = c$.

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6^{th} Topic

Differential Equations of First order and higher degree

PART-I: Equation solvable for p

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Now here we will discuss the differential equations of first order and higher degrees. If $\frac{dy}{dx}$ will occur in higher degrees, then it is convenient to denote $\frac{dy}{dx}$ = p. Such equations are of the form $f(x, y, p) = 0$.

Now there are three cases for discussion:

(i) Equation solvable for p:

(ii) Equation solvable for y:

(iii) Equation solvable for x:

Equation solvable for p:

In the beginning, write down the given differential equation of the first order and n^{th} degree, which is of the form (in general)

$$p^n + P_1 p^{n-1} + P_2 p^{n-2} + \ldots\ldots\ldots\ldots + P_n = 0, \qquad \text{(i)}$$

where $P_1, P_2, \ldots\ldots\ldots\ldots, P_n$ are function of x and y.

Step No.1: (Process of splitting)

Splitting up the left hand side of (i) into n linear factors, we have

$[p - f_1(x, y)].[p - f_2(x, y)]...........[p - f_n(x, y)] = 0$.

Step No.2:

Equating each of the factor to zero, we obtain

$p - f_1(x, y) = 0,\ p - f_2(x, y) = 0, ..., p - f_n(x, y) = 0$

$\Rightarrow p = f_1(x, y),\ p = f_2(x, y), ...p = f_n(x, y)$.

These are ODE of the first order and first degree.

Step No.3:

Solving each of these equations of the first order and first degree, we get the following set of solutions

$F_1(x, y, c) = 0,\ F_2(x, y, c) = 0, ..., F_n(x, y, c) = 0$.

These n solutions constitute the general solution of (i).

Step No.4:

Otherwise, the general solution of (i) may be written as

$F_1(x, y, c).F_2(x, y, c)...........F_n(x, y, c) = 0$.

Now let us solve few differential equations of first order and higher degree, which are solvable for p:

Q.No.1.: Solve the differential equation $\frac{dy}{dx} - \frac{dx}{dy} = \frac{x}{y} - \frac{y}{x}$.

Sol.: The given equation is $\frac{dy}{dx} - \frac{dx}{dy} = \frac{x}{y} - \frac{y}{x}$ $\Rightarrow p - \frac{1}{p} = \frac{x}{y} - \frac{y}{x}$, where $p = \frac{dy}{dx}$.

$$\Rightarrow p^2 + p\left(\frac{y}{x} - \frac{x}{y}\right) - 1 = 0.$$

Step No.1: (Process of splitting)

$$\left(p + \frac{y}{x}\right)\left(p - \frac{x}{y}\right) = 0.$$

Step No.2:

Equating each of the factors to zero, we obtain

$p+\frac{y}{x}=0$ (i) or $p-\frac{x}{y}=0$ (ii)

Step No.3:

Solving each of these equations of the first order and first degree.

From equation (i), $\frac{dy}{dx}+\frac{y}{x}=0 \Rightarrow xdy+ydx=0 \Rightarrow d(xy)=0$.

By integrating, we get $\boxed{xy=c}$.

From equation (ii), $\frac{dy}{dx}-\frac{x}{y}=0 \Rightarrow xdx-ydy=0$.

By integrating, we get $\boxed{x^2-y^2=c}$.

Thus $xy=c$ and $x^2-y^2=c$ constitute the required solution.

Step No.4:

Otherwise, combining these two results into one, the required solution can be written as

$$\boxed{(xy-c)\left(x^2-y^2-c\right)=0}.$$

Q.No.2.: Solve $p^2+2py\cot x=y^2$.

Sol.: The given equation is $p^2+2py\cot x=y^2$.

$\Rightarrow p^2+2py\cot x+(y\cot x)^2=y^2+y^2\cot^2 x \qquad \Rightarrow p+y\cot x=\pm y\operatorname{cosec}x$.

Thus we have $p=y(-\cot x+\operatorname{cosec}x)$ (i) or $p=y(-\cot x-\operatorname{cosec}x)$. (ii)

From (i), $\frac{dy}{dx}=y(-\cot x+\operatorname{cosec}x) \quad \Rightarrow \frac{dy}{y}=(\operatorname{cosec}x-\cot x)dx$.

Integrating, we get $\int \frac{dy}{y}=\int (\operatorname{cosec}x-\cot x)dx+c$.

$$\Rightarrow \log y=\log\tan\frac{x}{2}-\log\sin x+\log c=\log\left(\frac{c\tan\frac{x}{2}}{\sin x}\right) \Rightarrow y=\frac{c}{2\cos\frac{x^2}{2}} \Rightarrow y(1+\cos x)=c. \text{ (iii)}$$

From (ii), $\frac{dy}{dx}=-y(\cot x+\operatorname{cosec}x) \Rightarrow \frac{dy}{y}=-(\cot x+\operatorname{cosec}x)dx$.

Integrating, we get $\int \frac{dy}{y}=\int -(\cot x+\operatorname{cosec}x)dx+c$

$$\Rightarrow \log y = -\log \sin x - \log \tan\frac{x}{2} + \log c = \log\left(\frac{c}{\sin x \tan\frac{x}{2}}\right)$$

$$\Rightarrow y = \frac{c}{2\sin^2\frac{x}{2}} \Rightarrow y(1-\cos x) = c. \qquad \text{(iv)}$$

Thus combining (iii) and (iv), the required general solution is

$y(1\pm\cos x) = c$.

Q.No.3.: Solve the equation $y\left(\frac{dy}{dx}\right)^2 + (x-y)\frac{dy}{dx} - x = 0$.

Sol.: The given equation is $y\left(\frac{dy}{dx}\right)^2 + (x-y)\frac{dy}{dx} - x = 0$.

$\Rightarrow p^2y + (x-y)p - x = 0$, where $p = \frac{dy}{dx}$.

$$\Rightarrow p^2 + p\left(\frac{x}{y}-1\right) - \frac{x}{y} = 0 \quad \Rightarrow (p-1)p + \frac{x}{y}(p-1) = 0 \quad \Rightarrow (p-1)\left(p+\frac{x}{y}\right) = 0.$$

Thus, we have $p - 1 = 0$ (i) or $p + \frac{x}{y} = 0$. (ii)

From (i), $\frac{dy}{dx} - 1 = 0 \Rightarrow dy - dx = 0$.

Integrating, we get $y - x = c$.

From (ii), $\frac{dy}{dx} + \frac{x}{y} = 0 \Rightarrow ydy + xdx = 0$.

Integrating, we get $y^2 + x^2 = c$.

Thus, $x - y = c$ and $x^2 + y^2 = c$ constitute the required solution.

Otherwise, combining these into one, the required solution can be written as

$(x-y-c)(x^2+y^2-c) = 0$. Ans.

Q.No.4.: Solve the equation $p(p+y) = x(x+y)$.

Sol.: The given equation is $p(p+y) = x(x+y)$

$\Rightarrow p^2 + py - (x^2 + xy) = 0 \Rightarrow (p - x)(p + x + y) = 0$.

Thus, we have $p - x = 0$ (i) or $p + x + y = 0$ (ii)

From (i), $\frac{dy}{dx} - x = 0 \Rightarrow dy - xdx = 0$.

Integrating, we get $y - \frac{x^2}{2} = c \Rightarrow 2y - x^2 = c$. (iii)

From (ii), $\frac{dy}{dx} + y = -x$, which is a Leibnitz's linear equation in y,

$\therefore$ I. F. $= e^{\int dx} = e^x$.

Then solution is $y(\text{I. F.}) = \int (-x)(\text{I. F.})dx + c$

$\Rightarrow ye^x = \int -xe^x dx + c \Rightarrow ye^x = -\left[xe^x - \int e^x dx\right] + c \Rightarrow ye^x = -\left[xe^x - e^x\right] + c$

$\Rightarrow y + x - 1 = ce^{-x} \Rightarrow y + x + ce^{-x} - 1 = 0$. (iv)

Combining (iii) and (iv), we get

$(2y - x^2 + c)(y + x + ce^{-x} - 1) = 0$,

which is the required general solution.

Q.No.5.: Solve the equation $y = x\left[p + \sqrt{(1 + p^2)}\right]$.

Sol.: The given equation is $y = x\left[p + \sqrt{(1 + p^2)}\right]$

$\Rightarrow y = xp + \left(\sqrt{1 + p^2}\right)x \Rightarrow y - xp = \left(\sqrt{1 + p^2}\right)x$

$\Rightarrow y^2 + p^2x^2 - 2pxy = x^2(1 + p^2) \Rightarrow y^2 - x^2 - 2pxy = 0$

$\Rightarrow -2\frac{dy}{dx}xy = x^2 - y^2 \Rightarrow 2y\frac{dy}{dx} - \frac{y^2}{x} = -x$. (i)

Putting $y^2 = t$, so that $2y\frac{dy}{dx} = \frac{dt}{dx}$.

The equation (i) becomes $\frac{dt}{dx} - \frac{t}{x} = -x$. (ii)

which is Leibnitz's linear equation in t.

$\therefore$ I. F. $= e^{-\int \frac{1}{x}dx} = e^{-\log x} = \dfrac{1}{x}$.

Then solution is $y(\text{I. F.}) = \int (-x)(\text{I. F.})dx + c$

$\Rightarrow t\dfrac{1}{x} = -\int x\dfrac{1}{x}dx + c \quad \Rightarrow \dfrac{y^2}{x} = -x + c \quad \Rightarrow y^2 - -x^2 + cx \quad \Rightarrow x^2 + y^2 = cx$,

which is the required solution.

Q.No.6.: Solve the equation $xy\left(\dfrac{dy}{dx}\right)^2 - \left(x^2 + y^2\right)\dfrac{dy}{dx} + xy = 0$.

Sol.: The given equation is $xy\left(\dfrac{dy}{dx}\right)^2 - \left(x^2 + y^2\right)\dfrac{dy}{dx} + xy = 0$.

$\Rightarrow xyp^2 - \left(x^2 + y^2\right)\dfrac{dy}{dx} + xy = 0 \quad \Rightarrow p^2 - \left(\dfrac{x}{y} + \dfrac{y}{x}\right)p + 1 = 0 \quad \Rightarrow \left(p - \dfrac{x}{y}\right)\left(p - \dfrac{y}{x}\right) = 0$

Thus, we have $p - \dfrac{x}{y} = 0$ (i) or $p - \dfrac{y}{x} = 0$ (ii)

From (i), $\dfrac{dy}{dx} - \dfrac{x}{y} = 0 \quad \Rightarrow ydy - xdx = 0$.

Integrating, we get $y^2 - x^2 = c$. (iii)

From (ii), $\dfrac{dy}{dx} - \dfrac{y}{x} = 0 \quad \Rightarrow \dfrac{dy}{y} - \dfrac{dx}{x} = 0$.

Integrating, we get $\log y - \log x = \log c \quad \Rightarrow \dfrac{y}{x} = c$. (iv)

Combining (iii) and (iv), we get

$\left(x^2 - y^2 + c\right)(y - cx) = 0$,

which is the required solution.

Q.No.7.: Solve the equation $p^3 + 2xp^2 - y^2p^2 - 2xy^2p = 0$.

Sol.: The given equation is $p^3 + 2xp^2 - y^2p^2 - 2xy^2p = 0$

$\Rightarrow p\left[p^2 + p\left(p2x - y^2\right) - 2xy^2\right] = 0 \qquad \Rightarrow p\left[(p + 2x)\left(p - y^2\right)\right] = 0$

Thus, we have $p = 0$ (i), $p + 2x = 0$, (ii) or $p - y^2 = 0$ (iii)

From (i), $\frac{dy}{dx} = 0 \Rightarrow dy = 0$.

Integrating, we get $y = c$. (iv)

From (ii), $\frac{dy}{dx} - y^2 = 0 \Rightarrow dy + 2xdx = 0$.

Integrating, we get $y + \frac{2x^2}{2} = c \Rightarrow y + x^2 = c$. (v)

From (iii), $\frac{dy}{dx} - y^2 = 0 \Rightarrow dy - y^2dx = 0 \Rightarrow dx - \frac{1}{y^2}dy = 0$.

Integrating, we get $\int dx - \int \frac{1}{y^2}dy = c \Rightarrow x + \frac{1}{y} = c \Rightarrow xy + 1 = cy$. (vi)

Combining (iv), (v) and (vi), we get

$(y - c)(y + x^2 - c)(xy + cy + 1) = 0$,

which is the required general solution.

Q.No.8.: Solve the equation $xyp^2 + p(3x^2 - 2y^2) - 6xy = 0$.

Sol.: The given equation is $xyp^2 + p(3x^2 - 2y^2) - 6xy = 0$.

Solving the given equation for p, we have

$$p = \frac{-(3x^2 - 2y^2) \pm \sqrt{(3x^2 - 2y^2)^2 + 24x^2y^2}}{2xy}$$

$$\Rightarrow p = \frac{-(3x^2 - 2y^2) \pm (3x^2 + 2y^2)}{2xy} = \frac{2y}{x} \quad \text{or} \quad -\frac{3x}{y}.$$

Thus, we have $p = \frac{2y}{x}$ (i) or $p = -\frac{3x}{y}$ (ii)

From (i), $\frac{dy}{dx} = \frac{2y}{x} \Rightarrow \frac{dy}{y} - 2\frac{dx}{x} = 0$.

Integrating, we get $\log y - 2\log x = \log c \Rightarrow \frac{y}{x^2} = c \Rightarrow y = cx^2$. (iii)

From (ii), $\frac{dy}{dx} = -\frac{3x}{y} \Rightarrow ydx + 3xdx = 0$.

Integrating, we get $\frac{y^2}{2}+3\frac{x^2}{2}=c' \Rightarrow y^2+3x^2=c$. (iv)

Combining (iii) and (iv), we get

$$\left(y-cx^2\right)\left(y^2+3x^2-c\right)=0,$$

which is the required general solution.

Q.No.9.: Solve $x^2\left(\frac{dy}{dx}\right)^2+xy\frac{dy}{dx}-6y^2=0$.

Sol.: This is a first order, second degree, non-linear homogeneous differential equation (all the three terms are non-linear).

Introducing $\frac{dy}{dx}=p$, the given equation taken the form

$$x^2p^2+xyp-6y^2=0.$$

Factorizing, we get

$$(px+3y)(px-2y)=0 \Rightarrow (px+3y)=0 \text{ and } (px-2y)=0.$$

Solving

$$x\frac{dy}{dx}+3y=0,\quad \frac{dy}{dy}+3\frac{dx}{x}=0,\quad yx^3=c$$

$$x\frac{dy}{dx}-2y=0,\quad \frac{dy}{y}-\frac{2dx}{x}=0,\quad \frac{y}{x^2}=c.$$

The primitive of the given differential equation is

$$\left(yx^3-c\right)\left(y-cx^2\right)=0.$$

Q.No.10.: Solve

$$2p^3-(2x+4\sin x-\cos x).p^2-(x\cos x-4x\sin x+\sin 2x)p+x.\sin 2x=0.$$

Sol.: Observe that p = x satisfies the given differential equation. i.e.,

$$2x^3-(2x+4\sin x-\cos x).x^2-(x\cos x-4x\sin x+\sin 2x)x+x.\sin 2x=0.$$

Thus $(p-x)$ is factor.

Rewriting the given differential equation $(p-x)\left[2p^3-(4\sin x-\cos x).p-2\sin 2x\right]=0$

$$\Rightarrow (p-x)[2p(p-2\sin x)+\cos x(p-2\sin x)]=0.$$

$$\Rightarrow (p-x)(p-2\sin x)2p+\cos x=0.$$

Equating the three factors to zero.

$$p - x = 0 \Rightarrow \frac{dy}{dx} = x \Rightarrow y = \frac{x^2}{2} + c.$$

$$p - 2\sin x = 0 \Rightarrow dy - 2\sin x dx = 0 \Rightarrow y + 2\cos x = c.$$

$$2p + \cos x = 0 \Rightarrow 2dy + \cos x dx = 0 \Rightarrow 2y + \sin x = c.$$

The general solution is $(2y - x^2 - c)(y + 2\cos x - c)(2y + \sin x - c) = 0$.

Home Assignments

Q.No.1.: Solve the differential equation $p^2 - 5p + 6 = 0$.

Ans.: $(y - 3x - c)(y - 2x - c) = 0$.

Q.No.2.: Solve the differential equation $4y^2p^2 + 2pxy(3x+1) + 3x^3 = 0$.

Ans.: $(x^2 + 2y^2 - c)(x^3 + y^2 - c) = 0$.

Q.No.3.: Solve the differential equation

$$p^3 - \left(y + 2x - e^{x-y}\right)p^2 + \left(2xy - 2xe^{x-y} - ye^{x-y}\right)p + 2xye^{x-y} = 0.$$

Ans.: $\left(y - ce^x\right)\left(y - x^2 - c\right)\left(e^y + e^x - c\right) = 0$.

Q.No.4.: Solve the differential equation

$$p^4 - (x + 2y + 1)p^3 + (x + 2y + 2xy)p^2 - 2xyp = 0.$$

Ans.: $(y - c)(y - x - c)(2y - x^2 - c)\left(y - ce^{2x}\right) = 0$.

Q.No.5.: Solve the differential equation $xyp^2 + \left(x^2 + xy + y^2\right)p + x^2 + xy = 0$.

Ans.: $(2xy + x^2 - c)(x^2 + y^2 - c) = 0$.

Q.No.6.: Solve the differential equation $\left(x^2 + x\right)p^2 + \left(x^2 + x - 2xy - y\right)p + y^2 - xy = 0$.

Ans.: $[y - c(x+1)][y + x \ \log cx] = 0$.

Q.No.7.: Solve the differential equation $p^2 - 7p + 12 = 0$.

Ans.: $(y - 4x - c)(y - 3x - c) = c$.

Q.No.8.: Solve the differential equation $yp^2 + (x - y)p - x = 0$.

Ans.: $(y - x - c)(x^2 + y^2 - c) = 0$.

Q.No.9.: Solve the differential equation $x^2\left(\frac{dy}{dx}\right)^2 + 3xy\frac{dy}{dx} + 2y^2 = 0$.

Ans.: $(xy - c)\left(x^2y - c\right) = 0$.

Q.No.10.: Solve the differential equation $p^2 - 2p\sinh x - 1 = 0$.

Ans.: $\left(y - e^x - c\right)\left(y - e^{-x} - c\right) = 0$.

Q.No.11.: Solve the differential equation $p(p + y) = x(x + y)$.

Ans.: $\left(y - \frac{1}{2}x^2 + c\right)\left(y + x + ce^{-x} - 1\right) = 0$.

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6^{th} Topic

Differential Equations of First order and higher degree

PART- II: Equation solvable for y

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Equations solvable for y:

In the beginning, write down the given differential equation of the first order but of the n^{th} degree, which is of the form (in general)

$$p^n + P_1 p^{n-1} + P_2 p^{n-2} + \ldots\ldots\ldots\ldots + P_n = 0,$$

where $P_1, P_2, \ldots\ldots\ldots\ldots, P_n$ are function of x and y.

Step No.1:

The given equation, on solving for y, takes the form

$$y = f(x,\ p). \tag{i}$$

Step No.2:

Differentiating (i) w. r. t. x , we get an equation of the form

$$p = \frac{dy}{dx} = \phi\left(x, p, \frac{dp}{dx}\right). \tag{ii}$$

This equation is a differential equation of first order in p and x.

Step No.3:

Solve this new differential equation in x and p.

Suppose the solution of (ii) is $F(x, p, c) = 0$. (iii)

Step No.4:

Now the elimination of p from (i) and (iii) gives the required solution.

In case elimination of p is not possible, then we solve (i) and (ii) for x and y and obtain

$x = F_1(p, c), \quad y = F_2(p, c)$.

These two relations taken together, with parameter p, constitute the solution of the given equation.

Remarks: This method is useful for equations, which do not contain x.

Now let us solve few differential equations of first order and higher degree, which are solvable for y:

Q.No.1.: Solve $y - 2px = \tan^{-1}\left(xp^2\right)$.

Sol.: The given equation is $y - 2px = \tan^{-1}\left(xp^2\right)$.

Step No.1:

$$y = 2px + \tan^{-1}\left(xp^2\right). \qquad \text{(i)}$$

Step No.2:

Differentiating both sides w. r. t. x, we get

$$\frac{dy}{dx} = p = 2\left(p + x\frac{dp}{dx}\right) + \frac{p^2 + 2xp\frac{dp}{dx}}{1 + x^2p^4} \qquad \Rightarrow p + 2x\frac{dp}{dx} + \left(p + 2x\frac{dp}{dx}\right).\frac{p}{1 + x^2p^4} = 0$$

$$\Rightarrow \left(p + 2x\frac{dp}{dx}\right)\left(1 + \frac{p}{1 + x^2p^4}\right) = 0.$$

Step No.3:

We have $p + 2x\frac{dp}{dx} = 0 \Rightarrow \frac{dx}{x} + 2\frac{dp}{p} = 0$.

Integrating, we get $\int \frac{dx}{x} + 2\int \frac{dp}{p} = \log c$

$$\Rightarrow \log x + 2\log p = \log c \Rightarrow \log\left(xp^2\right) = \log c$$

$$\Rightarrow xp^2 = c \quad \Rightarrow p = \sqrt{\left(\frac{c}{x}\right)}. \tag{ii}$$

Step No.4:

Eliminating p from (i) and (ii), we get

$$y = 2\sqrt{\left(\frac{c}{x}\right)}x + \tan^{-1} c \Rightarrow y = 2\sqrt{(cx)} + \tan^{-1} c,$$

which is the general solution of (i).

Note: The significance of the factor $\left(1+\frac{p}{1+x^2p^4}\right) = 0$, which we did not consider, will not be considered here as it concerns 'singular solution' of (i), whereas we are interested only in finding general solution.

Singularity:

In mathematics, a **singularity** is in general a point at which a given mathematical object is not defined or a point of an exceptional set where it fails to be well-behaved in some particular way, such as differentiability.

For example:

- The function $f(x) = \frac{1}{x}$ on the real line has a singularity at x = 0, where it seems to "explode" to $\pm\infty$ and is not **defined**.
- The function $g(x) = |x|$ also has a singularity at x = 0, since it has not **differentiable** there.
- The graph defined by $y^2 = x$ also has a singularity at (0, 0). This time it has a 'corner' (**vertical tangent**) at that point.

Singular solution:

A **singular solution** $y_s(x)$ of an ordinary differential equation is a solution that is tangent to every solution from the family of general solutions. By *tangent* we mean that there is a point *x* where $y_s(x) = y_c(x)$ and $y'_s(x) = y'_c(x)$ where y_c is any general solution.

Usually, singular solutions appear in differential equations when there is a need to divide in a term that might be equal to zero. Therefore, when one is solving a differential equation and using division one must check what happens if the term is equal to zero, and whether it leads to a singular solution.

Q.No.2.: Solve $y = 2px + p^n$.

Sol.: Given equation is $y = 2px + p^n$. (i)

Differentiating both sides w. r. t. x, we get

$$\frac{dy}{dx} = p = 2p + 2x\frac{dp}{dx} + np^{n-1}\frac{dp}{dx} \quad \Rightarrow p\frac{dx}{dp} + 2x = -np^{n-1} \quad \Rightarrow \frac{dx}{dp} + \frac{2x}{p} = -np^{n-2}. \qquad \text{(ii)}$$

This is Leibnitz's linear equation in x and p.

$$\therefore \text{I. F.} = e^{\int \frac{2}{p}dp} = e^{\log p^2} = p^2.$$

$\therefore$ The solution of (ii) is $\quad x(\text{I. F.}) = \int \left(-np^{n-2}\right)(\text{I. F.})dp + c$

$$\Rightarrow xp^2 = -n\int p^n dp + c = -\frac{np^{n+1}}{n+1} + c$$

$$\Rightarrow x = cp^{-2} - \frac{np^{n-1}}{n+1}p^n. \qquad \text{(iii)}$$

Substituting this value of x in (i), we get $y = 2p\left[\frac{c}{p^2} - \frac{np^{n-1}}{n+1}p^n\right] + p^n$

$$\Rightarrow y = \frac{2c}{p} + p^n\left[-\frac{2n}{n+1} + 1\right] \Rightarrow y = \frac{2c}{p} + p^n\left[\frac{-2n+n+1}{n+1}\right]$$

$$\Rightarrow y = \frac{2c}{p} + \frac{1-n}{1+n}p^n. \qquad \text{(iv)}$$

The equation (iii) and (iv) taken together, with parameter p, constitute the general solution of (i).

Q.No.3.: Solve the equation $y = x + a\tan^{-1}p$.

Sol.: The given equation is $y = x + a\tan^{-1}p$. (i)

Differentiating (i) w. r. t. x, we get

$$\frac{dy}{dx} = p = 1 + \frac{a}{1+p^2}\frac{dp}{dx} \quad \Rightarrow \frac{dx}{a} = \frac{dp}{(p-1)\left(1+p^2\right)}$$

$$\Rightarrow \frac{dx}{a} = \frac{1}{2}\left[\frac{1}{p-1} - \frac{p+1}{\left(p^2+1\right)}\right]dp \quad \Rightarrow dx = \frac{a}{2}\left[\frac{1}{p-1} - \frac{1}{2}\frac{2p}{\left(p^2+1\right)} - \frac{1}{p^2+1}\right]dp.$$

Integrating both sides, we get

$$\int dx + c = \frac{a}{2}\left[\int \frac{dp}{p-1} - \frac{1}{2}\int \frac{2p}{\left(p^2+1\right)}dp - \int \frac{dp}{p^2+1}\right]$$

$$\Rightarrow x + c = \frac{a}{2}\left[\log\frac{p-1}{\sqrt{p^2+1}} - \tan^{-1}p\right], \qquad \text{(ii)}$$

with the given relation, constitute the required solution.

Q.No.4.: Solve the equation $y + px = x^4p^2$.

Sol.: The given equation is $y + px = x^4p^2$. (i)

Differentiating (i) w. r. t. x, we get

$$\frac{dy}{dx} + p + x\frac{dp}{dx} - x^4 2p\frac{dp}{dx} - p^2.4x^3 = 0 \quad \Rightarrow p + p + x\frac{dp}{dx} - 2px^4\frac{dp}{dx} - p^2 4x^3 = 0$$

$$\Rightarrow x\frac{dp}{dx} - 2px^4\frac{dp}{dx} + 2p - p^2 4x^3 = 0 \qquad \Rightarrow x\frac{dp}{dx}\left(1 - 2px^3\right) + 2p\left(1 - 2px^3\right) = 0$$

$$\Rightarrow \left(1 - 2px^3\right)\left(x\frac{dp}{dx} + 2p\right) = 0.$$

Thus we have $x\frac{dp}{dx} + 2p = 0$.

Separating the variables, we get $\frac{1}{2}\frac{dp}{p} = -\frac{dx}{x}$.

Integrating both sides, we get $\frac{1}{2}\int \frac{dp}{p} = -\int \frac{dx}{x} + \log c'$

$$\Rightarrow \frac{1}{2}\log p = -\log x + \log c' \Rightarrow \sqrt{p} = \frac{c'}{x} \quad \Rightarrow p = \frac{c'^2}{x^2}.$$

Putting the value of p in equation (i), we get

$$y + \frac{c'^2}{x^2}x = x^4.\frac{c'^4}{x^4} \Rightarrow y + \frac{c'^2}{x} = c'^4 \Rightarrow xy + c'^2 = xc'^4$$

$xy = c^2x + c$, (where $c'^2 = -c$)

which is the required general solution.

Q.No.5.: Solve the equation $x^2\left(\frac{dy}{dx}\right)^4 + 2x\frac{dy}{dx} - y = 0$.

Sol.: The given equation is $x^2\left(\frac{dy}{dx}\right)^4 + 2x\frac{dy}{dx} - y = 0$.

$\Rightarrow x^2p^4 + 2xp - y = 0 \qquad \Rightarrow y = x^2p^4 + 2px$. (i)

Differentiating (i) w. r. t. x, we get

$$\frac{dy}{dx} = p = x^2 4p^3\frac{dp}{dx} + p^4 2x + 2p + 2x\frac{dp}{dx} \quad \Rightarrow p + 2p^4x + 2x\frac{dp}{dx} + 4x^2p^3\frac{dp}{dx} = 0$$

$$\Rightarrow p\left(1 + 2xp^3\right) + 2x\frac{dp}{dx}\left(1 + 2xp^3\right) = 0 \qquad \Rightarrow \left(1 + 2xp^3\right)\left(p + 2x\frac{dp}{dx}\right) = 0.$$

Thus we have $p + 2x\frac{dp}{dx} = 0 \Rightarrow 2\frac{dp}{p} = -\frac{dx}{x}$.

Integrating both sides, we get $2\int \frac{dp}{p} = -\int \frac{dx}{x} + \log c$

$$\Rightarrow 2\log p = -\log x + \log c \quad \Rightarrow \log p^2 = \log\frac{c}{x} \quad \Rightarrow p^2 = \frac{c}{x} \quad \Rightarrow p = \sqrt{\frac{c}{x}}$$

Substituting the value of p in equation (i), we get

$$y = x^2\left(\frac{c^2}{x^2}\right) + 2\sqrt{\frac{c}{x}}x \quad \Rightarrow y = c^2 + 2\sqrt{cx},$$

which is the required general solution.

Q.No.6.: Solve the equation $xp^2 + x = 2yp$.

Sol.: The given equation is $xp^2 + x = 2yp$ (i)

$\Rightarrow y = \frac{xp^2 + x}{2p} \quad \Rightarrow 2y = xp + \frac{x}{p}$. (ii)

Differentiating (ii) w. r. t. x, we get $2\frac{dy}{dx} = \left(x\frac{dp}{dx} + p\right) + \left(\frac{p - x\frac{dp}{dx}}{p^2}\right)$

$$\Rightarrow 2p = x\frac{dp}{dx} + p + \frac{p - x\frac{dp}{dx}}{p^2} \qquad \Rightarrow p = x\frac{dp}{dx} + \frac{1}{p} - \frac{x}{p^2}\frac{dp}{dx}$$

$\Rightarrow x\dfrac{dp}{dx}-\dfrac{x}{p^2}\dfrac{dp}{dx}+\dfrac{1}{p}-p=0 \qquad \Rightarrow x\dfrac{dp}{dx}\left(1-\dfrac{1}{p^2}\right)+p\left(-1+\dfrac{1}{p^2}\right)=0$

$\Rightarrow x\dfrac{dp}{dx}\left(1-\dfrac{1}{p^2}\right)-p\left(1-\dfrac{1}{p^2}\right)=0 \Rightarrow \left(1-\dfrac{1}{p^2}\right)\left(x\dfrac{dp}{dx}-p\right)=0$.

Thus we have $\left(x\dfrac{dp}{dx}-p\right)=0 \Rightarrow \dfrac{dp}{p}=\dfrac{dx}{x}$.

Integrating both sides, we get $\displaystyle\int \frac{dp}{p}=\int \frac{dx}{x}+\log c$

$\Rightarrow \log p=\log x+\log c \;\Rightarrow \log p=\log(xc) \;\Rightarrow p=cx$.

Substituting the value of p in (i), we get $x\left(c^2x^2\right)+x=2y(cx)$

$\Rightarrow c^2x^3+x=2xyc \qquad \Rightarrow x^2c^2=2cy-1 \qquad \Rightarrow 2cy=c^2x^2+1$,

which is the required general solution.

Q.No.7.: Solve the equation $y=xp^2+p$.

Sol.: The given equation is $y=xp^2+p$. (i)

Differentiating (i) w. r. t. x, we get

$\dfrac{dy}{dx}=p=x2p\dfrac{dp}{dx}+p^2+\dfrac{dp}{dx} \quad \Rightarrow 2xp\dfrac{dp}{dx}+\dfrac{dp}{dx}+p^2-p=0$.

Dividing both sides by $\dfrac{dp}{dx}$, we get

$2xp+1+\dfrac{dx}{dp}\left(p^2-p\right)=0 \quad \Rightarrow \dfrac{dx}{dp}+\dfrac{2p}{\left(p^2-p\right)}x=\dfrac{1}{p-p^2} \quad \Rightarrow \dfrac{dx}{dp}+\dfrac{2}{(p-1)}x=\dfrac{1}{p-p^2}$, (ii)

which is Leibnitz's linear equation in x and p.

$\therefore$ I. F. $=e^{\int\frac{2}{p-1}dp}=e^{\int\frac{2dp}{(p-1)}}=e^{2\log[p-1]}=(p-1)^2$.

So required solution of equation (ii) is

$x(\text{I. F.})=\displaystyle\int \frac{1}{p-p^2}(\text{I. F.})dp+c$

$\Rightarrow x(p-1)^2=\displaystyle\int \frac{1}{p(1-p)}(p-1)^2dp+c \;=-\int \frac{p-1}{p}dp+c \;=\int \frac{1-p}{p}dp+c$

$$\Rightarrow x(p-1)^2 = \int \frac{1}{p}dp - \int dp + c \Rightarrow x(p-1)^2 = \log p - p + c$$

$\Rightarrow x = (\log p - p + c)(p-1)^{-2}$, with the given relation, constitute the required solution.

Q.No.8.: Solve the equation $y = p\sin p + \cos p$.

Sol.: The given equation is $y = p\sin p + \cos p$. (i)

Differentiating (i) w. r. t. x, we get

$$\frac{dy}{dx} = p = p\cos p\frac{dp}{dx} + \sin p\frac{dp}{dx} - \sin p\frac{dp}{dx} \quad \Rightarrow p = p\cos p\frac{dp}{dx}$$

$\Rightarrow dx = \cos p dp$.

Integrating both sides, we get

$\int dx = \int \cos p dp + c \quad \Rightarrow x = \sin p + c$, with the given relation, constitute the required solution.

i.e. $\sin p = x - c \quad \therefore \cos p = \sqrt{1-(x-c)^2}$.

Hence $y = (x-c)\sin^{-1}(x-c) + \sqrt{1-(x-c)^2}$,

which is the required general solution.

Q.No.9.: Find the general solution $3x^4p^2 - xp - y = 0$.

Sol.: This is a first order, second degree, non-linear homogeneous differential equation which can be solved for y.

Thus $y = 3x^4p^2 - xp$.

Differentiating w.r.t. x both sides, we get

$$\frac{dy}{dx} = 12x^3p^2 + 6x^4p\frac{dp}{dx} - p - x\frac{dp}{dx} \Rightarrow \left(2p - 12x^3p^2\right) + \left(x - 6x^4p\right)\frac{dp}{dx}$$

$$\Rightarrow \left(1 - 6x^3p\right)\left(2p + x\frac{dp}{dx}\right) = 0 .$$

Equating the second order to zero, we have

$$2p + x\frac{dp}{dx} = 0 \Rightarrow 2\frac{dx}{x} + \frac{dp}{p} \Rightarrow px^2 = c .$$

Eliminating p from the given differential equation by using $p = \frac{c}{x^2}$, we get

$$y = 3x^4\left(\frac{c}{x^2}\right)^2 - x.\frac{c}{x^2} = 3c^2 - \frac{c}{x}.$$

The require general solution is $xy = c(3cx - 1)$.

Q.No.10.: Solve $p\tan p - y + \log\cos p = 0$.

Sol.: Solving $y = p\tan p - y + \log\cos p$

Differentiating w.r.t. x both sides, we get

$$\frac{dy}{dx} = \tan p.\frac{dp}{dx} + p.\sec^2 p.\frac{dp}{dx} + \frac{1}{\cos p}.(-\sin p).\frac{dp}{dx} \Rightarrow p\left(1 - \sec^2 p.\frac{dp}{dx}\right) = 0.$$

Considering the second factor, $1 - \sec^2 p\frac{dp}{dx} = 0$.

Solving $dx = \sec^2 pdp \Rightarrow x = \tan p + c$.

Since p cannot be eliminated, the general solution in the parametric form (with parametric p) is $x = \tan p + c,\ y = p\tan p + \log\cos p$.

Home Assignments

Q.No.1.: Solve the differential equation $2p^2y - p^3x + 16x^2 = 0$.

Ans.: $2 + c^2y - c^3x^2 = 0$.

Q.No.2.: Solve the differential equation $yp + p^2 - x = 0$.

Ans.: $x = -\frac{p}{-p_1}\log(p + p_1) + \frac{cp}{p_1}$, $\quad y = -p - \frac{1}{p_1}\log(p + p_1) + \frac{c}{p_1}$, where $p_1 = \sqrt{p^2 - 1}$.

Q.No.3.: Solve the differential equation $2x + p^2 - y + px = 0$.

Ans.: $x = 2(2 - p) + ce^{-p/2}$, $y = 8 - p^2 + (2 + p)ce^{-p/2}$.

Q.No.4.: Solve the differential equation $p^3 + mp^2 - ay - amx = 0$.

Ans.: $ax = c + \frac{3p^2}{2} - mp + m^2\log(p + m)$ and

$$ay = -m\left[c + \frac{3}{2}p^2 - mp + m^2\log(p + m)\right] + mp^2 + p^3.$$

Q.No.5.: Solve the differential equation $xp^2 - 2yp + ax = 0$.

Ans.: $2y = cx^2 + \frac{a}{c}$.

Q.No.6.: Solve the differential equation $y - 2px = \tan^{-1}\left(xp^2\right)$.

Ans.: $y = 2\sqrt{cx} + \tan^{-1} c$.

Q.No.7.: Solve the differential equation $16x^2 + 2p^2y - p^3x = 0$.

Ans.: $16 + 2c^2y - c^3x^2 = 0$.

Q.No.8.: Solve the differential equation $y = x + 2\tan^{-1} p$.

Ans.: $x + c = \log\frac{p-1}{\sqrt{p^2+1}} - \tan^{-1} p$, with the given relation.

Q.No.9.: Solve the differential equation $y = 3x + \log p$.

Ans.: $y = 3x + \log\frac{3}{1 - ce^{3x}}$.

Q.No10.: Solve the differential equation $x - yp = ap^2$.

Ans.: $x = \frac{p}{\sqrt{1-p^2}}\left(c + a\sin^{-1} p\right)$, $y = \frac{1}{\sqrt{1-p^2}}\left(c + a\sin^{-1} p\right) - ap$.

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6^{th} Topic

Differential Equations of First order and higher degree

PART-III: Equation solvable for x

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Equations solvable for x:

In the beginning, write down the given differential equation of the first order but of the n^{th} degree, which is of the form (in general)

$$p^n + P_1 p^{n-1} + P_2 p^{n-2} + \dots\dots\dots\dots + P_n = 0,$$

where $P_1, P_2, \dots\dots\dots\dots, P_n$ are function of x and y.

Step No.1:

The given equation on solving for x, takes the form

$$x = f(y, p). \tag{i}$$

Step No.2:

Differentiating w. r. t. y, we get an equation of the form

$$\frac{1}{p} = \frac{dx}{dy} = \phi\left(y, p, \frac{dp}{dy}\right).$$

Step No.3:

Solve the new differential equation in y and p.

Let its solution be $F(y, p, c) = 0$. (ii)

Step No.4:

The elimination of p from (i) and (ii) gives the required solution.

In case the elimination is not feasible, (i) and (ii) may be expressed in terms of p and p may be regarded as a parameter.

Remarks: This method is especially useful for equations, which do not contain y.

Now let us solve few differential equations of first order and higher degree, which are solvable for x:

Q.No.1.: Solve $y = 2px + y^2p^3$.

Sol.: The given equation is $y = 2px + y^2p^3$

Step No.1: The given equation on solving for x, takes the form

$$x = \frac{y - y^2p^3}{2p}.$$

Step No.2: Differentiating it w. r. t. y, we get

$$\frac{dx}{dy}\left(= \frac{1}{p}\right) = \frac{1}{2}.\frac{p\left(1 - 2y.p^3 - y^2 3p^2\frac{dp}{dy}\right) - \left(y - y^2p^3\right)\frac{dp}{dy}}{p^2}$$

$$\Rightarrow 2p = p - 2yp^4 - 3y^2p^3\frac{dp}{dy} - y\frac{dp}{dy} + y^2p^3\frac{dp}{dy}$$

$$\Rightarrow p + 2yp^4 + 2y^2p^3\frac{dp}{dy} + y\frac{dp}{dy} = 0 \quad \Rightarrow p\left(1 + 2yp^3\right) + y\frac{dp}{dy}\left(1 + 2yp^3\right) = 0$$

$$\Rightarrow \left(p + y\frac{dp}{dy}\right)\left(1 + 2p^3y\right) = 0.$$

Step No.3: We have $p + y\frac{dp}{dy} = 0 \qquad \Rightarrow \frac{d}{dy}(py) = 0$.

Integrating, we get $\qquad py = c \Rightarrow p = \frac{c}{y}$. (i)

Step No.4:

Substitute the value of p in (i), we get

$$y = 2\frac{c}{y}x + \frac{c^3}{y^3}y^2 \quad \Rightarrow y^2 = 2cx + c^3,$$

which is the required general solution.

Q.No.2.: Solve the equation $p^3 - 4xyp + 8y^2 = 0$.

Sol.: The given equation is $p^3 - 4xyp + 8y^2 = 0$. (i)

$$\Rightarrow 4xyp = p^3 + 8y^2 \quad \Rightarrow x = \frac{p^3 + 8y^2}{4yp}.$$

Differentiating w. r. t. y, we get

$$\frac{dx}{dy} = \frac{d}{dy}\left(\frac{p^3 + 8y^2}{4yp}\right) = \frac{1}{4}\frac{yp\left(3p^2\frac{dp}{dy} + 16y\right) - \left(p^3 + 8y^2\right)\left(y\frac{dp}{dy} + p\right)}{(yp)^2}$$

$$\Rightarrow \frac{1}{p} = \frac{1}{4}\left[\frac{yp\left(3p^2\frac{dp}{dy} + 16y\right) - \left(p^3 + 8y^2\right)\left(y\frac{dp}{dy} + p\right)}{y^2p^2}\right]$$

$$\Rightarrow 4y^2p^2 = p\left[yp\left(3p^2\frac{dp}{dy} + 16y\right) - p^3y\frac{dp}{dy} - p^4 - 8y^3\frac{dp}{dy} - 8py^2\right]$$

$$\Rightarrow 4y^2p = py\left[3p^2\frac{dp}{dy} + 16y\right] - p^3y\frac{dp}{dy} - p^4 - 8y^3\frac{dp}{dy} - 8py^2$$

$$\Rightarrow 4y^2p = 3p^3y\frac{dp}{dy} + 16py^2 - p^3y\frac{dp}{dy} - p^4 - 8y^3\frac{dp}{dy} - 8py^2$$

$$\Rightarrow 4py^2 - p^4 - 8y^3\frac{dp}{dy} + 2p^3y\frac{dp}{dy} = 0 \qquad \Rightarrow 4py^2 - 8y^3\frac{dp}{dy} - p^4 + 2p^3y\frac{dp}{dy} = 0$$

$$\Rightarrow 4y^2\left[p - 2y\frac{dp}{dy}\right] - p^3\left[p - 2y\frac{dp}{dy}\right] = 0 \qquad \Rightarrow \left[p - 2y\frac{dp}{dy}\right]\left[4y^2 - p^3\right] = 0.$$

Thus we have $p - 2y\frac{dp}{dy} \Rightarrow 2y\frac{dp}{dy} = p \Rightarrow 2\frac{dp}{p} = \frac{dy}{y}$.

Integrating, we get $\qquad 2\int\frac{dp}{p} = \int\frac{dy}{y} + \log c'$

$$\Rightarrow 2\log p = \log y + \log c' \Rightarrow \log\left(p^2\right) = \log(yc') \Rightarrow \left(p^2\right) = (yc') \Rightarrow p = \sqrt{yc'}$$

Substituting the value of p in (i), we get

$$\left(\sqrt{yc'}\right)^3 - 4xy\left(\sqrt{yc'}\right) + 8y^2 = 0 \Rightarrow c'\left(\sqrt{c'}\right) - 4xy\left(\sqrt{c'}\right) + 8\sqrt{y} = 0$$

$$\Rightarrow 8\sqrt{y} = \sqrt{c'}[4x - c'] \qquad \Rightarrow \sqrt{y} = \frac{1}{4}\sqrt{\frac{c'}{4}}[4x - c'] \;\Rightarrow \sqrt{y} = \sqrt{\frac{c'}{4}}\left[x - \frac{c'}{4}\right]$$

$$\Rightarrow \sqrt{y} = \sqrt{c}[x - c], \quad \text{where } c = \frac{c'}{4}$$

which is the required solution.

Q.No.3.: Solve the equation $p^2y + 2px = y$.

Sol.: The given equation is $p^2y + 2px = y$. (i)

$$\Rightarrow x = \frac{y - p^2y}{2p}.$$

Differentiating w. r. t. y, we get

$$\frac{dx}{dy} = \frac{1}{p} = \frac{1}{2}\left[\frac{p\left[1 - \left(p^2 + y.2p\frac{dp}{dy}\right)\right] - \left[\left(y - p^2y\right)\frac{dp}{dy}\right]}{p^2}\right]$$

$$\Rightarrow 2p = p - p^3 - 2p^2y\frac{dp}{dy} - y\frac{dp}{dy} + p^2y\frac{dp}{dy}$$

$$\Rightarrow -p - p^3 - 2p^2y\frac{dp}{dy} - y\frac{dp}{dy} + p^2y\frac{dp}{dy} = 0 \Rightarrow -p - p^3 - p^2y\frac{dp}{dy} - y\frac{dp}{dy} = 0$$

$$\Rightarrow -p\left(1 + p^2\right) - y\frac{dp}{dy}\left(1 + p^2\right) = 0 \quad \Rightarrow \left(p + y\frac{dp}{dy}\right)\left(1 + p^2\right) = 0.$$

Thus we have $p + y\frac{dp}{dy} = 0 \;\Rightarrow pdy + ydp = 0 \;\Rightarrow \frac{d}{dy}(py) = 0$.

Integrating, we get $py = c \;\Rightarrow p = \frac{c}{y}$.

Substitute the value of p in (i), we get

$$\frac{c^2}{y^2}.y + 2\frac{c}{y}.x = y \quad \Rightarrow y^2 = 2cx + c^2,$$

which is the required solution.

Q.No.4.: Solve the equation $x - yp = ap^2$.

Sol.: The given equation is $x - yp = ap^2$.

$$\Rightarrow x = yp + ap^2 \,. \qquad \text{(i)}$$

Differentiating w. r. t. y, we get

$$\Rightarrow \frac{dx}{dy} = y\frac{dp}{dy} + p + a2p.\frac{dp}{dy} \qquad \Rightarrow \frac{1}{p} = y\frac{dp}{dy} + p + 2ap\frac{dp}{dy}$$

$$\Rightarrow \left(p - \frac{1}{p}\right) + y\frac{dp}{dy} + 2ap\frac{dp}{dy} = 0 \quad \Rightarrow \frac{dp}{dy}(y + 2ap) + \frac{1-p^2}{p} = 0$$

$$\Rightarrow \frac{dp}{dy}(y + 2ap) = \frac{1-p^2}{p} \qquad \Rightarrow \frac{dy}{dp} = \frac{p(y+2ap)}{1-p^2}$$

$$\Rightarrow \frac{dy}{dp} + \left(\frac{p}{p^2-1}\right)y = \left(\frac{2ap^2}{1-p^2}\right),$$

which is Leibnitz's linear equation, in y

$$\therefore \text{I. F.} = e^{\int \frac{p}{p^2-1}dp} = e^{\frac{1}{2}\int \frac{2p}{p^2-1}dp} = e^{\frac{1}{2}\log\left(p^2-1\right)} = e^{\log\sqrt{p^2-1}} = \sqrt{p^2-1}\,.$$

$\therefore$ The solution is $\quad y(\text{I. F.}) = \int Q(\text{I. F.})dp + c$

$$\Rightarrow y\sqrt{p^2-1} = \int \frac{2ap^2}{1-p^2}.\sqrt{p^2-1}dp + c \qquad \Rightarrow y\sqrt{p^2-1} + \int \frac{2ap^2}{\sqrt{p^2-1}}dp = c$$

$$\Rightarrow y\sqrt{p^2-1} + 2a\left[\int \sqrt{p^2-1}dp + \int \frac{1}{\sqrt{p^2-1}}dp\right] = c$$

$$\Rightarrow y\sqrt{p^2-1} + 2a\left[\left(\frac{p\sqrt{\left(p^2-1\right)}}{2} - \frac{1}{2}\cosh^{-1}p\right) + \cosh^{-1}p\right]$$

$$\Rightarrow (y + ap)\sqrt{p^2-1} + a\cosh^{-1}p = c,$$

with the given relation, constitute the general solution.

Q.No.5.: Solve the equation $p = \tan\left(x - \frac{p}{1+p^2}\right)$.

Sol.: The given equation is $p = \tan\left(x - \frac{p}{1+p^2}\right) \Rightarrow x = \tan^{-1} p + \frac{p}{1+p^2}$. (i)

Differentiating w. r. t. y, we get

$$\frac{dx}{dy} = \frac{1}{p} = \frac{1}{1+p^2}\cdot\frac{dp}{dy} + \frac{\left(1+p^2\right) - 2p^2}{\left(1+p^2\right)^2}\cdot\frac{dp}{dy}$$

$$\Rightarrow \frac{1}{p} = \frac{2\left(1+p^2\right) - 2p^2}{\left(1+p^2\right)^2}\frac{dp}{dy} \qquad \Rightarrow dy = \frac{2p}{\left(1+p^2\right)^2}dp.$$

Integrating both sides, we get $\int dy = \int \frac{2p}{\left(1+p^2\right)^2}dp + c$

Putting $1+p^2 = t$, so that $2pdp = dt$

$$\Rightarrow y = \int \frac{dt}{t^2} + c \quad \Rightarrow y = -\frac{1}{t} + c \quad \Rightarrow y = -\frac{1}{1+p^2} + c$$

$$\Rightarrow y + \left(1+p^2\right)^{-1} = c. \qquad \text{(ii)}$$

Thus the equations (i) and (ii) taken together constitute the general solution.

Q.No.6.: Find the primitive of $p^2 - xp + y = 0$.

Sol.: Solving for x, we obtain $x = \frac{p^2 + y}{p}$.

Differentiating w.r.t. y, we get

$$\frac{dx}{dy} = \frac{1}{p} = \frac{dp}{dy} + \frac{\left(p - y\frac{dp}{dy}\right)}{p^2} \Rightarrow \left(p^2 - y\right)\frac{dp}{dy} = 0.$$

So $\frac{dp}{dy} = 0$ with solution $p = c =$ constant.

Eliminating p from the given equation

$c^2 - xc + y = 0 \Rightarrow y = cx - c^2$, which is the required primitive.

Q.No.7.: Solve $xp^2 - yp - y = 0$.

Sol.: Solving for x, we obtain $x = \frac{y(1+p)}{p^2}$.

Differentiating w.r.t. y, we get

$$\frac{dx}{dy} = \frac{1}{p} = \frac{1+p}{p^2} + \left(\frac{p^2 - (1+p)2p}{p^4}\right)\frac{dp}{dy} \Rightarrow \frac{dp}{dy} = \frac{p}{2+p} \cdot \frac{1}{y}.$$

Separating the variables and integrating $\frac{2+p}{p}dp = \frac{dy}{y}$

$$\Rightarrow \log p^2 + p = \log\ y + c_1 \Rightarrow \log\left(p^2 e^p\right) = \log\ y + c_1$$

$$\Rightarrow y = cp^2 e^p \text{ and } x = \frac{1+p}{p^2}.y = c(1+p)e^p,$$

which is required primitive in the parametric form.

Home Assignments

Q.No.1.: Obtain the primitive for the equation $6p^2y^2 - y + 3px = 0$.

Ans.: $y^3 = 3cx + 6c^2$.

Q.No.2.: Obtain the primitive for the equation $4y^2 + p^3 = 2xyp$.

Ans.: $2y = c(c-x)^2$.

Q.No.3.: Obtain the primitive for the equation $3py + 4x = p^3y$.

Ans.: $y = c.P$, $x = \frac{1}{4}cp\left(p^2 - 3\right)P$, where $P^{-1} = -\left(p^2 - 4\right)^{9/10}.\left(p^2 + 1\right)^{3/5}$.

Q.No.4.: Obtain the primitive for the equation $y^2p^3 - y + 2px = 0$.

Ans.: $y^2 = 2cx + c^3$.

Q.No.5.: Obtain the primitive for the equation $3px - y + 6p^2y^2 = 0$.

Ans.: $y^3 = 3c(x + 2c)$.

Q.No.6.: Obtain the primitive for the equation $p^2 + y - c = 0$.

Ans.: $x = c - 2[p + \log(p-1)]$, $y = c - 2\left[\frac{1}{2}p^2 + p + \log(p-1)\right]$.

Q.No.7.: Solve the differential equation $y = 2px + p^2 y$.

Ans.: $y^2 = 2cy + c^2$.

Q.No.8.: Solve the differential equation $y^2 \log y = xyp + p^2$.

Ans.: $\log y = cx + c^2$.

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7^{th} Topic

Differential Equations of First order and higher degree

"Clairaut's Equation"

(Equation solvable for y)

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Clairaut's Equation:

Alexis Claude de Clairault (or Clairaut) (3 May 1713 – 17 May 1765)
He was a prominent French mathematician, astronomer, geophysicist, and intellectual.

In mathematics, a Clairaut's equation is a differential equation of the form

$$y(x) = x\frac{dy}{dx} + f\left(\frac{dy}{dx}\right) \Rightarrow y = px + f(p)$$

This equation was named after Alexis Clairaut, who introduced it in 1734. This is a first order and higher degree differential equation, linear in y and x.

Note:

A first-order partial differential equation is also known as Clairaut's equation or Clairaut equation:

$$u(x,y) = xu_x + yu_y + f\left(u_x, u_y\right).$$

Solution: Given differential equation is $y = px + f(p)$. (i)

Differentiating w.r.t. x, we have $p = p + x\dfrac{dp}{dx} + f'(p)\dfrac{dp}{dx}$

$\Rightarrow [x + f'(p)]\dfrac{dp}{dx} = 0$.

Discarding the factor $x + f'(p) = 0$, we have $\dfrac{dp}{dx} = 0$.

Integrating, we get $p = c$. (ii)

Substituting the value of p in (i), we get $y = cx + f(c)$, (iii)

which is the general (or complete) solution of (i).

This is one parameter family of straight lines; with c as parameter.

Hence, the solution of the Clairaut's equation is obtained on replacing p by c.

Note: The significance of the factor $\left(1 + \dfrac{p}{1 + x^2p^4}\right) = 0$, which we did not consider, will not be considered here as it concerns 'singular solution' of (i), whereas we are interested only in finding general solution.

Singularity:

In mathematics, a **singularity** is in general a point at which a given mathematical object is not defined or a point of an exceptional set where it fails to be well-behaved in some particular way, such as differentiability.

Examples:

- The function $f(x)=\frac{1}{x}$ on the real line has a singularity at x = 0, where it seems to "explode" to $\pm\infty$ and is not **defined.**
- The function $g(x)=|x|$ also has a singularity at x = 0, since it has not **differentiable** there.
- The graph defined by $y^2=x$ also has a singularity at (0, 0). This time it has a 'corner' (**vertical tangent**) at that point.

Singular solution:

A **singular solution** $y_s(x)$ of an ordinary differential equation is a solution that is tangent to every solution from the family of general solutions. By *tangent* we mean that there is a point *x* where $y_s(x) = y_c(x)$ and $y'_s(x) = y'_c(x)$ where y_c is any general solution.

Usually, singular solutions appear in differential equations when there is a need to divide in a term that might be equal to zero. Therefore, when one is solving a differential equation and using division one must check what happens if the term is equal to zero, and whether it leads to a singular solution.

Singular solution" of Clairaut's equation:

Clairaut's equation is $y=px+f(p)$. (i)

General (or complete) solution of (i) is $y=cx+f(c)$, (iii)

Besides the complete solution (iii), one may obtain a "singular solution" of Clairaut's equation, which satisfies the Clairaut's equation (i), but not obtained from the complete solution (iii) for any value of c.

Method of solution:

If we eliminate p from $x+f'(p)=0$ and (i), we get an equation involving no constant. This is the singular solution of (i), which gives the envelope of the family of straight lines (iii).

To obtain the singular solution, we proceed as follows:

(i) Find the general solution by replacing p by c, i.e., we get (iii) means $y=cx+f(c)$.

(ii) Differentiate this w.r.t. 'c' giving $x+f'(c)=0$. (iv)

(iii) Eliminate c from (iii) and (iv) which will be the singular solution.

Remarks:

Equations, which are not in the Clairaut's form can be reduced to Clairaut's form by suitable substitutions (transformations).

Now let us solve some differential equations, which are solvable for y and of Clairaut's form:

Q.No.1.: Solve $p = \sin(y - xp)$. Also find its **singular solution**.

Sol.: The given equation is $\sin^{-1} p = y - xp$.

$\Rightarrow y = px + \sin^{-1} p$, which is of Clairaut's form.

$\therefore$ Its general solution is $y = cx + \sin^{-1} c$. (i)

To find the singular solution:

Differentiating (i) w.r.t. c, we get

$$0 = x + \frac{1}{\sqrt{1 - c^2}} \Rightarrow \frac{1}{x^2} = 1 - c^2 \Rightarrow c^2 = \frac{x^2 - 1}{x^2} \Rightarrow c = \frac{\sqrt{x^2 - 1}}{x}. \quad \text{(ii)}$$

Now substituting this value of c in (i), we get

$$y = \sqrt{\left(x^2 - 1\right)} + \sin^{-1}\left\{\frac{\sqrt{\left(x^2 - 1\right)}}{x}\right\},$$

which is the desired singular solution.

Q.No.2.: Find the **general and singular solution** of the equations:

(i) $y = xp + \frac{a}{p}$, **(ii)** $p = \log(px - y)$,

(iii) $y = px - \sqrt{1 + p^2}$, **(iv)** $\sin(px - y) = p$

Sol.: (i): The given equation is $y = xp + \frac{a}{p}$,

which is of Clairaut's form.

$\therefore$ Its general solution is $y = xc + \frac{a}{c}$. Ans. (i)

To find the singular solution:

Differentiating (i) w.r.t. c, we get

$$0 = x - \frac{a}{c^2} \quad \Rightarrow x = \frac{a}{c^2} \quad \Rightarrow c = \sqrt{\frac{a}{x}}.$$

Now substituting this value of c in (i), we get

$$y = x\sqrt{\frac{a}{x}} + \frac{a}{\sqrt{\frac{a}{x}}} \quad \Rightarrow y = \sqrt{ax} + \sqrt{ax} \quad \Rightarrow y = 2\sqrt{ax} \quad \Rightarrow y^2 = 4ax,$$

which is the required singular solution.

(ii): The given equation is $p = \log(px - y)$

$$\Rightarrow e^p = px - y \quad \Rightarrow y = px - e^p,$$

which is of Clairaut's form.

$\therefore$ Its general solution is $y = xc - e^c \;\Rightarrow c = \log(cx - y)$. Ans. (i)

To find the singular solution:

Differentiating (i) w.r.t. c, we get

$$1 = \frac{1}{cx - y}.x \quad \Rightarrow cx - y = x \quad \Rightarrow cx = x + y \quad \Rightarrow c = \frac{x + y}{x}$$

Now substituting this value of c in (i), we get

$$\frac{x+y}{x} = \log\left[x.\frac{x+y}{x} - y\right] \quad \Rightarrow \frac{x+y}{x} = \log[x] \quad \Rightarrow \frac{y}{x} = \log x - 1$$

$$\Rightarrow y = x(\log x - 1),$$

which is the required singular solution.

(iii): The given equation is $y = px - \sqrt{1 + p^2}$,

which is of Clairaut's form.

$\therefore$ Its general solution is $y = cx - \sqrt{1 + c^2}$. Ans. (i)

To find the singular solution:

Differentiating (i) w.r.t. c, we get

$$0 = x - \frac{1}{2}\left(1 + c^2\right)^{-1/2}.2c \Rightarrow x - \frac{c}{\sqrt{1 + c^2}} = 0 \Rightarrow \frac{c}{\sqrt{1 + c^2}} = x \Rightarrow \frac{c^2}{1 + c^2} = x^2$$

$$\Rightarrow c^2 = x^2 + x^2c^2 \Rightarrow c^2\left[1 - x^2\right] = x^2 \;\Rightarrow c = \sqrt{\frac{x^2}{1 - x^2}} = \frac{x}{\sqrt{1 - x^2}}.$$

Now substituting this value of c in (i), we get

$$y = \frac{x}{\sqrt{1-x^2}}.x - \sqrt{1+\frac{x^2}{1-x^2}} \quad \Rightarrow y\sqrt{\left(1-x^2\right)} = x^2 - \sqrt{1-x^2+x^2} = x^2 - 1 = -\left(1-x^2\right)$$

$$\Rightarrow y + \sqrt{1-x^2} = 0,$$

which is the required singular solution.

(iv): The given equation is $\sin(px - y) = p$

$$\Rightarrow px - y = \sin^{-1} p \quad \Rightarrow y = px - \sin^{-1} p,$$

which is of Clairaut's form.

$\therefore$ Its general solution is $y = cx - \sin^{-1} c$. Ans. (i)

To find the singular solution:

Differentiating (i) w.r.t. c, we get

$$0 = x - \frac{1}{\sqrt{1-c^2}} \quad \Rightarrow \frac{1}{\sqrt{1-c^2}} = x \quad \Rightarrow \frac{1}{1-c^2} = x^2 \quad \Rightarrow 1 = x^2 - c^2x^2$$

$$\Rightarrow 1 - x^2 = -c^2x^2 \quad \Rightarrow c^2x^2 = x^2 - 1 \quad \Rightarrow c = \frac{\sqrt{x^2-1}}{x}.$$

Now substituting this value of c in (i), we get

$$y = \frac{\sqrt{x^2-1}}{x}.x - \sin^{-1}\frac{\sqrt{x^2-1}}{x} \quad \Rightarrow y = \sqrt{x^2-1} - \sin^{-1}\frac{\sqrt{x^2-1}}{x},$$

which is the required singular solution.

Q.No.3.: Find the **general and singular solution** of $y = xy' - (y')^2$.

Sol.: The given equation is $y = xy' - (y')^2$.

$\Rightarrow y = xp - p^2$, which is of Clairaut's form.

Its general solution is obtained by replacing p by a constant c.

$\therefore$ Its general solution is $y = xc - c^2$.

To find the singular solution:

To obtain the singular solution, differentiate the general solution w.r.t. 'c', we get

$$0 = x - 2c \qquad \therefore c = \frac{x}{2}.$$

Eliminating c form the GS, we get

$$y = cx - c^2 = \frac{x}{2}.x - \frac{x^2}{4} = \frac{x^2}{4} \Rightarrow x^2 = 4y.$$

Thus, the singular solution $x^2 = 4y$ (which is a parabola) is the envelop of the one parameter family of straight lines $y = cx - c^2$ (representing the general solution).

Q.No.4.: Solve $y + 2\left(\frac{dy}{dx}\right)^2 = (x+1)\frac{dy}{dx}$.

Sol.: The given equation is $y + 2\left(\frac{dy}{dx}\right)^2 = (x+1)\frac{dy}{dx}$.

$$\Rightarrow y + 2p^2 = (x+1)p \quad \Rightarrow y = -2p^2 + (x+1)p \quad \Rightarrow y = px + \left(p - 2p^2\right),$$

which is of Clairaut's form.

$\therefore$ Its general solution is $y = cx + \left(c - 2c^2\right)$. Ans.

Q.No.5.: Solve $(y - px)(p-1) = p$.

Sol.: The given equation is $(y - px)(p-1) = p \quad \Rightarrow y = px + \frac{p}{p-1}$,

which is of Clairaut's form.

$\therefore$ Its general solution is $y = cx + \frac{c}{c-1} \quad (y - cx)(c-1) = c$. Ans.

Q.No.6.: Solve $(x-a)\left(\frac{dy}{dx}\right)^2 + (x-y)\frac{dy}{dx} - y = 0$.

Sol.: The given equation is $(x-a)\left(\frac{dy}{dx}\right)^2 + (x-y)\frac{dy}{dx} - y = 0$

$$\Rightarrow (x-a)p^2 + xp - yp - y = 0 \quad \Rightarrow x(p+1)p - y(p+1) - ap^2 = 0$$

$\Rightarrow y = xp - \frac{ap^2}{p+1}$, which is of Clairaut's form.

$\therefore$ Its general solution is $y = xc - \frac{ac^2}{c+1} \quad \Rightarrow (y - cx)(c+1) + ac^2 = 0$. Ans.

Q.No.7.: Find the primitive $y = 4xp - 16y^3p^2$.

Sol.: The equation is not in the Clairaut's form.

Multiplying the given equation by y^3, we have $y^4 = 4xy^3p - 16y^6p^2$.

Put $y^4 = v$ so $4y^3\frac{dy}{dx} = \frac{dv}{dx}$.

Then $v = x\frac{dv}{dx} - \left(\frac{dv}{dx}\right)^2$,

which is a Clairaut's equation in v.

Its general solution is obtained by replacing $\frac{dv}{dx}$ by a constant c.

Thus, the general solution is

$v = xc - c^2 \Rightarrow y^4 = cx - c^2$.

Equations reducible to Clairaut's form:

Q.No.8.: Solve the following differential equation by reducing it to Clairaut's form by suitable substitution $x^2(y - px) = yp^2$.

Sol.: The given equation is $x^2(y - px) = yp^2$. (i)

Putting $x^2 = u$ and $y^2 = v$ (ii)

so that $2xdx = du$ and $2ydy = dv$.

$$\therefore p = \frac{dy}{dx} = \frac{\frac{dv}{2y}}{\frac{du}{2x}} = \frac{x}{y}\frac{dv}{du} = \frac{\sqrt{u}}{\sqrt{v}}P, \quad \text{where } P = \frac{dv}{du}$$

Then the given equation becomes

$$\Rightarrow u\left[\sqrt{v} - \frac{\sqrt{u}}{\sqrt{v}}.P\sqrt{u}\right] = \sqrt{v}\frac{u}{v}P^2 \quad \Rightarrow \sqrt{v} - \frac{u}{\sqrt{v}}P = \frac{1}{\sqrt{v}}P^2$$

$\Rightarrow v - uP = P^2 \quad \Rightarrow v = uP + P^2$,

which is of Clairaut's form.

$\therefore$ Its general solution is $v = uc + c^2 \Rightarrow y^2 = cx^2 + c^2$. Ans.

Q.No.9.: Solve the following differential equation by reducing it to Clairaut's form by suitable substitution $(px - y)(x + py) = 2p$.

Sol.: The given equation is $(px-y)(x+py)=2p$ (i)

Now putting $u=x^2$ and $v=y^2$,

so that $du=2xdx$ and $dv=2ydy$.

$\therefore p=\frac{dy}{dx}=\frac{x}{y}\frac{dv}{du}=\frac{\sqrt{u}}{\sqrt{v}}P$, where $P=\frac{dv}{du}$.

Then, the given equation becomes

$$\left[\frac{\sqrt{u}}{\sqrt{v}}.P.\sqrt{u}-\sqrt{v}\right]\left[\sqrt{u}+\frac{\sqrt{u}}{\sqrt{v}}P\sqrt{v}\right]=2\frac{\sqrt{u}}{\sqrt{v}}P \Rightarrow (Pu-v)(P+1)=2P$$

$$Pu-v=\frac{2P}{P+1} \quad \Rightarrow v=Pu-\frac{2P}{P+1},$$

which is of Clairaut's form.

$\therefore$ Its general solution is $\Rightarrow v=cu-\frac{2c}{c+1} \quad \Rightarrow y^2=cx^2-\frac{2c}{c+1}$. Ans.

Q.No.10.: Solve the following differential equation by reducing it to Clairaut's form by suitable substitution $(px-y)(py+x)=a^2p$.

Sol.: Put $x^2=u$ and $y^2=v$, so that $2xdx=du$ and $2ydy=dv$

$$\therefore P=\frac{dy}{dx}=\frac{\frac{dv}{y}}{\frac{du}{x}}=\frac{x}{y}P, \text{ where } P=\frac{dv}{du}.$$

Then, the given equation becomes $\left(\frac{xP}{y}.x-y\right)\left(\frac{xP}{y}.y+x\right)=a^2\frac{xP}{y}$

$$\Rightarrow (uP-v)(P+1)=a^2P \Rightarrow uP-v=\frac{a^2P}{P+1} \Rightarrow v=uP-\frac{a^2P}{(P+1)},$$

which is a Clairaut's form.

$\therefore$ Its general solution is $v=uc-\frac{a^2c}{(c+1)} \Rightarrow y^2=cx^2-\frac{a^2c}{(c+1)}$.

Q.No.11.: Solve the following differential equation by reducing it to Clairaut's form by suitable substitution $(px+y)^2=py^2$.

Sol.: The given equation is $(px+y)^2=py^2$.

Now putting $u = y$ and $v = xy$,

so that $du = dy$, and $dv = ydx + xdy = udx + \frac{v}{u}du$

$$\Rightarrow dv - \frac{v}{u}du = udx \quad \Rightarrow dx = \frac{udv - vdu}{u^2}$$

$$\therefore p = \frac{dy}{dx} = \frac{du}{\frac{udv - vdu}{u^2}} = \frac{u^2}{uP - v}, \text{ where } P = \frac{du}{dv}.$$

Then the given equation becomes

$$\left[\frac{u^2}{uP - v}\cdot\frac{v}{u} + u\right]^2 = \frac{u^2}{uP - v}.u^2 \quad \Rightarrow [uv + u(uP - v)]^2 = u^4(uP - v)$$

$$\Rightarrow \left[uv + u^2P - uv\right]^2 = u^4(uP - v) \quad \Rightarrow u^4P^2 = u^4(uP - v)$$

$$\Rightarrow uP - v = P^2 \quad \Rightarrow v = uP - P^2,$$

which is of Clairaut's form.

$\therefore$ Its general solution is $\Rightarrow v = uc - c^2 \quad \Rightarrow xy = cy - c^2$. Ans.

Q.No.12.: Solve the following differential equation by reducing it to Clairaut's form by suitable substitution $e^{4x}(p - 1) + e^{2y}p^2 = 0$.

Sol.: The given equation is $e^{4x}(p - 1) + e^{2y}p^2 = 0$ (i)

Now putting $X = e^{2x}$ and $Y = e^{2y}$,

so that $dX = 2e^{2x}dx$ and $dY = 2e^{2y}dy$.

$$\therefore p = \frac{dy}{dx} = \frac{e^{2x}}{e^{2y}}\frac{dY}{dX} = \frac{X}{Y}P, \text{ where } P = \frac{dY}{dX}.$$

Then, the given equation becomes

$$X^2\left(\frac{X}{Y}P - 1\right) + Y.\frac{X^2}{Y^2}P^2 = 0 \quad \Rightarrow XP - Y + P^2 = 0$$

$\Rightarrow Y = PX + P^2$, which is of Clairaut's form.

$\therefore$ Its general solution is $Y = cX + c^2 \Rightarrow e^{2y} = ce^{2x} + c^2$. Ans.

Q.No.13.: Solve the following differential equation by reducing it to Clairaut's form by suitable substitution $(p-1)e^{3x}+p^3e^{2y}=0$.

Sol.: To reduce the Clairaut's form, put $e^x=X$, $e^y=Y$.

$\Rightarrow e^x dx = dX,\ e^y dy = dY$

$\Rightarrow dx = \dfrac{dX}{e^x},\ dy = \dfrac{dY}{e^y}$

$\Rightarrow dx = \dfrac{dX}{X},\ dy = \dfrac{dY}{Y}$

Then $p=\dfrac{dy}{dx}=\dfrac{XdY}{YdX}=\dfrac{X}{Y}P$.

Substituting for $p=\dfrac{Y}{X}P$, the given equation reduces to

$$\left(\frac{X}{Y}P-1\right)X^3+\frac{X^3}{Y^3}P^3Y^2=0 \Rightarrow Y=XP+P^3,$$

which is the Clairaut's differential equation.

$\therefore$ Its general solution is $Y=Xc+c^2 \Rightarrow e^y=ce^x+c^3$.

Q.No.14.: Solve the following differential equation by reducing it to Clairaut's form by

suitable substitution $\sin y\cos^2 x=\cos^2 yp^2+\sin x.\cos x\cos yp$.

Sol.: To reduce to Clairaut's form, put $\sin y=u$ and $\sin x=v$.

$$p=\frac{dy}{dx}=\frac{du/\cos y}{dv/\cos x}=\frac{du\cos x}{dv\cos y}$$

Then $\sin y\cos^2 x=\cos^2 yp^2+\sin x.\cos x\cos yp$

$$\Rightarrow u\cos^2 x=\cos^2 y\left(\frac{du\cos x}{dv\cos y}\right)^2+v\cos x\cos y\left(\frac{du\cos x}{dv\cos y}\right)$$

$$\Rightarrow u=\left(\frac{du}{dv}\right)^2+v\left(\frac{du}{dv}\right)\Rightarrow u=P^2+vP,\ \text{ where } P=\frac{du}{dv}.$$

$\Rightarrow u=vP+P^2$

which is the Clairaut's form.

$\therefore$ Its general solution is $u=cv+c^2 \Rightarrow \sin y=c.\sin x+c^2$.

Q.No.15.: Solve the following differential equation by reducing it to Clairaut's form by

suitable substitution $\left(x^2+y^2\right)(1+p)^2=2(x+y)(1+p)(x+yp)-(x+yp)^2$.

Sol.: Rewriting $x^2+y^2=\dfrac{2(x+y)(x+yp)}{(1+p)}-\left(\dfrac{x+yp}{1+p}\right)^2$.

Put $x^2+y^2=v$ so $2x+2yp=\dfrac{dv}{dx}$ and put $x+y=u$ so $1+p=\dfrac{du}{dx}$.

Then $\dfrac{dv}{du}=\dfrac{2(x+yp)}{(1+p)}$.

Substituting the given equation reduces to $v=u\dfrac{dv}{du}-\dfrac{1}{4}\left(\dfrac{dv}{du}\right)^2$,

which is a Clairaut's form.

$\therefore$ Its general solution is $v=uc-\dfrac{1}{4}c^2 \Rightarrow x^2+y^2=c(x+y)-\dfrac{1}{4}c^2$.

Home Assignments

Q.No.1.: Find the general solution (GS) and singular solution (SS) of the Clairaut's equation $y=xp+\dfrac{ap}{\sqrt{1+p^2}}$.

Ans.: GS: $y=cx+\dfrac{ac}{\sqrt{1+c^2}}$; SS: $x^{2/3}+y^{2/3}=a^{2/3}$.

Q.No.2.: Find GS and SS of $p=px+p^3$.

Ans.: GS: $y=cx+c^3$; SS: $27y^2=-4x^3$.

Q.No.3.: Find GS and SS of $(x^2-1)p^2-2xyp+y^2-1=0$.

Ans.: GS: $(x^2-1)c^2-2xyc+y^2-1=0$. SS: $x^2+y^2=1$.

Q.No.4.: Solve the differential equation $y=px+\log p$.

Ans.: GS: $y=cx+\log c$.

Q.No.5.: Solve the differential equation $p^2y+px^3-x^3y=0$.

Ans.: $y^2=cx^2+c^2$.

Q.No.6.: Solve the differential equation $y = xy' + \dfrac{a}{2y'}$.

Ans.: GS: $y = cx + \dfrac{a}{2c}$, one parameter family of straight lines. SS: $y^2 = 2ax$ parabola, which is the envelop of the straight lines.

Q.No.7.: Find GS and SS of $y = px + 2p^2$.

Ans.: GS: $y = cx + c^2$, SS: $x^2 - 8y$ (Parabola).

Q.No.8.: Find GS and SS of $y = 3px + 6p^2y^2$.

Ans.: GS: $y^3 = cx + \dfrac{2}{3}c^2$, SS: $8y^3 + 3x^2 = 0$ (semicubical parabola).

Q.No.9.: Solve the following differential equation by reducing it to Clairaut's form by suitable substitution $a^2p = (py + x)(px - y)$.

Ans.: $y^2 = \dfrac{cx^2 - a^2c}{(c+1)}$.

Hint: Use $x^2 = u, y^2 = v, P = \dfrac{dv}{du}, v = Pu - \dfrac{a^2P}{P+1}$.

Q.No.10.: Solve the differential equation $y + y^2 + p(2y - 2xy - x + 2) + xp^2(x-2) = 0$.

Ans.: GS: $(y - cx + 2c)(y - cx + 1) = 0$.

Q.No.11.: Solve the following differential equation by reducing it to Clairaut's form by suitable substitution $(px^2 + y^2)px + y = (p+1)^2$.

Ans.: $c^2(x + y) - cxy - 1 = 0$.

Hint: Use $x + y = u, xy = v, P = \dfrac{dv}{du}, v = Pu - \dfrac{1}{P}$.

Q.No.12.: Solve the differential equation $y = px + \sqrt{a^2p^2 + b^2}$.

Ans.: $y = cx + \sqrt{a^2c^2 + b^2}$.

Q.No.13.: Solve the differential equation $p^2(x^2 - 1) - 2pxy + y^2 - 1 = 0$.

Ans.: $(y - cx)^2 = 1 + c^2$.

Q.No.14.: Solve the differential equation $e^{3x}(p-1) + p^3e^{2y} = 0$.

Ans.: $e^y = ce^x + c^2$.

Q.No.15.: Solve the differential equation $(y + px)^2 = x^2 p$.

Ans.: $xy = cx - c^2$.

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Applications

of

Ordinary Differential Equations of first order

"Orthogonal Trajectories"

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Orthogonal trajectories:

Orthogonal: right angle, 90^0, perpendicular

Trajectory (*Latin word*): cut across

(i) Trajectory: **A curve, which cuts every member of a given family of curves according to some definite law, is called a trajectory of the family.**

(ii) Orthogonal trajectory: **A curve, which cuts every member of a given family of curves at right angles, is called an orthogonal trajectory of the family.**

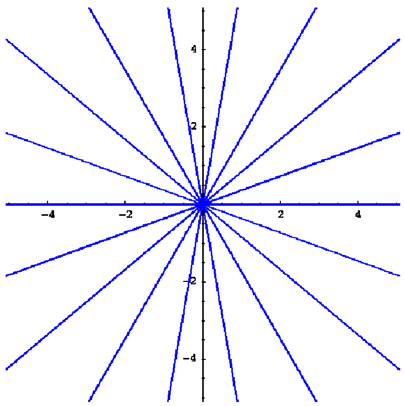
(iii) Orthogonal trajectories: **Two families of curves are said to be orthogonal if every member of either family cuts each member of the other family at right angles.**

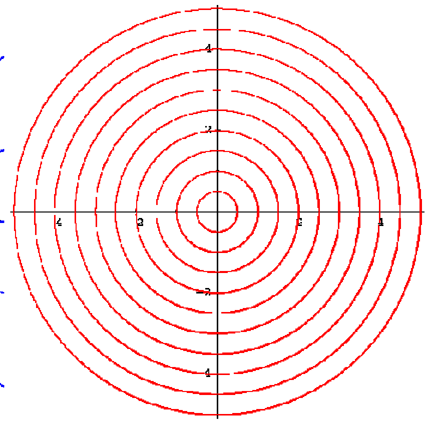
In other words, orthogonal trajectories are a family of curves in the plane that intersect a given family of curves at right angles.

In applied mathematics, we frequently come across families of curves related in this manner.

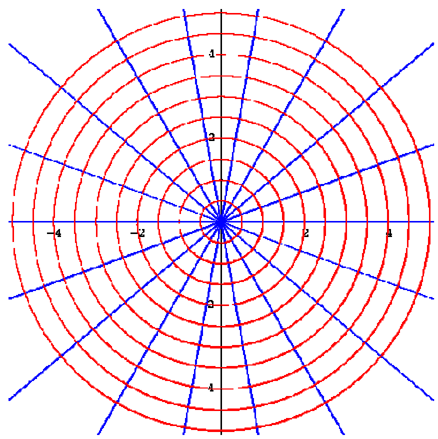
Classical examples of Orthogonal trajectories are:

1. **Meridians and parallels on world globe.**
2. **Curves of steepest descent and contour lines on a map.**
3. **Curves of electric force and equipotential lines (constant voltage).**
4. **Streamlines and equipotential lines (of constant velocity potential)**
5. **Lines of heat flow and isothermal curves.**





If we draw the two families together on the same graph we get



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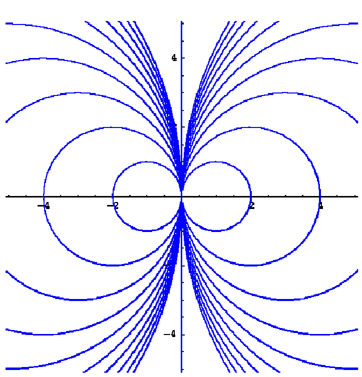
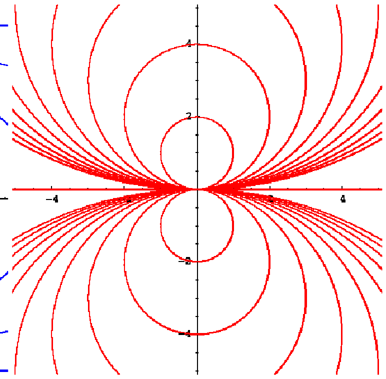
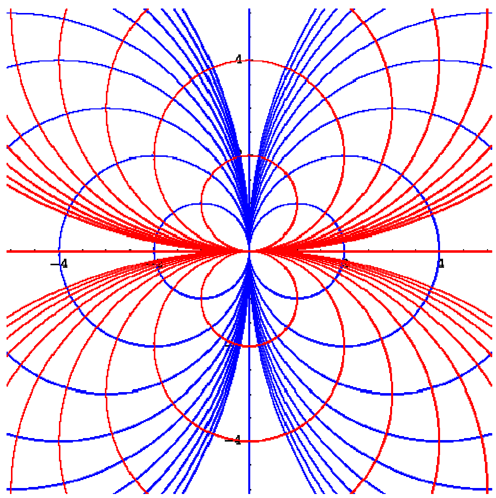
 

If we put both families together, we appreciate better the orthogonality of the curves (see the picture below).



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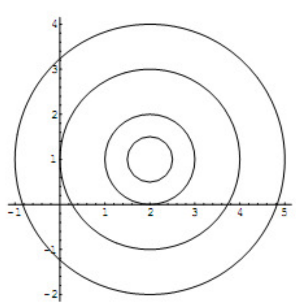
The one-parameter family of curves

$$(x-2)^2 + (y-1)^2 = C \qquad (C \geq 0) \tag{a}$$

is a family of circles with center at the point $(2, 1)$ and radius $\sqrt{C}$.



If we differentiate this equation with respect to x, we get

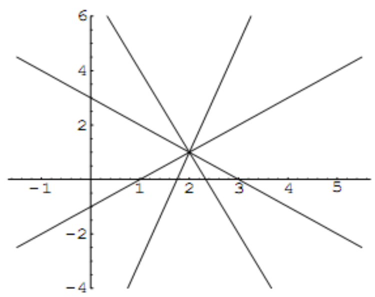
$$2(x-2)+2(y-1)\,y'=0$$

and

$$y'=-\frac{x-2}{y-1} \tag{b}$$

Now consider the family of straight lines passing through the point $(2,1)$:

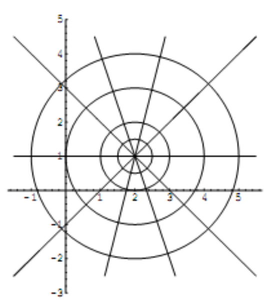
$$y-1=K(x-2). \tag{c}$$



The differential equation for this family is

$$y'=\frac{y-1}{x-2} \qquad \text{(verify this)} \tag{d}$$

Comparing equations (b) and (d) we see that right side of (b) is the negative reciprocal of the right side of (d). Therefore, we can conclude that if $P\,(x_0, y_0)$ is a point of intersection of one of the circles and one of the lines, then the line and the circle are perpendicular (orthogonal) to each other at P. The following figure shows the two families drawn in the same coordinate system.



A curve that intersects each member of a given family of curves at right angles (orthogonally) is called an *orthogonal trajectory* of the family. Each line in (c) is an orthogonal trajectory of the family of circles (a) [and conversely, each circle in (a) is an orthogonal trajectory of the family of lines (c)]. In general, if

$$F(x, y, c) = 0 \quad \text{and} \quad G(x, y, K) = 0$$

are one-parameter families of curves such that each member of one family is an orthogonal trajectory of the other family, then the two families are said to be *orthogonal trajectories*.

Method of finding the equation of orthogonal trajectories of a family of curves:

(a) Cartesian curve $f(x,y,c)=0$. (i)

Step No. 1.: To find the differential equation of given family of curves.

- Differentiate given family of curves (i).
- Eliminate the arbitrary constant c between (i) and the resulting equation.

That gives the differential equation of the family (i).

Let it be $F\left(x,y,\frac{dy}{dx}\right)=0$. (ii)

Step No. 2.: To find the differential equation of the orthogonal trajectories.

Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$.

Purpose: The product of their slopes at each point of intersection is -1.

Then the differential equation of the orthogonal trajectories is $F\left(x,y,-\frac{dx}{dy}\right)=0$. (iii)

Step No. 3.: To find the equation of the required orthogonal trajectories.

Solve the differential equation of the orthogonal trajectories (iii),

we get the equation of the required orthogonal trajectories.

(b) Polar curves $f(r,\theta,c)=0$. (i)

Step No. 1.: To find the differential equation of given family of curves.

- Differentiate given family of curves (i).
- Eliminate the arbitrary constant c between (i) and the resulting equation.

That gives the differential equation of the family (i).

Let it be $F\left(r,\theta,\frac{dr}{d\theta}\right)=0$ (ii)

Step No. 2.: To find the differential equation of the orthogonal trajectories.

Replace, in this differential equation, $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$

or $r\frac{d\theta}{dr}$ by $-\frac{1}{r}\frac{dr}{d\theta}$.

The differential equation of the orthogonal trajectories is $F\left(r,\theta,-r^2\frac{d\theta}{dr}\right)=0$. (iii)

Step No. 3.: To find the equation of the required orthogonal trajectories.

Solve the differential equation of the orthogonal trajectories (iii),

we get the equation of the required orthogonal trajectories.

Self-Orthogonal:

A given family of curves is said to be 'self-orthogonal' if its family of orthogonal trajectories is the same as the given family.

Now let evaluate the orthogonal trajectories of the family of the following curves:

Cartesian curves

Q.No.1.: Find the orthogonal trajectories of the family of parabolas $y^2=4ax$.

Sol.: Given family of parabolas is, $y^2=4ax$. (i)

Step No. 1.: To find the differential equation of given family of parabolas.

Differentiating (i) w. r. t. x, we get

$$2y\frac{dy}{dx}=4a \Rightarrow y\frac{dy}{dx}=2a. \qquad \text{(ii)}$$

Eliminating a between (i) and (ii), we get

$$y^2=4\left(\frac{y}{2}\frac{dy}{dx}\right)x \Rightarrow y=2x\frac{dy}{dx}, \qquad \text{(iii)}$$

which is the differential equation of the given family (i).

Step No. 2.: To find the differential equation of the orthogonal trajectories.

For orthogonal trajectories, replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in (iii), we get

$$y = -2x\frac{dx}{dy} \Rightarrow ydy + 2xdx = 0, \qquad \text{(iv)}$$

which is the differential equation of the family of orthogonal trajectories.

Step No. 3.: To find the equation of the required orthogonal trajectories.

Integrating (iv), we get

$$\frac{y^2}{2} + x^2 = c \Rightarrow 2x^2 + y^2 = 2c \Rightarrow 2x^2 + y^2 = c',$$

which is the equation of required orthogonal trajectories of (i).

Q.No.2.: If the stream lines (paths of fluid particles) of a flow around a corner are $xy = $ constant, find their orthogonal trajectories.

Sol.: Taking the axis as the walls, the stream lines of the flow around the corner of the walls is $xy = c$. (i)

Step No. 1.: To find the differential equation of given family of curves.

Differentiating (i) w. r. t. x, we get $x\frac{dy}{dx} + y = 0$, (ii)

which is the differential equation of the given family (i).

Step No. 2.: To find the differential equation of the orthogonal trajectories.

For orthogonal trajectories, replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in (ii), we get

$$x\left(-\frac{dx}{dy}\right) + y = 0 \Rightarrow xdx - ydy = 0, \qquad \text{(iii)}$$

which is the differential equation of the orthogonal trajectories.

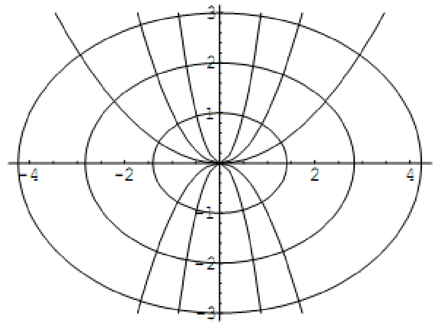
Step No. 3.: To find the equation of the required orthogonal trajectories.

Integrating (iii), we get $x^2 - y^2 = c'$,

which is the required equation of the orthogonal trajectories of (i).

Q.No.3.: Find the orthogonal trajectories of the family of parabolas $y = ax^2$.

Sol.: Given family of parabolas is, $y = ax^2$. (i)



Step No. 1.: To find the differential equation of given family of curves.

Differentiating (i) w. r. t. x, we get

$$\frac{dy}{dx} = 2ax\,. \tag{ii}$$

Eliminating a between (i) and (ii), we get

$$\frac{dy}{dx} = 2.\left(\frac{y}{x^2}\right).x \Rightarrow \frac{dy}{dx} = 2.\frac{y}{x}\,, \tag{iii}$$

which is the differential equation of the given family (i).

Step No. 2.: To find the differential equation of the orthogonal trajectories.

For orthogonal trajectories, replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in (iii), we get

$$-\frac{dx}{dy} = 2.\frac{y}{x} \Rightarrow -xdx = 2ydy\,, \tag{iv}$$

which is the differential equation of the family of orthogonal trajectories.

Step No. 3.: To find the equation of the required orthogonal trajectories.

Integrating (iv), we get

$$-\frac{x^2}{2} = \frac{2y^2}{2} + c' \Rightarrow x^2 + 2y^2 = -2c'$$

$\therefore$ Replace $-2c'$ by c^2, we get $\qquad\qquad \left[\text{since } x^2 + 2y^2 = \text{non-negative}\right]$

$$\Rightarrow x^2 + 2y^2 = c^2\,,$$

which is the equation of required orthogonal trajectories of (i).

Q.No.4.: Find the orthogonal trajectories of the family of semi-cubical parabolas

$$ay^2 = x^3.$$

Sol.: The given equation of the family of semi-cubical parabolas is $ay^2 = x^3$. (i)

Step No. 1.: To find the differential equation of given family of curves.

Differentiating (i) w. r. t. x, we get

$$2ay\frac{dy}{dx} = 3x^2 \Rightarrow \frac{dy}{dx} = \frac{3x^2}{2ay}. \quad \text{(ii)}$$

Eliminating a between (i) and (ii), we get

$$\frac{dy}{dx} = \frac{3x^2}{\frac{2x^3y}{y^2}} = \frac{3}{2}.\frac{y}{x}, \quad \text{(iii)}$$

which is the differential equation of the given family (i).

Step No. 2.: To find the differential equation of the orthogonal trajectories.

For orthogonal trajectories, replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in (iii), we get

$$-\frac{dx}{dy} = \frac{3}{2}\frac{y}{x} \Rightarrow -2xdx = 3ydy, \quad \text{(iv)}$$

which is the differential equation of the family of orthogonal trajectories.

Step No. 3.: To find the equation of the required orthogonal trajectories.

Integrating (iv), we get

$$-2\int xdx = 3\int ydy \Rightarrow -\frac{2x^2}{2} = \frac{3y^2}{2} + c_1^2 \Rightarrow -2x^2 = 3y^2 + c_2^2$$

$$\Rightarrow 2x^2 + 3y^2 = c^2,$$

which is the equation of required orthogonal trajectories of (i).

Q.No5.: Find the orthogonal trajectories of the family of curves (Confocal conics)

$$\frac{x^2}{a^2} + \frac{y^2}{b^2+\lambda} = 1, \text{ where } \lambda \text{ is a parameter.}$$

Sol.: The equation of the family of given curves is $\frac{x^2}{a^2} + \frac{y^2}{b^2+\lambda} = 1$. (i)

Step No. 1.: To find the differential equation of given family of curves.

Differentiating (i) w. r. t. x, we get

$$\frac{2x}{a^2} + \frac{2y}{b^2+\lambda}\frac{dy}{dx} = 0 \Rightarrow \frac{x}{a^2} + \frac{y}{b^2+\lambda}\frac{dy}{dx} = 0. \quad \text{(ii)}$$

To eliminate the parameter λ, we equate the values of $b^2+\lambda$ from (i) and (ii).

Since from (i), $\frac{y^2}{b^2+\lambda}=1-\frac{x^2}{a^2}=\frac{a^2-x^2}{a^2}\Rightarrow b^2+\lambda=\frac{a^2y^2}{a^2-x^2}$,

and from (ii), $b^2+\lambda=-\frac{a^2y}{x}\frac{dy}{dx}$.

$$\therefore \frac{a^2y^2}{a^2-x^2}=-\frac{a^2y}{x}\frac{dy}{dx}\Rightarrow \frac{xy}{a^2-x^2}+\frac{dy}{dx}=0, \qquad \text{(iii)}$$

which is the differential equation of the given family (i).

Step No. 2.: To find the differential equation of the orthogonal trajectories.

For orthogonal trajectories, replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in (iii), we get

$$\frac{xy}{a^2-x^2}-\frac{dx}{dy}=0\Rightarrow ydy-\left(\frac{a^2-x^2}{x}\right)dx=0, \qquad \text{(iv)}$$

which is the differential equation of the orthogonal trajectories.

Step No. 3.: To find the equation of the required orthogonal trajectories.

Integrating (iv), we get $\int ydy-\int\left(\frac{a^2}{x}-x\right)dx=c$

$$\Rightarrow \frac{y^2}{2}-a^2\log x+\frac{x^2}{2}=c\Rightarrow x^2+y^2=2a^2\log x+C,$$

which is the required equation of orthogonal trajectories of (i).

Q.No.6.: Prove that the system of confocal and coaxial parabolas $y^2=4a(x+a)$ is a self-orthogonal.

or

Find the orthogonal trajectories of a system of confocal and coaxial parabolas.

Sol.: The equation of the family of given parabolas is $y^2=4a(x+a)$. (i)

Step No. 1.: To find the differential equation of given family of curves.

Differentiating (i) w. r. t. x, we get

$$2y\frac{dy}{dx}=4a\Rightarrow y\frac{dy}{dx}=2a. \qquad \text{(ii)}$$

Eliminating a between (i) and (ii), we get

$$y^2 = 4.\frac{y}{2}\frac{dy}{dx}\left[x + \frac{y}{2}.\frac{dy}{dx}\right] = 2xy\frac{dy}{dx} + y^2\left(\frac{dy}{dx}\right)^2 \Rightarrow y\left(\frac{dy}{dx}\right)^2 + 2x\frac{dy}{dx} - y = 0, \qquad \text{(iii)}$$

which is the differential equation of the given family (i).

Step No. 2.: To find the differential equation of the orthogonal trajectories.

For orthogonal trajectories, replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in (iii), we get

$$y\left(\frac{dx}{dy}\right)^2 - 2x\frac{dx}{dy} - y = 0 \Rightarrow y - 2x\frac{dy}{dx} - y\left(\frac{dy}{dx}\right)^2 = 0 \Rightarrow y\left(\frac{dy}{dx}\right)^2 + 2x\frac{dy}{dx} - y = 0, \qquad \text{(iv)}$$

which is the differential equation of the orthogonal trajectories.

Since (iv) is same as (iii)

$\Rightarrow$ The system of confocal and coaxial parabolas is self-orthogonal,

i.e. each member of (i) cuts every other member orthogonally.

Q.No.7. Find the orthogonal trajectories of the family of coaxial circles

$$x^2 + y^2 + 2\lambda y + c = 2, \ \lambda \text{ being the parameter.}$$

Sol.: The given equation of the family of coaxial circles is $x^2 + y^2 + 2\lambda y + c = 2$. (i)

Step No. 1.: To find the differential equation of given family of curves.

Differentiating (i) w. r. t. x, we get

$$2x + 2y\frac{dy}{dx} + 2\lambda\frac{dy}{dx} = 0. \qquad \text{(ii)}$$

Eliminating λ between (i) and (ii), we get

$$2x + 2y\frac{dy}{dx} + \left(\frac{2 - c - x^2 - y^2}{y}\right)\frac{dy}{dx} = 0$$

$$\Rightarrow 2xy + \left(y^2 - x^2 - c + 2\right)\frac{dy}{dx} = 0,$$

which is the differential equation of the given family (i).

Step No. 2.: To find the differential equation of the orthogonal trajectories.

For orthogonal trajectories, replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in (iii), we get

$$-\left(y^2 - x^2 - c + 2\right)\frac{dx}{dy} = -2xy \Rightarrow \left(y^2 - x^2 - c + 2\right)dx - 2xydy = 0, \qquad \text{(iii)}$$

which is the differential equation of the family of orthogonal trajectories.

Step No. 3.: To find the equation of the required orthogonal trajectories.

Now from (iv): $M = y^2 - x^2 - c + 2$, $N = -2xy$.

Now since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow$ The above equation is not exact.

Now $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = -\frac{2}{x} = f(x)$ alone.

$\therefore$ I.F. $= e^{\int -\frac{2}{x}dx} = \frac{1}{x^2}$.

Multiplying (iii) by I.F., we obtain

$$\frac{\left(y^2 - x^2 - c + 2\right)dx}{x^2} - \frac{2xydy}{x^2} = 0. \qquad \text{(iv)}$$

Now here $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow$ The above equation is exact.

$\therefore$ Solution is $\int \frac{y^2 - x^2 - c + 2}{x^2}dx - \int 0dy = \mu$

$$-\frac{y^2}{x} - x + \frac{c}{x} - \frac{2}{x} = \mu \Rightarrow -y^2 - x^2 + c - 2 = \mu x$$

$$\Rightarrow -y^2 - x^2 + c - 2 = \mu x \Rightarrow -y^2 - x^2 + c' = 2 + \mu x$$

$$\Rightarrow x^2 + y^2 + \mu x + c_1 = 0,$$

which is the required equation of orthogonal trajectories of (i).

Q.No.8.: The electric lines of force of two opposite charges of the same strength at $(\pm 1,\ 0)$ are circles (through these points) of the form $x^2 + y^2 - ay = 1$, find their equipotential lines (orthogonal trajectories).

Sol.: The given equation of the family of circles is $x^2 + y^2 - ay = 1$. (i)

Step No. 1.: To find the differential equation of given family of curves.

Differentiating (i) w. r. t. x, we get

$$2x + (2y - a)\frac{dy}{dx} = 0 \Rightarrow 2x + (2y - a)P = 0. \qquad \left(\text{Here } P = \frac{dy}{dx}\right) \qquad \text{(ii)}$$

Eliminating a from (i) and (ii), we get

$$x^2+y^2-\left(\frac{2x}{P}+2y\right)y=1 \Rightarrow x^2-y^2-1=\frac{2xy}{P} \Rightarrow P=\frac{2xy}{x^2-y^2-1}$$

$$\Rightarrow \frac{dy}{dx}=\frac{2xy}{x^2-y^2-1}, \qquad \text{(iii)}$$

which is the differential equation of family of circles given in (i)

Step No. 2.: To find the differential equation of the orthogonal trajectories.

For orthogonal trajectories, replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in (iii), we get

$$-\frac{dx}{dy}=\frac{2xy}{x^2-y^2-1} \Rightarrow 2xydy+\left(x^2-y^2-1\right)dx=0$$

$$\left(x^2-y^2-1\right)dx+2xydy=0, \qquad \text{(iv)}$$

which is the differential equation of the family of orthogonal trajectories.

Step No. 3.: To find the equation of the required orthogonal trajectories.

Now $\left(x^2-y^2-1\right)dx+2xydy=0 \Rightarrow Mdx+Ndy=0$

Here $M=x^2-y^2-1, \qquad N=2xy$

Now $\frac{\partial M}{\partial y}=-2y$ and $\frac{\partial N}{\partial x}=2y$

$$\Rightarrow \frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x} \Rightarrow \text{The above equation is not exact.}$$

Now since $\dfrac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=\frac{-4y}{2xy}=-\frac{2}{x}=f(x)$ alone.

$$\therefore \text{I. F.} = e^{\int -\frac{2}{x}dx}=e^{-2\log x}=\frac{1}{x^2}.$$

Multiplying by $\frac{1}{x^2}$ throughout (iv), we get

$$\frac{1}{x^2}\left(x^2-y^2-1\right)dx+\frac{2xy}{x^2}dy=0 \left(1-\frac{y^2}{x^2}-\frac{1}{x^2}\right)dx+\frac{2y}{x}dy=0,$$

Now here $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow$ The above equation is exact.

$\therefore$ The solution is

$$\int_{y\ \text{constant}} \left(1 - \frac{y^2}{x^2} - \frac{1}{x^2}\right) dx + \int\ 0 dy = c' \Rightarrow x + \frac{y^2}{x} + \frac{1}{x} = c' \Rightarrow x^2 + y^2 - c'x + 1 = 0$$

$$\Rightarrow x^2 + y^2 + cx + 1 = 0,$$

which is the required equation of orthogonal trajectories of (i).

Q.No.9.: Find the orthogonal trajectories of one parameter family of curves $xy = c$

Sol.: Given family of parabolas is, $xy = c$. (i)

Step No. 1.: To find the differential equation of given family of curves.

Differentiating (i) w.r.t. x, we get

$$y + x\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-y}{x}, \qquad \text{(ii)}$$

which is the differential equation of the given family (i).

Step No. 2.: To find the differential equation of the orthogonal trajectories.

For orthogonal trajectories, replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in (ii), we get

$$\frac{dy}{dx} = \frac{x}{y}, \qquad \text{(iii)}$$

which is the differential equation of the family of orthogonal trajectories.

Step No. 3.: To find the equation of the required orthogonal trajectories.

Integrating (iii), we get

$$y^2 - x^2 = 2c,$$

which is the equation of required orthogonal trajectories of (i).

Q.No.10.: Find the orthogonal trajectories of one parameter family of curves

$$e^x + e^{-y} = c.$$

Sol.: Given family of parabolas is, $e^x + e^{-y} = c$. (i)

Step No. 1.: To find the differential equation of given family of curves.

Differentiating (i) w.r.t. x, we get

$$e^x = e^{-y}.y' = 0 \Rightarrow y' = e^{x+y}, \qquad \text{(ii)}$$

which is the differential equation of the given family (i).

Step No. 2.: To find the differential equation of the orthogonal trajectories.

For orthogonal trajectories, replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in (ii), we get

$$y' = -e^{-(x+y)} \Rightarrow \frac{dy}{dx} = \frac{-e^{-x}}{e^{y}} \Rightarrow e^{y}dy + e^{-x}dx = 0, \tag{iii}$$

which is the differential equation of the family of orthogonal trajectories.

Step No. 3.: To find the equation of the required orthogonal trajectories.

Integrating (iii), we get

$$e^{y} - e^{-y} = k,$$

which is the equation of required orthogonal trajectories of (i).

Q.No.11.: Find the orthogonal trajectories of one parameter family of curves $y^2 = cx$.

Sol.: Given family of parabolas is, $y^2 = cx$. (i)

Step No. 1.: To find the differential equation of given family of curves.

Differentiating (i) w.r.t. x, we get

$$2yy' = c. \tag{ii}$$

Eliminating c between (i) and (ii), we get

$$2yy' = c = \frac{y^2}{x} \Rightarrow y' = \frac{y}{2x}, \tag{iii}$$

which is the differential equation of the given family (i).

Step No. 2.: To find the differential equation of the orthogonal trajectories.

For orthogonal trajectories, replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in (iii), we get

$$y' = -\frac{2x}{y}, \tag{iv}$$

which is the differential equation of the family of orthogonal trajectories.

Step No. 3.: To find the equation of the required orthogonal trajectories.

Integrating (iv), we get

$$\frac{y^2}{2} + x^2 = c,$$

which is the equation of required orthogonal trajectories of (i).

Q.No.12.: Show that family of curves $x^2 + 4y^2 = c_1$ and $y = c_2 x^4$ are (mutually) orthogonal (to each other).

Sol.: Slope of tangent of any curve of the first family of curves is obtained by differentiating it w.r.t . x

i.e. $2x + 8yy' = 0$

$$\Rightarrow y_1' = -\frac{2x}{8y} = -\frac{x}{4y} \qquad (*)$$

Similarly for the second family by eliminating c_2

$$dydx = 4x^3 c_2 = 4x^3 . \frac{y}{x^4} = \frac{4y}{x}$$

i.e. $$y_2' = \frac{4y}{x} \qquad (**)$$

The given two families are orthogonal of the product of their slopes is -1. From (*) and (**), we get

$$y_1' . y_2' = \left(\frac{-x}{4y}\right)\left(\frac{4y}{x}\right) = -1,$$ hence the result.

Q.No.13.: Find particular member of orthogonal trajectories of $x^2 + cy^2 = 1$ passing through the point (2, 1).

Sol.: DE $2x + 2cyy' = 0$

Eliminating c

$$x + yy'c = 0 . x + yy' . \left(\frac{1-x^2}{y^2}\right) = 0$$

i.e. $$y' = \frac{xy}{x^2 - 1}$$

DE corresponding to OT

$$y' = \frac{1-x^2}{xy}$$

Solving $$ydy = \frac{1-x^2}{x}dx = \frac{dx}{x} - xdx$$

$$\frac{y^2}{2}+\frac{x^2}{2}=\ln x+c_1$$

$$x^2=c_2e^{x^2+y^2} \qquad (*)$$

To find the particular member of this OT

Put x = 2 when y = 1 in (*)

$$4=c_2e^5 \qquad c_2=4e^{-5}$$

Thus the particular curve of OT passing through the point (2, 1)

$$x^2=4e^{-5}e^{x^2+y^2}.$$

Q.No.14.: Show that the family of parabolas $y^2=4cx+4c^2$ is **self-orthogonal**.

Sol.: DE $2yy'=4c+0$

Substituting $c=\frac{yy'}{2}$ in given equation, we get

$$y^2=4x.\left(\frac{yy'}{2}\right)+4\left(\frac{yy'}{2}\right)$$

$$y^2=2xyy'+y^2y'^2 \qquad (*)$$

Put $p=y'$ so that

$$y^2=2xyp+y^2p^2 \qquad (**)$$

This is the DE of the given family of parabolas.

In order to get DE corresponding to the OT replace y' by $-\frac{1}{p}$ in (*). Then

$$y^2=2xy\left(-\frac{1}{p}\right)+y^2\left(-\frac{1}{p}\right)^2$$

$$p^2y^2=2xy(-p)+y^2$$

Rewriting

$y^2=2xyp+p^2y^2$, which is same as equation (**). Thus (*) is DE for the given family and its orthogonal trajectories. Hence the given family is self-orthogonal.

Polar curves

Q.No.1.: Find the orthogonal trajectory of the cardioids $r = a(1-\cos\theta)$.

Sol.: The equation of the family of given cardioids is $r = a(1-\cos\theta)$. (i)

Step No. 1.: To find the differential equation of given family of curves.

Differentiating (i) w. r. t θ, we get $\dfrac{dr}{d\theta} = a\sin\theta$. (ii)

Dividing (ii) by (i) to eliminate a,

$$\frac{1}{r}\cdot\frac{dr}{d\theta} = \frac{\sin\theta}{1-\cos\theta} \Rightarrow \frac{1}{r}\frac{dr}{d\theta} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}} = \cot\frac{\theta}{2}\ , \qquad \text{(iii)}$$

which is the differential equation of the given family (i)

Step No. 2.: To find the differential equation of the orthogonal trajectories.

For orthogonal trajectories, replacing $\dfrac{dr}{d\theta}$ by $-r^2\dfrac{d\theta}{dr}$ in (iii), we get

$$\frac{1}{r}\left(-r^2\frac{d\theta}{dr}\right) = \cot\frac{\theta}{2} \Rightarrow r\frac{d\theta}{dr} + \cot\frac{\theta}{2} = 0 \Rightarrow \frac{dr}{r} + \tan\frac{\theta}{2}d\theta = 0, \qquad \text{(iv)}$$

which is the differential equation of the family of orthogonal trajectories.

Step No. 3.: To find the equation of the required orthogonal trajectories.

Integrating (iv), we get

$$\log r - 2\log\cos\frac{\theta}{2} = \log c \Rightarrow \log r = \log c + \log\cos^2\frac{\theta}{2} = \log c\cos^2\frac{\theta}{2}$$

$$\Rightarrow r = c\cos^2\frac{\theta}{2} = \frac{c}{2}(1+\cos\theta) \Rightarrow r = C(1+\cos\theta),$$

which is the required equation of orthogonal trajectories of (i).

Q.No.2.: Find the orthogonal trajectories of the family of cardioids $r = a(1+\cos\theta)$.

Sol.: The given equation of the family of cardioids is $r = a(1+\cos\theta)$. (i)

Step No. 1.: To find the differential equation of given family of curves.

Differentiating (i) w. r. t. θ, we get

$$\frac{dr}{d\theta} = -a\sin\theta \Rightarrow \frac{dr}{d\theta} = \frac{-r\sin\theta}{1+\cos\theta} \Rightarrow \frac{dr}{d\theta}\left(\frac{1}{r}\right) = \frac{-\sin\theta}{1+\cos\theta}, \qquad \text{(ii)}$$

which is the differential equation of the given family (i).

Step No. 2.: To find the differential equation of the orthogonal trajectories.

For orthogonal trajectories, replace $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$ in (ii), we get.

$$r\frac{d\theta}{dr} = \frac{\sin\theta}{1+\cos\theta} \Rightarrow \frac{dr}{r} = (\operatorname{cosec}\theta + \cot\theta)d\theta\,, \tag{iii}$$

which is the differential equation of the family of orthogonal trajectories.

Step No. 3.: To find the equation of the required orthogonal trajectories.

Integrating (iii), we get

$$\log r = \log|\operatorname{cosec}\theta - \cot\theta| + \log(\sin\theta) + \log c \qquad \text{[where log c is constant]}$$

$$\Rightarrow r = c(\operatorname{cosec}\theta - \cot\theta)\sin\theta \Rightarrow r = c(1-\cos\theta),$$

which is the required equation of orthogonal trajectories of (i).

Q.No.3.: Find the orthogonal trajectories of the family of confocal and coaxial parabolas

$$r = \frac{2a}{(1+\cos\theta)}$$

Sol.: The given equation of the family of confocal and coaxial parabolas is

$$r = \frac{2a}{1+\cos\theta} = a\sec^2\frac{\theta}{2}. \tag{i}$$

Step No. 1.: To find the differential equation of given family of curves.

Differentiating (i) w. r. t. θ, we get $\frac{dr}{d\theta} = \frac{2a\sin\theta}{(1+\cos\theta)^2} = a\sec^2\frac{\theta}{2}\tan\frac{\theta}{2}.$ (ii)

Dividing (ii) by (i), we get $\frac{\left(\frac{dr}{d\theta}\right)}{r} = r\tan\frac{\theta}{2},$ (iii)

which is the differential equation of the given family (i).

Step No. 2.: To find the differential equation of the orthogonal trajectories.

For orthogonal trajectories, replace $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$, we get $\frac{d\theta}{\tan\frac{\theta}{2}} = \frac{-rdr}{r^2},$ (iv)

which is the differential equation of the family of orthogonal trajectories.

Step No. 3.: To find the equation of the required orthogonal trajectories.

Integrating (iv), we get

$$\log\sin^2\frac{\theta}{2}=\log\frac{1}{r}+\log c \Rightarrow \sin^2\frac{\theta}{2}=\frac{c}{r}$$

$$\Rightarrow r=\frac{c}{\frac{(1-\cos\theta)}{2}} \Rightarrow r=\frac{2b}{1-\cos\theta}, \quad (\text{where } b=2c)$$

which is the required equation of orthogonal trajectories of (i).

Q.No.4.: Find the orthogonal trajectories of the family of curves $r^2=a^2\cos 2\theta$.

Sol.: The given equation of the family of the curve is $r^2=a^2\cos 2\theta$. (i)

Step No. 1.: To find the differential equation of given family of curves.

Differentiating (i) w. r. t. θ, we get

$$2r\frac{dr}{d\theta}=-a^2 2\sin 2\theta \Rightarrow r\frac{dr}{d\theta}=-a^2\sin 2\theta. \quad \text{(ii)}$$

Also from (i), we have $a^2=\dfrac{r^2}{\cos 2\theta}$. (iii)

From (ii) and (iii), we get

$$r\frac{dr}{d\theta}=-r^2\tan 2\theta, \quad \text{(iv)}$$

which is the differential equation of the given family (i).

Step No. 2.: To find the differential equation of the orthogonal trajectories.

For orthogonal trajectories, replace $\dfrac{dr}{d\theta}$ by $-r^2\dfrac{d\theta}{dr}$, we get

$$-r^3\frac{d\theta}{dr}=-r^2\tan 2\theta \Rightarrow r\frac{d\theta}{dr}=\tan 2\theta \Rightarrow \frac{d\theta}{\tan 2\theta}=\frac{dr}{r}, \quad \text{(v)}$$

which is the differential equation of the family of orthogonal trajectories.

Step No. 3.: To find the equation of the required orthogonal trajectories.

Integrating (v), we get

$$\int\frac{d\theta}{\tan 2\theta}=\int\frac{dr}{r} \Rightarrow \frac{\log\sin 2\theta}{2}=\log r+\log c_1 \Rightarrow \log\sin 2\theta=2\log rc_1$$

$$\Rightarrow \log\sin 2\theta=\log(rc_1)^2 \Rightarrow \sin 2\theta=(rc_1)^2$$

$$\Rightarrow r^2=\frac{\sin 2\theta}{c_1^2}\, r^2=c^2\sin 2\theta, \qquad \left(\text{where } c^2=\frac{1}{c_1^2}\right)$$

which is the required equation of orthogonal trajectories of (i).

Q.No.5.: Find the orthogonal trajectories of the family of curves $r^n = a^n \sin n\theta$.

Sol.: The given equation of the family of curves is $r^n = a^n \sin n\theta$. (i)

Step No. 1.: To find the differential equation of given family of curves.

Differentiating (i) w. r. t. θ, we get

$$nr^{n-1}\frac{dr}{d\theta} = a^n \cos n\theta . n \Rightarrow \frac{r^{n-1}}{\cos n\theta}\frac{dr}{d\theta} = a^n. \qquad \text{(ii)}$$

From (i) and (ii), we get

$$r^n = \frac{r^{n-1}}{\cos n\theta}\frac{dr}{d\theta}\sin n\theta \Rightarrow r = \frac{dr}{d\theta}\tan n\theta \Rightarrow r\frac{d\theta}{dr} = \tan n\theta, \qquad \text{(iii)}$$

which is the differential equation of the given family (i).

Step No. 2.: To find the differential equation of the orthogonal trajectories.

For orthogonal trajectories, replace $r\frac{d\theta}{dr}$ by $-\frac{1}{r}\frac{dr}{d\theta}$, we get

$$-\frac{1}{r}\frac{dr}{d\theta} = \tan n\theta \Rightarrow -\frac{1}{r}dr = \tan n\theta d\theta, \qquad \text{(iv)}$$

which is the differential equation of the family of orthogonal trajectories.

Step No. 3.: To find the equation of the required orthogonal trajectories.

Integrating (iv), we get

$$\int -\frac{1}{r}dr = \int \frac{\sin n\theta}{\cos n\theta}d\theta + \log b$$

Now put $\cos n\theta = t \Rightarrow -\sin n\theta . n d\theta = dt \Rightarrow \sin n\theta d\theta = -\frac{dt}{n}$

$$\therefore \int -\frac{1}{r}dr = \int -\frac{dt}{nt} + \log b \Rightarrow \log r = \frac{1}{n}\log t + \log b \Rightarrow \log r = \log\left(t^{1/n}b\right)$$

Taking antilog, we get

$$r = t^{1/n}b \Rightarrow r^n = tb^n \Rightarrow r^n = \cos n\theta b^n,$$

which is the required equation of orthogonal trajectories of (i).

Q.No.6.: Find the orthogonal trajectories (OT) of the family of curves $r = a\cos^2\theta$.

Sol.: The given equation of the family of curves is $r = a\cos^2\theta$. (i)

Step No. 1.: To find the differential equation of given family of curves.

Differentiating (i) w. r. t. θ, we get

$dr = 2a\cos\theta(-\sin\theta)d\theta$. (ii)

From (i) and (ii), we get

$$dr = -2.\frac{r}{\cos^2\theta}.\cos\theta.\sin\theta.d\theta \left[\because a = \frac{r}{\cos^2\theta}\right]$$

$$r\frac{d\theta}{dr} = \frac{1}{2\tan\theta}. \qquad \text{(iii)}$$

which is the differential equation of the given family (i).

Step No. 2.: To find the differential equation of the orthogonal trajectories.

For orthogonal trajectories, replace $r\frac{d\theta}{dr}$ by $-\frac{1}{r}\frac{dr}{d\theta}$, we get

$$r\frac{d\theta}{dr} = 2\tan\theta, \qquad \text{(iv)}$$

which is the differential equation of the family of orthogonal trajectories.

Step No. 3.: To find the equation of the required orthogonal trajectories.

$$\text{(iv)} \Rightarrow \frac{d\theta}{\tan\theta} = 2\frac{dr}{r}. \qquad \text{(v)}$$

Integrating (v), we get

$$\sin\theta = 2\log r + c \Rightarrow r^2 = b\sin\theta,$$

which is the required equation of orthogonal trajectories of (i).

Q.No.7.: Find the orthogonal trajectories (OT) of the family of curves $r^2 = a\sin 2\theta$.

Sol.: The given equation of the family of curves is $r^2 = a\sin 2\theta$. (i)

Step No. 1.: To find the differential equation of given family of curves.

Differentiating (i) w. r. t. θ, we get

$2rdr = 2a\cos 2\theta d\theta$. (ii)

From (i) and (ii), we get

$$rdr = \frac{r^2}{\sin 2\theta}.\cos 2\theta.d\theta \left[\because a = \frac{r^2}{\sin 2\theta}\right]$$

$$\Rightarrow r\frac{d\theta}{dr} = \tan 2\theta, \qquad \text{(iii)}$$

which is the differential equation of the given family (i).

Step No. 2.: To find the differential equation of the orthogonal trajectories.

For orthogonal trajectories, replace $r\frac{d\theta}{dr}$ by $-\frac{1}{r}\frac{dr}{d\theta}$, we get

$$r\frac{d\theta}{dr} = -\cot 2\theta, \tag{iv}$$

which is the differential equation of the family of orthogonal trajectories.

Step No. 3.: To find the equation of the required orthogonal trajectories.

$$\text{Solving } -\frac{\sin 2\theta}{\cos 2\theta} d\theta = \frac{dr}{r}. \tag{v}$$

Integrating (v), we get

$$\log(\cos 2\theta) = 2\ \log r + c \Rightarrow r^2 = b\cos 2\theta,$$

which is the required equation of orthogonal trajectories of (i).

Isogonal Trajectories:

Definition: Two families of curves such that every member of either family cuts each member of the other family at constant angle α (say) are called isogonal trajectories of each other. The slopes $m,\ m'$ of the tangent to corresponding curves at each point are connected by the relation,

$$\frac{m - m'}{1 + mm'} = \tan\alpha = \text{constant.}$$

Q.No.1.: Find the isogonal trajectories of the family of circles $x^2 + y^2 = a^2$, which intersects at $45°$.

Sol.: The given equation is $x^2 + y^2 = a^2$. (i)

Step No. 1.: To find the differential equation of given family of curves.

Differentiating (i) w. r. t. x, we get

$$2x + 2y\frac{dy}{dx} = 0 \Rightarrow 2y\frac{dy}{dx} = -2x \Rightarrow \frac{dy}{dx} = -\frac{x}{y}, \tag{ii}$$

which is the differential equation of family of circles given in (i)

Step No. 2.: To find the differential equation of the family of isogonal trajectories.

Now isogonal trajectories are two families of curves such that every member of either family cuts each member of the other family at constant angle α (say). The slopes $m,\ m'$ of the tangent to corresponding curves at each point are connected by the relation,

$$\frac{m-m'}{1+mm'}=\tan\alpha=\text{constant}\,. \tag{iii}$$

Now here $\alpha=45°$ and $m=\dfrac{dy}{dx}$.

$\therefore$ (iii) becomes

$$\frac{\frac{dy}{dx}-m'}{1+\left(\frac{dy}{dx}\right)m'}=\tan 45°=1\ \Rightarrow \frac{dy}{dx}-m'=1+\left(\frac{dy}{dx}\right)m'\Rightarrow\frac{dy}{dx}-1=\left(\frac{dy}{dx}+1\right)m'$$

$$m'=\frac{\frac{dy}{dx}-1}{\frac{dy}{dx}+1}.$$

Replacing $\dfrac{dy}{dx}$ by $m'=\dfrac{\frac{dy}{dx}-1}{\frac{dy}{dx}+1}$ in equation (ii), we get

$$\frac{\frac{dy}{dx}-1}{\frac{dy}{dx}+1}=-\frac{x}{y}\Rightarrow y\frac{dy}{dx}-y=-x\frac{dy}{dx}-x\Rightarrow(y+x)\frac{dy}{dx}=y-x\ \Rightarrow\frac{dy}{dx}=\frac{y-x}{y+x}, \tag{iv}$$

which is the differential equation of the family of isogonal trajectories.

Step No. 3.: To find the equation of the required isogonal trajectories.

Now since (iv) is homogenous in x and y

To solve (iv), we have put $y=tx\Rightarrow\dfrac{dy}{dx}=t+x\dfrac{dt}{dx}$. (v)

$$\therefore\ \text{(iv)}\Rightarrow\ \ t+x\frac{dt}{dx}=\frac{tx-x}{tx+x}\Rightarrow t+x\frac{dt}{dx}=\frac{t-1}{t+1}\Rightarrow x\frac{dt}{dx}=\frac{t-1}{t+1}-t$$

$$\Rightarrow x\frac{dt}{dx}=\frac{t-1-t^2-t}{t+1}\ \Rightarrow\frac{-(t+1)}{1+t^2}dt=\frac{dx}{x}. \tag{vi}$$

Integrating (vi), we get

$$-\int\frac{(t+1)}{1+t^2}dt=\int\frac{dx}{x}+\log c_1\Rightarrow-\frac{1}{2}\int\frac{2t}{1+t^2}dt-\int\frac{1}{1+t^2}dt=\log x+\log c_1$$

$$\Rightarrow-\frac{1}{2}\log\left(1+t^2\right)-\tan^{-1}(t)=\log x+\log c_1$$

$$\Rightarrow -\log\sqrt{1+t^2} + \left(-\tan^{-1} t\right) = \log x + \log c_1 \qquad \left[\because n\log m = \log m^n\right]$$

Substituting $t = \frac{y}{x}$, we get

$$-\log\sqrt{1+\frac{y^2}{x^2}} - \tan^{-1}\left(\frac{y}{x}\right) = \log x + \log c_1 \Rightarrow -\log\sqrt{\frac{x^2+y^2}{x^2}} - \tan^{-1}\left(\frac{y}{x}\right) = \log x + \log c_1$$

$$\Rightarrow \log\sqrt{x^2+y^2} + \tan^{-1}\left(\frac{y}{x}\right) = \log c_1$$

which is the required equation of isogonal trajectories of (i).

Home Assignments

Cartesian curves

Q.No.1.: Find the orthogonal trajectories of one parameter family of curves $x - 4y = c$.

Ans.: $4x + y = k$.

Q.No.2.: Find the orthogonal trajectories of one parameter family of curves $x^2 + y^2 = c^2$.

Ans.: $y = kx$.

Q.No.3.: Find the orthogonal trajectories of one parameter family of curves $x^2 - y^2 = c$.

Ans.: $xy = k$.

Q.No.4.: Find the orthogonal trajectories of one parameter family of curves $y^2 = cx^3$.

Ans.: $(x+1)^2 + y^2 = a^2$.

Q.No.5.: Find the orthogonal trajectories of one parameter family of curves

$$y = c(\sec x + \tan x).$$

Ans.: $y^2 = 2(k - \sin x)$.

Q.No.6.: Find the orthogonal trajectories of one parameter family of curves

$$x^2 - y^2 = cx.$$

Ans.: $y\left(y^2 + 3x^2\right) = k$.

Q.No.7.: Find the orthogonal trajectories of one parameter family of curves $y^2 = \frac{x^3}{a-x}$.

Ans.: $\left(x^2+y^2\right)^2 = b\left(2x^2+y^2\right)$.

Q.No.8.: Find the orthogonal trajectories of one parameter family of curves

$$(a+x)y^2 = x^2(3a-x).$$

Ans.: $\left(x^2+y^2\right)^5 = cy^3\left(5x^2+y^2\right)$.

Q.No.9.: Find the orthogonal trajectories of one parameter family of curves $y = cx^2$.

Ans.: $\frac{x^2}{2} + y^2 = c^*$.

Q.No.10.: Find the orthogonal trajectories of circles through origin with centres on the x-axis.

Ans.: Circles through origin with centres bon y-axis.

Q.No.11.: Find the orthogonal trajectories of Family of parameters through origin and foci on y-axis.

Ans.: Ellipses with centres at origin and foci on x-axis.

Q.No.12.: Find the orthogonal trajectories of the family of ellipses having centre at the origin, a focus at the point (c, 0) and semi-major axis of length 2c.

Ans.: $y = cx^{\frac{4}{3}}$.

Q.No.13.: Given $x^2 + 3y^2 = cy$, find that member of the orthogonal trajectories which passes through the point (1, 2).

Ans.: $y^2 = x^2(3x+1)$.

Q.No.14.: Given $y = ce^{-2x} + 3x,$ find that member of the OT which passes through point (0, 3).

Ans.: $9x - 3y + 5 = -4e^{6(3-y)}$.

Q.No.15.: Find constant 'e' such that $y^3 = c_1 x$ and $x^2 + ey^2 = c_2$ are orthogonal to each other .

Ans.: $e = \frac{1}{3}$.

Q.No.16.: Find the value of constant d such that the parabolas $y = c_1 x^2 + d$ are the orthogonal trajectories of the family of ellipses $x^2 + 2y^2 - y = c_2$.

Ans.: $d = \frac{1}{4}$.

Q.No.17.: Show that the family of parabolas $y^2 = 2cx + c^2$ is self-orthogonal.

Q.No.18.: Show that the family of confocal conics $\frac{x^2}{a} + \frac{y^2}{(a-b)} = 1$ is self orthogonal. Here a is an arbitrary constant.

Q.No.19.: Show that the family of confocal conics $\frac{x^2}{a^2 + c} + \frac{y^2}{b^2 + c} = 1$ is self-orthogonal. Here a and b are given constants.

Q.No.20.: Find the orthogonal trajectories of $x^p + cy^p = 1$, p = constant.

Ans.: $y^2 = \frac{2x^{2-p}}{2-p} - x^2 + c$, if $p \neq 2$

$c_1 x^2 = e^{x^2 + y^2}$ if $p = 2$

Q.No.21.: Show that the two families of parameter family of curves $u(x, y) = c_1$ and $v(x, y) = c_2$ are mutually orthogonal provided they satisfy the b(Cauchy-Riemann) equation $u_x = v_y$ and $u_y = -v_x$

Q.No.22.: Find the orthogonal trajectories of the family of confocal conics

$\frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} = 1$, where λ being the parameter.

Q.No.23.: Prove that the system of confocal conics $\frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} = 1$, λ being the parameter, is self-orthogonal.

Q.No.24.: Show that the family of the parabolas $x^2 = 4a(y + a)$ is self orthogonal.

Q.No.25.: Find the orthogonal trajectories of the family of coaxial

$x^2 + y^2 + 2gx + c = 0$, g being the parameter.

Ans.: $x^2 + y^2 + 2fy - c = 0$.

Polar curves

Q.No.1.: Find the orthogonal trajectories of family of curves

$$\left(r^2 + \frac{k^2}{r}\right)\cos\theta = d, \text{ d being a parameter.}$$

Ans.: $\left(r^2 - k^2\right)\sin\theta = cr$.

Q.No.2.: Find the orthogonal trajectories of family of curves $r = 2a(\sin\theta + \cos\theta)$.

Ans.: $r = 2b(\sin\theta - \cos\theta)$.

Q.No.3.: Find the orthogonal trajectories of family of curves $r = 4a\sec\theta.\tan\theta$.

Ans.: $r^2\left(1 + \sin^2\theta\right) = b^2$.

Q.No.4.: Find the orthogonal trajectories of family of curves $r = a\left(1 + \sin^2\theta\right)$.

Ans.: $r^2 = b\cos\theta.\cot\theta$.

Q.No.5.: Find the orthogonal trajectories of family of curves: Cissoid $r = a\sin\theta.\tan\theta$.

Ans.: $r^2 = b\left(1 + \cos^2\theta\right)$.

Q.No.6.: Find the orthogonal trajectories of family of curves $r = \dfrac{k}{1 + 2\cos\theta}$.

Ans.: $r^2\sin^3\theta = b(1 + \cos\theta)$.

Q.No.7.: Find the orthogonal trajectories of family of curves: Stophoids

$$r = a(\sec\theta + \tan\theta).$$

Ans.: $r = be^{-\sin\theta}$

Q.No.8.: Find the orthogonal trajectory of the family of the curve $r^n = a^n\cos n\theta$.

Ans.: $r^n = c^n\sin n\theta$.

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Applications

of

Ordinary Differential Equations of first order

“Physical Applications”

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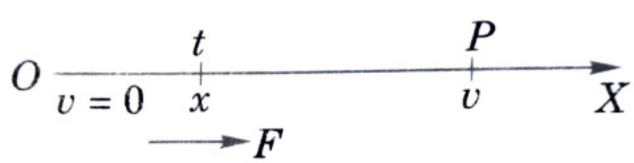
Physical applications:

I: Let a body of mass m start moving from a fixed point O along a straight line OX under the action of a force F. Let P be the position of the body at any instant t, where OP = x, then



(i) its velocity (v) = $\frac{dx}{dt}$

(ii) its acceleration (a) $= \frac{dv}{dt}$ or $\frac{d^2x}{dt^2}$ or $v\frac{dv}{dx}$.

Also, by Newton's second law of motion, $F = ma = m\frac{dv}{dt}$ **or** $m\frac{d^2x}{dt^2}$ **or** $mv\frac{dv}{dx}$**,** **where F is the effective force.**

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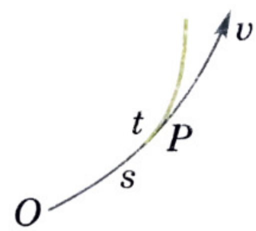
Let a body of mass m start moving from a fixed point O along a curve then



(i) **its velocity (v) =** $\frac{ds}{dt}$

(ii) **its acceleration (a)** $= \frac{dv}{dt}$ **or** $\frac{d^2s}{dt^2}$ **or** $v\frac{dv}{ds}$**.**

The quantity mv is called the momentum.

II: **Newton's second law states that** $F = \frac{d}{dt}(mv)$ **.**

If m is constant, then $F = m\frac{dv}{dt} = ma$**,**

i.e., net force = mass x acceleration.

Q.No.1.: [Resisted motion]:

A moving body is **opposed** by a force per unit mass of value cx and resistance per unit mass of value bv^2, where x and v are the displacement and velocity of the particle at that instant. Show that the velocity of the particle, if it starts from rest, is given by $v^2 = \frac{c}{2b^2}\left(1 - e^{-2bx}\right) - \frac{cx}{b}$.

Sol.: Mathematical model of given problem:

By Newton's second law, the equation of motion of the body is

$$v\frac{dv}{dx} = -cx - bv^2$$

$$\Rightarrow v\frac{dv}{dx} + bv^2 = -cx, \qquad \text{(i)}$$

This is Bernoulli's equation.

$$\boxed{\frac{dy}{dx}+Py=Qy^n} \quad \boxed{y^{-n}\frac{dy}{dx}+Py^{1-n}=Q}.$$

$\therefore$ Putting $v^2=z$ and $2v\frac{dv}{dx}=\frac{dz}{dx}$.

Then (i) becomes $\frac{1}{2}\frac{dz}{dx}+bz=-cx \Rightarrow \frac{dz}{dx}+2bz=-2cx$, (ii)

This is Leibnitz's linear equation in z, i.e. $\frac{dz}{dx}+Pz=Q$.

$\therefore$ I. F. $=e^{\int 2bdx}=c^{2bx}$.

$\therefore$ The solution of (ii) is $z(\text{I.F.})=\int .Q(\text{I.F.})dx+c$

$$z.e^{2bx}=\int -2cx.e^{2bx}dx+c_1=-2c\int xe^{2bx}dx+c_1$$

$$\Rightarrow v^2.e^{2bx}=-\frac{cx}{b}e^{2bx}+\frac{c}{2b^2}e^{2bx}+c_1 \Rightarrow v^2=-\frac{cx}{b}+\frac{c}{2b^2}+c_1e^{-2bx} \quad \text{(iii)}$$

Initial value problem:

Initially, when x = 0: v = 0.

$$\therefore \frac{c}{2b^2}+c_1=0 \Rightarrow c_1=-\frac{c}{2b^2}.$$

Substituting the value of c_1 in (iii), we get

$$v^2=-\frac{cx}{b}+\frac{c}{2b^2}-\frac{c}{2b^2}e^{-2bx}$$

$$\Rightarrow v^2=\frac{c}{2b^2}\left(1-e^{-2bx}\right)-\frac{cx}{b},$$

which is the required velocity of the particle

Q.No.2.: A particle of mass m is projected vertically upward under gravity, the resistance of the air being mk times the velocity. Show that the greatest height attained by the particle is $\frac{V^2}{g}[\lambda-\log(1+\lambda)]$, where V is the greatest velocity which the above mass will attain when it falls freely and λV is the initially velocity.

Sol.: Let v be the velocity of the particle at time t. The forces acting on the particle are:

(i) its weight mg acting vertically downwards.

(ii) the resistance mkv of the air acting vertically downwards.

Accelerating force on the particle $= -mg - mkv$

$\therefore$ By Newton's second law, the equation of the motion of the particle is

$$mv\frac{dv}{dx} = -mg - mkv \Rightarrow v\frac{dv}{dx} = -g - kv. \qquad \text{(i)}$$

When the particle falls freely (under gravity), equation (i) becomes(changing g to $-g$).

$$v\frac{dv}{dx} = g - kv. \qquad \text{(ii)}$$

When the particle attains the greatest velocity V, its acceleration is zero.

$\therefore$ From (ii), $0 = g - kV \Rightarrow k = \frac{g}{V}$

Putting this value of k in (i), we have

$$v\frac{dv}{dx} = -g - \frac{g}{V}v = -\frac{g}{V}(V + v) \Rightarrow \frac{v}{V+v}dv = -\frac{g}{V}dx.$$

Integrating, we get $\int \frac{v}{V+v}dv = -\frac{g}{V}\int dx + c \Rightarrow \int \left(1 - \frac{V}{V+v}\right)dv = -\frac{g}{V}x + c$

$$\Rightarrow v - V\log(V + v) = -\frac{g}{V}x + c \qquad \text{(iii)}$$

Initially, when $x = 0$, $v = \lambda V$

From (iii), we have $\lambda V - V\log(V + \lambda V) = c \Rightarrow c = V[\lambda - \log V(1+\lambda)]$

Substituting the value of c in (iii), $v - V\log(V + v) = -\frac{g}{V}x + V[\lambda - \log V(1+\lambda)]$ (iv)

Let h be the greatest height attained by the particle, then $x = h$ when $v = 0$

$\therefore$ From (iv), we have $-V\log V = -\frac{g}{V}h + V[\lambda - \log V(1+\lambda)]$

$$\Rightarrow \frac{g}{V}h = V\lambda - V[\log V(1+\lambda) - \log V] = V\lambda - V\log\frac{V(1+\lambda)}{V}$$

$$\Rightarrow h = \frac{V^2}{g}[\lambda - \log(1+\lambda)], \text{ Ans.}$$

which is the required greatest height attained by the particle.

Q.No.3.: [Resisted vertical motion]

A body of mass m, falling from rest, is subject to the force of gravity and an air resistance proportional to the square of the velocity (i.e. kv^2). If it falls through a distance x and possesses a velocity v at that instant, prove that $\frac{2kx}{m} = \log\frac{a^2}{a^2 - v^2}$, where $mg = ka^2$.

Sol.: The forces acting on the body are

(i) its **weight** 'mg' acting vertically downwards.

(ii) The **resistance** 'kv^2' of the air acting vertically upward.

Accelerating force on the body $= mg - kv^2 = ka^2 - kv^2 = k\left(a^2 - v^2\right)$. $\left[\because mg = ka^2\right]$

If the body be moving with the velocity 'v' after having fallen through a distance 'x', then its equation of motion is

$$mv\frac{dv}{dx} = k\left(a^2 - v^2\right).$$

$$\Rightarrow \frac{v}{a^2 - v^2}dv = \frac{k}{m}dx \qquad \text{(i)}$$

Integrating on both sides, we get

$$\int \frac{v}{a^2 - v^2}dv = \frac{k}{m}\int dx + c \Rightarrow -\frac{1}{2}\int \frac{-2v}{a^2 - v^2}dv = \frac{k}{m}x + c$$

$$\Rightarrow -\frac{1}{2}\log\left(a^2 - v^2\right) = \frac{k}{m}x + c \qquad \text{(ii)}$$

Initially, when $x = 0,\ v = 0$ $\quad \therefore$ (ii) $\Rightarrow -\frac{1}{2}\log a^2 = c$

$\therefore$ From (i), we have

$$-\frac{1}{2}\log\left(a^2 - v^2\right) = \frac{k}{m}x - \frac{1}{2}\log a^2$$

$$\Rightarrow \frac{2kx}{m} = \log a^2 - \log\left(a^2 - v^2\right) \Rightarrow \frac{2kx}{m} = \log\left(\frac{a^2}{a^2 - v^2}\right).$$

Hence proved.

Q.No.4.: A paratrooper and his parachute weight 50 kg. At the instant parachute opens, he is traveling vertically downward at the speed of 20 m/s. If the air resistance varies directly as the instantaneous velocity and it is 20 Newtons when the velocity is 10m/s, find the limiting velocity, the position and the velocity of the paratrooper at any time t.

Sol.: Let v m/s be the velocity of the paratrooper t seconds after the parachute opens. The forces acting on the paratrooper are

(iii) its weight 50 kg acting vertically downwards.

(iv) the resistance kv of the air acting vertically upwards.

Accelerating force $= (50 - kv)N$.

$\therefore$ By Newton's second law, the acceleration force is $m\frac{dv}{dt} = \frac{W}{g}\frac{dv}{dt} = \frac{50}{g}\frac{dv}{dt}$.

$\therefore$ The equation of motion is $\frac{50}{g}\frac{dv}{dt} = 50 - kv$. (i)

When v = 10 m/s, the air resistance kv = 20 N.

$\therefore k.10 = 20 \Rightarrow k = 2$

Substituting the value of k in (i), we have

$$\frac{50}{g}\frac{dv}{dt} = 50 - 2v \Rightarrow \frac{dv}{25 - v} = \frac{g}{25}dt.$$

Integrating, we get $-\log(25 - v) = \frac{gt}{25} + c$. (ii)

When $t = 0$, $v = 25$ $\therefore c = -\log 5$.

Substituting the value of c in (ii), we get $-\log(25 - v) = \frac{gt}{25} - \log 5$

$$\Rightarrow \log\frac{25 - v}{5} = -\frac{gt}{25} \Rightarrow \frac{25 - v}{5} = e^{-\frac{gt}{25}} \Rightarrow v = 5\left(5 - e^{-\frac{gt}{25}}\right), \text{ Ans.} \quad \text{(iii)}$$

which gives the velocity of paratrooper at time t.

The limiting velocity is the velocity when $t \to \infty$.

$\therefore$ From (ii), the limiting velocity $= 25\ m/s$.

$\therefore$ From (iii), we have $\frac{dx}{dt} = 5\left(5 - e^{-\frac{gt}{25}}\right) \Rightarrow dx = 5\left(5 - e^{-\frac{gt}{25}}\right)dt$.

Integrating, we get $x = 5\left(5t + \frac{25}{g}e^{-\frac{gt}{25}}\right) + c'$. (iv)

Initially, when $t = 0$, $x = 0$ $\quad \therefore c' = -\frac{125}{g}$

Substituting the value of c' in (iv), we get

$x = 25t - \frac{125}{g}\left(1 - e^{-\frac{gt}{25}}\right)$, Ans.

which is the position of the paratrooper at any time t.

Q.No.5.: Velocity of escape from the earth

Determine the least velocity with which a particle must be projected vertically upwards so that it does not return to the earth. Assume that it is acted upon by the gravitational attraction of the earth only.

or

Find the initial velocity of a particle which is fired in radial direction from the earth's centre and is supposed to escape from the earth. Assume that it is acted upon by the gravitational attraction of the earth only.

Sol.: Let r be the variable distance of the particle from the earth' centre.

By Newton's law of gravitation, the acceleration a of the particle is proportional to $\frac{1}{r^2}$.

$$\therefore a = v\frac{dv}{dr} = -\frac{\mu}{r^2} \qquad \text{(i)}$$

where v is the velocity of the particle when its distance from the earth's centre is r.

On the surface of the earth, r = R, the radius of earth and $a = -g$.

$$\therefore -g = -\frac{\mu}{R^2} \Rightarrow \mu = gR^2$$

$\therefore$ From (i), $\quad v\frac{dv}{dr} = -\frac{gR^2}{r^2}$

Separating the variables, we get $\quad vdv = -\frac{gR^2}{r^2}dr.$

Integrating, we get $\int vdv = -gR^2 \int \frac{dr}{r^2} + c$

$$\Rightarrow \frac{v^2}{2} = \frac{gR^2}{r} + c$$

$$\Rightarrow v^2 = \frac{2gR^2}{r} + 2c \qquad \text{(ii)}$$

Let v_0 be the velocity of projection from the surface of the earth.

Then $v = v_0$ when $r = R$.

$\therefore$ From (ii), $v_0^2 = 2gR + 2c$. (iii)

Subtracting (iii) from (ii) [to eliminate c], we get

$$v^2 - v_0^2 = \frac{2gR^2}{r} - 2gR \Rightarrow v^2 = \frac{2gR^2}{r} + \left(v_0^2 - 2gR\right)$$

The particle will never return to earth if its velocity v during ascent remains positive. When v vanishes, the particle stops and the velocity will change from positive to negative and particle returns to the earth.

Now, as the particle rises upwards, $\frac{2gR^2}{r}$ goes on decreasing.

The velocity will remain positive if $v_0^2 - 2gR \geq 0$. i.e. if $v_0 \geq \sqrt{2gR}$.

$\therefore$ The least velocity of projection $v_0 = \sqrt{2gR}$.

A particle projected with this velocity will never return to the earth, i. e. will escape from the earth. This velocity is called the **velocity of escape** from the earth.

Q.No.6.: Motion of a boat across stream

A boat is rowed with a velocity u across a stream of width a. If the velocity of current is directly proportional to the product of the distances from the two banks, find the equation of the path of the boat and the distance down stream to the point where it lands.

Sol.: Let the point from where the boat starts be taken as the origin O and the axis as shown in the figure. At any time t, let the boat be at P(x, y), then its distances from the two banks are y and $(a - y)$.

$\frac{dx}{dt}$ = velocity of the current $= ky(a-y)$, $\frac{dy}{dt}$ = velocity of the boat = u

$$\therefore \frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{u}{ky(a-y)} \qquad \text{(i)}$$

This is the direction of the resultant velocity of the boat at P and hence the direction of the tangent to the path of the boat at P.

From (i), $y(a-y)dy = \frac{u}{k}dx$.

Integrating, we get $\frac{ay^2}{2} - \frac{y^2}{3} = \frac{u}{k}x + c$.

When $x = 0$, $y = 0$, so that c = 0.

$\therefore$ The equation of the path of the boat is $x = \frac{k}{6u}y^2(3a-2y)$. (ii)

The distance down-stream to the point where the boat lands, is obtained by putting y = a in (ii).

Thus $x = \frac{ka^3}{6u}$, Ans.

Which is the required distance AB, down stream where the boat lands

Q.No.7.: Atmospheric Pressure

Find the atmospheric pressure p kg/m^2 at a height z meters above the sea-level.

or

Find the atmospheric pressure $p\ell b$ per feet at a height z feet above the sea level, both when the temperature is constant and variable.

Sol.: Consider a vertical column of air of unit cross-section. Let an element of this column be bounded by two horizontal planes at height z and $z+\delta z$ above the sea level. Let p and $p+\delta p$ be the pressure at heights z and $z+\delta z$ respectively. Let ρ be the average density of the element.

The thin column δz of air is in equilibrium under the action of forces (i) p kg upward, (ii) $(p+\delta)$kg downwards, (iii) the weight $\rho g\,\delta z$ kg downwards.

$\therefore p = p + \delta p + \rho g \delta z \Rightarrow \delta p + \rho g \delta z = 0 \Rightarrow \dfrac{\delta p}{\delta z} = -\rho g$.

Taking the limit as $\delta z \to 0$, we have $\dfrac{dp}{dz} = -\rho g$, (i)

which is the differential equation giving atmospheric pressure at any height z.

Now, we discuss equation (i) under two assumptions:

(i) when the temperature is constant.

(ii) When the temperature varies.

Case 1. When the temperature is constant.

By Boyle's Law, $p = k\rho \Rightarrow \rho = \dfrac{p}{k}$.

Substituting this value of ρ in (i), we get $\dfrac{dp}{dz} = -\dfrac{pg}{k} \Rightarrow \dfrac{dp}{p} = -\dfrac{g}{k} dz$

Integrating, we get $\int \dfrac{dp}{p} = -\dfrac{g}{k} \int dz + c \Rightarrow \log p = -\dfrac{g}{k} z + c$ (ii)

At the sea-level, $z = 0$, $p = p_0$ (say), then $c = \log p_0$

$\therefore$ From (ii), $\log p = -\dfrac{g}{k} z + \log p_0 \Rightarrow \log \dfrac{p}{p_p} = -\dfrac{g}{k} z$

$\therefore p = p_0 e^{-\frac{gz}{k}}$. Ans.

Case I1. When the temperature varies.

Let $p = k\rho^n$, $n \neq 1$ or $\rho = \left(\dfrac{p}{k}\right)^{1/n}$.

Substituting this value of ρ in (i), we get $\dfrac{dp}{dz} = -\left(\dfrac{p}{k}\right)^{1/n}$

$\Rightarrow p^{-\frac{1}{n}} dp = -gk^{-\frac{1}{n}} dz$

Integrating, we get $\int p^{-\frac{1}{n}} dp = -gk^{-\frac{1}{n}} \int dz + c \Rightarrow \dfrac{n}{n-1} p^{1-\frac{1}{n}} = -gk^{-\frac{1}{n}} z + c$. (iii)

At the sea-level, $z = 0$, $p = p_0$ (say), then $c = \dfrac{n}{n-1} p_0^{1-\frac{1}{n}}$.

$\therefore$ From (iii), $\frac{n}{n-1}p^{1-\frac{1}{n}} = -gk^{-\frac{1}{n}}z + \frac{n}{n-1}p_0^{1-\frac{1}{n}}$

$\Rightarrow \frac{n}{n-1}\left(p_0^{1-\frac{1}{n}} - p^{1-\frac{1}{n}}\right) = gk^{-\frac{1}{n}}.z$. Ans.

Q.No.8.: Rotating cylinder containing liquid

A cylindrical tank of radius r is filled with water. When the tank is rotated rapidly about its axis with angular velocity ω, centrifugal force tends to drive the water outwards from the centre of the tank. Under steady conditions of uniform rotation, find the equation of the curve in which the free surface of the water is intersected by a plane through the axis of the cylinder.

or

A cylindrical tank of radius r is filled with water to a depth h. When the tank is rotated with angular velocity ω about its axis, centrifugal force tends to drive the water outwards from the centre of the tank. Under steady conditions of uniform rotation, show that the section of the water by a plane through the axis, is the curve $y = \frac{\omega^2}{2g}\left(x^2 - \frac{r^2}{2}\right) + h$.

Sol.: Consider a particle of mass m at the point P(x, y) on the plane curve cut out of the surface by an axial section. The forces acting on the particle are:

(i) the weight mg acting vertically downwards.

(ii) The centrifugal force $m\omega^2 x$ acting horizontally outwards.

Since the motion is steady, P cannot move either up or down the surface of the water. Thus P moves just on the surface of the water and, therefore, there is no force along the tangent to the curve. The resultant R of mg and $m\omega^2 x$ is along the outward normal to the curve,

From the figure, $R\cos\psi = mg$ and $R\sin\psi = m\omega^2 x$

$$\therefore \frac{dy}{dx} = \tan\psi = \frac{m\omega^2 x}{mg} = \frac{\omega^2 x}{g}, \qquad \text{(i)}$$

which is the differential equation of the surface of the rotating liquid.

From (i), $dy = \frac{\omega^2 x}{g} dx$

Integrating, we get $\int dy = \frac{\omega^2}{g}\int xdx + c \Rightarrow y = \frac{\omega^2 x^2}{2g} + c$ (ii)

To determine c, we use the fact that the volume of the liquid remains the same before and after rotation.

Thus the shaded area in the two figures remain the same.

Now, the shaded area in non-rotational case $= 2rh$.

Also, the shaded area in rotational case $= 2\int_0^r ydx = 2\int_0^r \left(\frac{\omega^2 x^2}{2g} + c\right) = 2\left(\frac{\omega^2 r^3}{2g} + cr\right)$

$\therefore 2\left(\frac{\omega^2 r^3}{6g} + cr\right) = 2rh \Rightarrow c = h - \frac{\omega^2 r^2}{6g}$

$\therefore$ From (ii), we have $y = \frac{\omega^2 x^2}{2g} + h - \frac{\omega^2 r^2}{6g} \Rightarrow y = \frac{\omega^2}{2g}\left(x^2 - \frac{r^2}{3}\right) + h$,

which is the required equation of the curve.

Q.No.9.: Discharge of water through a small hole

If the velocity of flow of water through a small hole is $0.6\sqrt{2gy}$ where g is the gravitational acceleration and y is the height of water level above the hole, find the time required to empty a tank having the shape of a right circular cone of base radius a and height h filled completely with water and having a hole of area A_0 in the base.

Sol.: A any time t, let the height of the water level be y and radius of its surface be r so that

$\frac{h-y}{r} = \frac{h}{a} \Rightarrow r = a\frac{(h-y)}{h}$

$\therefore$ Surface area of the liquid $= \pi r^2 = \pi a^2\left(1 - \frac{y}{h}\right)^2$

Volume of water drained through the hole per unit time

$$= 0.6\sqrt{(2gy)}A_0 = 4.8\sqrt{y}A_0 \qquad [\because g = 32]$$

$\therefore$ Rate of fall of liquid level $= 4.8A_0\sqrt{y} + \pi a^2\left(1-\frac{y}{h}\right)^2 \Rightarrow \frac{dy}{dt} = -\frac{4.8A_0\sqrt{y}}{\pi^2\left(1-\frac{y}{h}\right)^2}$

[Negative is taken since the water level decreases]

Hence time to empty the tank (= t)

$$= -\int_h^0 \frac{\pi a^2\left(1-\frac{y}{h}\right)^2}{4.8A_0\sqrt{y}}dy = \frac{\pi a^2}{4.8A_0}\int_0^h \left(y^{-1/2} - \frac{2y^{1/2}}{h} + \frac{y^{3/2}}{h^2}\right)dy$$

$$= \frac{\pi a^2}{4.8A_0}\left|2y^{1/2} - \frac{4}{3h}y^{3/2} + \frac{2}{5h^2}y^{5/2}\right|_0^h = 0.2\pi a^2\frac{\sqrt{h}}{A_0}. \text{ Ans.}$$

Q.No.10.: A particle of mass m moves under gravity in a medium whose resistance is k times its velocity, where k is a constant. If the particle is projected vertically upwards with a velocity v, show that the time to reach the highest point is $\frac{m}{k}\log_e\left(1+\frac{kv}{mg}\right)$.

Sol.: Let u be the velocity of the particle at time t. The forces acting on the particle are:

(v) its weight mg acting vertically downwards.

(vi) the resistance ku of the air acting vertically downwards.

Accelerating force on the particle $= -mg - ku$

$\therefore$ By Newton's second law, the equation of the motion of the particle is

$$m\frac{du}{dt} = -mg - ku \Rightarrow du = -\left(g + \frac{k}{m}u\right)dt \Rightarrow \frac{du}{\left(g+\frac{k}{m}u\right)} = -dt. \qquad \text{(i)}$$

Integrating (i) both sides, we get

$$\frac{m}{k}\log\left(g + \frac{k}{m}u\right) = -t + c. \qquad \text{(i)}$$

When $t = 0,\quad u = v \qquad \therefore$ (i) $\Rightarrow \frac{m}{k}\log\left(g + \frac{k}{m}u\right) = c.$

Putting the value of c in (i), we get

$$\frac{m}{k}\log_e\left(g+\frac{k}{m}u\right)=-t+\frac{m}{k}\log_e\left(g+\frac{k}{m}v\right)\Rightarrow \log_e\left(g+\frac{k}{m}u\right)=-\frac{kt}{m}+\log_e\left(g+\frac{k}{m}v\right)$$

$$\Rightarrow \log_e\left(g+\frac{k}{m}u\right)=\log_e\left(e^{-\frac{kt}{m}}\right)+\log_e\left(g+\frac{k}{m}v\right)$$

$$\Rightarrow \log_e\left(g+\frac{k}{m}u\right)=\log_e\left[\left(e^{-\frac{kt}{m}}\right)\left(g+\frac{k}{m}v\right)\right]\Rightarrow \left(g+\frac{k}{m}u\right)=e^{-\frac{kt}{m}}\left(g+\frac{k}{m}v\right)$$

$$\Rightarrow u=\frac{m}{k}\left[e^{-\frac{kt}{m}}\left(g+\frac{k}{m}v\right)-g\right].$$

At maximum height $u = 0$.

$$e^{-\frac{kt}{m}}\left(g+\frac{k}{m}v\right)=g\Rightarrow e^{-\frac{kt}{m}}=\frac{g}{\left(g+\frac{kv}{m}\right)}$$

$$\Rightarrow -\frac{k}{m}t=\log_e\left[\frac{g}{g+\frac{kv}{m}}\right]\Rightarrow t=-\frac{m}{k}\log_e\left[\frac{g}{g+\frac{kv}{m}}\right]$$

$\therefore$ The magnitude of tine required is

$$t=\frac{m}{k}\log_e\left[\frac{g}{g+\frac{kv}{m}}\right], \text{ Ans.}$$

which is the time to reach the highest point.

Q.No.11.: A body of mass m falls through a medium that opposes its fall with a force proportional to the square of its velocity so that its equation of motion is

$m\frac{d^2x}{dt^2}=mg-kv^2$, $v = \dot{x}$, where x is the distance of fall and v is the velocity.

If, at $t = 0$, $x = 0$ and $v = 0$, find the velocity of the body and the distance it has fallen in t seconds.

Sol.: Given $m\frac{d^2x}{dt^2}=mg-kv^2\Rightarrow m\frac{dv}{dt}=mg-kv^2\Rightarrow \frac{dv}{dt}=g-\frac{kv^2}{m}$

$$\Rightarrow \frac{dv}{\left(g-\frac{kv^2}{m}\right)} = dt. \qquad \text{(i)}$$

Integrating (i) both sides, we get

$$t = \int \frac{dv}{\left(\sqrt{g}+\sqrt{\frac{k}{m}}v\right)\left(\sqrt{g}-\sqrt{\frac{k}{m}}v\right)} + c = \frac{1}{2\sqrt{g}}\int \frac{\left(\sqrt{g}+\sqrt{\frac{k}{m}}v\right)+\left(\sqrt{g}-\sqrt{\frac{k}{m}}v\right)}{\left(\sqrt{g}+\sqrt{\frac{k}{m}}v\right)\left(\sqrt{g}-\sqrt{\frac{k}{m}}v\right)} + c$$

$$= \frac{1}{2\sqrt{g}}\left[\frac{\log\left(\sqrt{g}-\sqrt{\frac{k}{m}}v\right)}{-\sqrt{\frac{k}{m}}} + \frac{\log\left(\sqrt{g}+\sqrt{\frac{k}{m}}v\right)}{\sqrt{\frac{k}{m}}}\right] + c$$

$$\Rightarrow t = \frac{\sqrt{m}}{2\sqrt{gk}}\log\left(\frac{\sqrt{g}+\sqrt{\frac{k}{m}}v}{\sqrt{g}-\sqrt{\frac{k}{m}}v}\right) + c$$

But at $t = 0$, $\quad c = 0$.

$$\therefore t = \frac{1}{2}\sqrt{\frac{m}{gk}}\log\left(\frac{\sqrt{g}+\sqrt{\frac{k}{m}}v}{\sqrt{g}-\sqrt{\frac{k}{m}}v}\right) \Rightarrow e^{2t\sqrt{\frac{gk}{m}}} = \left(\frac{\sqrt{g}+\sqrt{\frac{k}{m}}v}{\sqrt{g}-\sqrt{\frac{k}{m}}v}\right)$$

By componendo and dividendo, we get

$$\frac{e^{2t\sqrt{\frac{gk}{m}}}-1}{e^{2t\sqrt{\frac{gk}{m}}}+1} = \frac{2\sqrt{\frac{k}{m}}v}{2\sqrt{g}} \Rightarrow \tanh\left(\sqrt{\frac{gk}{m}}t\right) = \sqrt{\frac{kv}{mg}}.v$$

$$\therefore v = \sqrt{\frac{mg}{k}}\tanh\left(\sqrt{\frac{gk}{m}}t\right) \Rightarrow \frac{dx}{dt} = \sqrt{\frac{mg}{k}}\tanh\left(\sqrt{\frac{gk}{m}}t\right), \text{ Ans.}$$

which is the required velocity of the body in t seconds.

Now, integrating again, we get

$$\Rightarrow \int dx = \sqrt{\frac{mg}{k}}\int \tanh\left(\sqrt{\frac{gk}{m}}t\right)dt + c$$

$$\therefore x = \sqrt{\frac{mg}{k}}\int \frac{e^{\sqrt{\frac{gk}{m}}t} - e^{-\sqrt{\frac{gk}{m}}t}}{e^{\sqrt{\frac{gk}{m}}t} + e^{-\sqrt{\frac{gk}{m}}t}}dt + c = \sqrt{\frac{mg}{k}}\log\frac{e^{\sqrt{\frac{gk}{m}}t} + e^{-\sqrt{\frac{gk}{m}}t}}{\sqrt{\frac{gk}{m}}} + c$$

$$= \frac{m}{k}\log\left(e^{\sqrt{\frac{gk}{m}}t} + e^{-\sqrt{\frac{gk}{m}}t}\right) + c.$$

At $t = 0, \quad x = 0$

$$\therefore 0 = \frac{m}{k}\log 2 + c \Rightarrow c = -\frac{m}{k}\log 2.$$

$$x = \frac{m}{k}\log\left(e^{\sqrt{\frac{gk}{m}}t} + e^{-\sqrt{\frac{gk}{m}}t}\right) - \frac{m}{k}\log 2 = \frac{m}{k}\log\left(\frac{e^{\sqrt{\frac{gk}{m}}t} + e^{-\sqrt{\frac{gk}{m}}t}}{2}\right)$$

$$x = \frac{m}{k}\log\left(\frac{e^{\sqrt{\frac{gk}{m}}t} + e^{-\sqrt{\frac{gk}{m}}t}}{2}\right), \text{ Ans.}$$

which is the required distance in t seconds.

Q.No.12.: A body of mass m falls from rest under gravity in a field whose resistance is mk times the velocity of the body. Find the terminal velocity of the body and also the time taken to acquire one half of its limiting speed.

Sol.: Let v be the velocity of the body at time t. The forces acting on the body are:

(i) its weight mg acting vertically downwards.

(ii) the resistance mkv of the medium acting vertically downwards.

Accelerating force on the body $= -mg - mkv$

$\therefore$ By Newton's second law, the equation of the motion of the particle is $ma = -mg - mkv \Rightarrow a = g - kv$,

when body falls with terminal velocity its acceleration is zero

$\therefore a = g - kv = 0 \Rightarrow kv = g \Rightarrow v = \frac{g}{k}$, Ans.

which is the required terminal velocity.

When the velocity of the body reaches to half of the terminal velocity, then time can be found by solving the differential equation under appropriate limits.

$$\frac{dv}{dt} = g - kv \Rightarrow \frac{dv}{(g - kv)} = dt.$$

Integrating both sides and the limits of v varies from $0 \to \frac{g}{2k}$ and time t varies from $0 \to t$, we get

$$\int_0^{\frac{g}{2k}} \frac{dv}{(g - kv)} = \int_0^t dt \Rightarrow \left| -\frac{1}{k}\log(g - kv) \right|_0^{\frac{g}{2k}} = t \Rightarrow -\frac{1}{k}\left[\log\left(g - \frac{kg}{2k}\right)\right] + \frac{1}{k}\log g = t$$

$$\Rightarrow -\frac{1}{k}\left[\log\frac{g}{2}\right] + \frac{1}{k}\log g = t \Rightarrow -\frac{1}{k}\log g + \frac{1}{k}\log 2 + \frac{1}{k}\log g = t$$

$$\Rightarrow t = \frac{1}{k}\log 2.$$

Hence time taken to acquire one half of its limiting speed is $\frac{1}{k}\log 2$. Ans.

Home Assignments

Q.No.1.: A particle is projected with velocity v along a smooth horizontal plane in the medium whose resistant per unit mass is μ times the cube of the velocity.

Show that the distance it has described in time t is $\frac{1}{\mu v}\left(\sqrt{1 + 2\mu v^2 t} - 1\right)$.

Sol.:

Q.No.2.: When a bullet is fired in a sand tank, its retardation is proportional to the square root of its velocity. How long will it take to come to rest if it enters the sand bank with velocity v_0?

Sol.:

Q.No.3.: A particle of mass m is attached to the lower end of the light spring (upper end is fixed) and is released. Express the velocity v as the function of the stretch x feet.

Sol.:

Q.No.4.: A chain coiled up near the edge of a smooth table just start to fall over the edge. The velocity v when a length x has fallen is given by $xv\frac{dv}{dx}+v^2=gx$. Show that $v=8\sqrt{\left(\frac{x}{3}\right)}$ ft./sec.

Sol.:

Q.No.5.: A toboggan weighing $200\ \ell b$, descends from rest on a uniform slope of 5 in 13 which is 15 yards long. If the co-efficient of friction is $\frac{1}{10}$ and the air resistance varies as the square of the velocity and is $3\ \ell b$ weight when the velocity is 10 ft/sec.; prove that its velocity at the bottom is 38.6 ft/sec and show that however long, the slope is the velocity can not exceed 44 ft/sec.

Sol.:

Q.No.6.: Show that a particle projected from the earth's surface with a velocity of 7 miles/sec. will not return to the earth.[taking earth' radius = 3960 miles and $g=32.17\text{ft}/\text{sec}^2$.

Sol.:

Q.No.7.: A cylindrical 1.5 m. high stands on its circular base of diameter 1 m. and is initially filled with water. At the bottom of the tank there is a hole of diameter 1 cm., which is opened at some instant, so that the water starts draining under gravity. Find the height of water in the tank at any time t sec. Find the time which the tank is one-half full, one-quarter full, and empty.

Sol.:

Q.No.8.: The rate at which water flows from a small hole at the bottom of a tank is proportional to the square root of the depth of the water. If half the water flows from the cylindrical tank (with vertical axis) in 5 minutes, find the time required to empty the tank.

Sol.:

Q.No.9.: A conical cistern of height h and semi-vertical angle α is filled with water and is held in vertical position with vertex downward. Water leaks out from the bottom at the rate of kx^2 cubic cms per second, k is a constant and x is the height of water level from the vertex. Prove that the cistern will be empty in $\dfrac{\left(\pi h \tan^2 \alpha\right)}{k}$ seconds.

Sol.:

Q.No.10.: Up to a certain height in the atmosphere, it is found that the pressure p and the density ρ are connected by the relation $p = k\rho^n \ (n > 1)$. If this relation continued to hold up any height, show that the density would vanish at a finite height.

Sol.:

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Applications

of

Ordinary Differential Equations of first order

"Application to Electric Circuit"

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Simple Electric Circuits:

As we know heat flows from one point to the other due to a temperature difference.

Electric current flows due to a difference in the electric potential or voltage measured in volts (v). Electric current is a flow of charges, measured in amperes (A).

Electric current:

Electric current means, a flow of electric charge (a phenomenon) or the rate of flow of electric charge (a quantity).

This flowing electric charge is typically carried by moving electrons, in a conductor such as wire; in an electrolyte, it is instead carried by ions, and, in a plasma, by both.

The SI unit for measuring the rate of flow of electric charge is the ampere, which is charge flowing through some surface at the rate of one coulomb per second.

Electric current is measured using an ammeter.

An electric circuit consists of a source of electric energy (electromotive force) and elements such as resistors, inductors or voltage and capacitors.

A mathematical model of an electric circuit is represented by linear (first or second order) differential equations.

Formation of Equation:

To form such an equation, the following relationships are needed:

1. The **voltage drop** E_R across a **resistor** is proportional to the instantaneous current I(t) through it:

   $E_R = RI$ **(Ohm's law)** (i)

   Here t is time and the constant of proportionality R is known as the **resistance** of the resistor, measured in ohms (Ω). Resistor uses (consumes) energy.

2. The **voltage drop** E_L across an **inductor** is proportional to the instantaneous time rate of change of the current:

   $$E_L = L\frac{dI}{dt}. \quad \text{(ii)}$$

   Here the constant of proportionality L is known as **inductance** of the inductor and is measured in henrys (H).

3. The **voltage drop** E_c across a **capacitor** is proportional to the instantaneous electric charge Q on the capacitor

   $$E_c = \frac{1}{c}Q. \quad \text{(iii)}$$

   Here c is called the **capacitance** and is measured in farads (F). The charge Q is measured in coulombs.

   Since current is the time rate of change of charge,

   $$I(t) = \frac{dQ}{dt}. \quad \text{(iv)}$$

   Equation (iii) may be written as

   $$E_c = \frac{1}{c}\int_{t_0}^{t} I(t)\,dt. \quad \text{(v)}$$

To determine the current I(t) in an electric circuit, a differential equation is formed using the Kirchoff's law (KVL) which states that "***the algebraic sum of all the instantaneous***

voltage drops around any closed loop is zero or the voltage impressed on a closed loop is equal to the sum of the voltage drops in the rest of the loop".

[Gustav Robert Kirchoff (1824-1887), German Physicist]

Consider two simple cases of series or one-loop electric circuits.

RL-Circuit

By (i), the voltage drop across the resistor is RL by (ii), the voltage drop across the inductor is $L\frac{dI}{dt}$.

Now applying Kirchoff's law to the RL-circuit, the sum of the two voltage drops must be equal to electromotive force E(t).

Thus, the current I(t) in the RL-circuit is determined by the first order linear differential equation.

$$\boxed{L\frac{dI}{dt}+RI=E(t)}. \tag{vi}$$

Rewriting (vi), we get

$$\frac{dI}{dt}+\frac{R}{L}I=\frac{E(t)}{L}.$$

This is a Leibnitz's Linear equation.

Integrating factor $= e^{\int \frac{R}{L}dt}=e^{\frac{R}{L}t}=e^{\alpha t}$, where $\alpha=\frac{R}{L}$.

Then, the general solution of (vi) is

$$I(t).e^{\alpha t}=\int \frac{E(t)}{L}.e^{\alpha t}dt+c \Rightarrow I(t)=e^{-\alpha t}\left[\int \frac{E(t)}{L}.e^{\alpha t}dt+c\right]. \tag{vii}$$

Case (a):

Suppose $E = E_0 =$ constant.

Then (vii) becomes

$$I(t)=e^{-\alpha t}\left[\int \frac{E(t)}{L}.e^{\alpha t}dt+c\right] \Rightarrow I(t)=e^{-\alpha t}\left[\frac{E_0}{L}.\frac{e^{\alpha t}}{\alpha}+c\right]=\frac{E_0}{R}+ce^{-\alpha t}. \tag{viii}$$

As $t\to\infty$, $I(t)\to\frac{E_0}{R}=\text{constant}$.

Here $\frac{L}{R}=\frac{1}{\alpha}$ is known as **inductive time constant**.

Case (b):

Suppose $E=E_0\sin\omega t$.

Then (vii) becomes

$$I(t)=e^{-\alpha t}\left[\int \frac{E(t)}{L}.e^{\alpha t}dt+c\right]\Rightarrow I(t)=e^{-\alpha t}\left[\frac{E_0}{L}\int e^{\alpha t}\sin\omega t dt+c\right],$$

$$\left[\because \int e^{at}\sin bt dt=\frac{e^{at}}{\left(a^2+b^2\right)}\left(a\sin bt-b\cos bt\right)\right],$$

$$\Rightarrow I(t)=ce^{-\frac{R}{L}t}+\frac{E_0}{L\left(\frac{R^2}{L^2}+\omega^2\right)}\left(\frac{R}{L}\sin\omega t-\omega.\cos\omega t\right)$$

$$\Rightarrow I(t)=ce^{-\frac{R}{L}t}+\frac{E_0}{R^2+\omega^2L^2}\left(R\sin\omega t-\omega L.\cos\omega t\right)$$

Using $a\cos x+b\sin x=\sqrt{a^2+b^2}\sin(x\pm\theta)$, where $\tan\theta=\frac{\sin\theta}{\cos\theta}=\pm\frac{a}{b}$.

The trigonometric terms can be expressed in phase angle from as

$$I(t)=ce^{\frac{R}{L}t}+\frac{E_0}{\sqrt{R^2+\omega^2L^2}}\sin(\omega t-\theta), \qquad \text{(ix)}$$

where $\theta=\tan^{-1}\frac{\omega L}{g}$.

The current I in (ix) is expressed as the sum of an exponential and sinusoidal terms.

As $t\to\infty$, the first term tends to zero. It is known as the **transient** term.

The second sinusoidal term corresponds to the steady-state, which is free of e^{-t}. The period is $\frac{2\pi}{\omega}$ and phase is θ. The steady state solution is permanent, periodic and has the same period as that of the applied external force.

Case (c):

Suppose $E=E_0\cos\omega t$.

Then (vii) becomes

$$I(t)=e^{-\alpha t}\left[\int \frac{E(t)}{L}.e^{\alpha t}dt+c\right] \Rightarrow I(t)=e^{\alpha t}\left[\frac{E_0}{L}\int e^{\alpha t}\cos\omega t dt+c\right].$$

$$\left[\because \int e^{at}\cos bt dt=\frac{e^{at}}{\left(a^2+b^2\right)}\left(a\cos bt+b\sin bt\right)\right]$$

$$\Rightarrow I(t)=ce^{-\frac{R}{L}t}+\frac{E_0}{R^2+\omega^2L^2}\left(R\cos\omega t+\omega L.\sin\omega t\right).$$

Using $a\cos x+b\sin x=\sqrt{a^2+b^2}\cos(x\pm\theta)$, where $\tan\theta=\frac{\sin\theta}{\cos\theta}=\mp\frac{b}{a}$.

We have

$$I(t)=ce^{-\frac{R}{L}t}+\frac{E_0}{\sqrt{R^2+\omega^2L^2}}\cos(\omega t-\theta), \tag{x}$$

where $\theta=\tan^{-1}\frac{\omega L}{R}$.

RC-Circuit:

Using (i), (v) and Kirchoff's law, we get the integro-differential equation.

$$RI+\frac{1}{c}\int I(t)dt=E(t),$$

which on differentiation reduces to

$$R\frac{dI}{dt}+\frac{1}{c}I=\frac{dE}{dt} \tag{xi}$$

$$\Rightarrow \frac{dI}{dt}+\frac{1}{Rc}I=\frac{1}{R}\frac{dE}{dt}.$$

This is a first order linear differential equation with integrating factor $e^{\int\frac{1}{cR}dt}=e^{\frac{1}{cR}}$.

The general solution of (xi) is

$$I(t).e^{\frac{1}{cR}}=\int \frac{1}{R}\frac{dE}{dt}.e^{\frac{1}{cR}}.dt+c \Rightarrow I(t)=e^{-\frac{1}{cR}}\left[\int \frac{1}{R}\frac{dE}{dt}.e^{\frac{1}{cR}}.dt+c\right]. \tag{xii}$$

Case (a):

Suppose $E=E_0=$ constant.

Then (xii) simplifies as

$$I(t) = ce^{-\frac{1}{cR}}. \tag{xiii}$$

Here RC is known as capacitive time constant of the circuit.

Case (b):

Suppose $E = E_0 \sin \omega t$.

Then (vii) reduces to $\frac{dE}{dt} = \omega E_0 \cos \omega t$.

From (xii), we get

$$I(t) = ce^{-\frac{1}{cR}} + \frac{1}{R}\int e^{-\frac{1}{cR}}.\omega E_0.\cos \omega t dt$$

$$= ce^{-\frac{1}{cR}} + \frac{\omega E_0}{R}\frac{e^{-\frac{1}{cR}}}{\left(\frac{1}{cR}\right)^2 + \omega^2}\left[\frac{1}{cR}\cos \omega t + \omega \sin \omega t\right]$$

$$= ce^{-\frac{1}{cR}} + \frac{\omega E_0 c}{1 + (\omega cR)^2}[\cos \omega t + \omega Rc.\sin \omega t]$$

$$\Rightarrow I(t) = ce^{-\frac{1}{cR}} + \frac{\omega E_0 c}{1 + (\omega cR)^2}\sin[\omega t - \theta] \tag{xiv}$$

where $\tan \theta = -\frac{1}{\omega Rc}$.

The first term which is transient tends to zero as $t \to \infty$ while the second sinusoidal term corresponds to steady state.

Problems on RL-circuit

Q.No.1.: Find the current at any time t > 0 in a circuit having in series a constant electromotive force (E) 40 V, a resistor (R) $10\,\Omega$ and an inductor (L) 0.2 H, given that initial current is zero. Find the current when $E(t) = 150\cos 200t$.

Sol.: **Mathematical model:**

The current I(t) in the RL-circuit is determined by the first order linear differential equation.

$$\boxed{L\frac{dI}{dt}+RI=E(t)}.$$

Case (a): L =0.2, R = 10, E = 40, so equation is

$$0.2\frac{dI}{dt}+10I=40 \Rightarrow \frac{dI}{dt}+50I=200.$$

Solution: Its general solution is [Leibnitz's linear equation in I]

$$I(t).e^{50t}=\int 200.e^{50t}dt+c=200.\frac{e^{50t}}{50}+c \Rightarrow I(t)=e^{-50t}\left[4e^{50t}+c\right].$$

At t = 0, I = 0, so 0 = [4 + c] $\Rightarrow c=-4$.

The current I(t) is given by $I(t)=4\left(1-e^{-50t}\right)$.

Case (b): $E(t)=150\cos 200tv$, so equation is

$$\frac{dI}{dt}+50I=750\cos 200t.$$

Solution: The general solution is [Leibnitz's linear equation in I]

$$I(t).e^{50t}=750\int e^{50t}.\cos 200t dt+c=c+750\frac{e^{50t}}{(2500+40000)}(50\cos 200t+200\sin 200t)$$

At t = 0, I = 0 so

$$0=\frac{3}{170}50+c \Rightarrow c=\frac{15}{17}.$$

The current I(t) is given by

$$I(t)=\frac{3}{170}(50\cos 200t+200\sin 200t)-\frac{15}{17}e^{-50t}.$$

Q.No.2.: A constant electromotive force E volts is applied to a circuit containing a constant resistance R ohms in series and a constant inductance L henries. If the initial current is zero, show that the current builds up to half its theoretical maximum in $\frac{(L\log 2)}{R}$ seconds.

Sol.: Let I be the current in the circuit at any time t.

Mathematical model:

The current I(t) in the RL-circuit is determined by the first order linear differential equation.

$$\boxed{L\frac{dI}{dt}+RI=E}.$$

Now $L\frac{dI}{dt}+RI=E \Rightarrow \frac{dI}{dt}+\frac{R}{L}I=\frac{E}{L}$. (i)

This is a Leibnitz's linear differential equation.

Its I.F. $=e^{\int \frac{R}{L}dt}=e^{\frac{Rt}{L}}$.

Solution: $\therefore$The solution is $I.(I.F.)=\int \frac{E}{L}(I.F.)dt+c$

$\Rightarrow I.e^{Rt/L}=\int \frac{E}{L}.e^{Rt/L}dt+c=\frac{E}{L}.\frac{L}{R}e^{Rt/L}+c$

$\Rightarrow I=\frac{E}{R}+ce^{-Rt/L}$. (ii)

Initial value problem: Initially when $t=0,\ I=0$ so that $c=-\frac{E}{R}$

Thus (ii) becomes, $I=\frac{E}{R}\left(1-e^{-Rt/L}\right)$. (iii)

This equation gives the current at any time t.

Clearly, I increases as t increases.

From (iii), theoretical maximum of I is $I=\frac{E}{R}$.

To show: The time, when the current builds up to half its theoretical maximum is

$\frac{(L\log 2)}{R}$ seconds.

This means that, we have to evaluate the time when current builds up to half its theoretical maximum.

Let T be the time when the current builds up to half its theoretical maximum.

Then $\frac{1}{2}.\frac{E}{R}=\frac{E}{R}\left(1-e^{-RT/L}\right)\Rightarrow e^{-RT/L}=\frac{1}{2}\Rightarrow -\frac{RT}{L}=\log\frac{1}{2}=-\log 2$

$\therefore T=\frac{(L\log 2)}{R}$. Ans.

Q.No.3.: Show that the differential equation for the current I in an electric circuit containing an inductance L and a resistance R in series and acted on by an electromotive force $E\sin\omega t$ satisfy the equation $L\frac{dI}{dt}+RI=E\sin\omega t$.

Find the value of the current at any time t, if initially there is no current in the circuit.

Sol.: **Mathematical model:**

The current I(t) in the RL-circuit is determined by the first order linear differential equation.

$$\boxed{L\frac{dI}{dt}+RI=E}.$$

In this problem, we have given sum of the voltage drop across R and L is $E\sin\omega t$.

$$L\frac{dI}{dt}+RI=E\sin\omega t.$$

This is the required differential equation which can be written as $\frac{dI}{dt}+\frac{R}{L}I=\frac{E}{L}\sin\omega t$.

This is a Leibnitz's equation.

Its I. F. $=e^{\int\frac{R}{L}dt}=e^{\frac{Rt}{L}}$.

Solution: $\therefore$The solution is $I.(I.\,F.)=\int\frac{E}{L}\sin\omega t.(I.F.)dt+c$

$$\Rightarrow Ie^{Rt/L}=\frac{E}{L}\int e^{Rt/L}\sin\omega t dt+c=\frac{E}{L}\frac{e^{Rt/L}}{\left[\sqrt{\left[\left(\frac{R}{L}\right)^2\right]+\omega^2}\right]}\sin\left(\omega t-\tan^{-1}\frac{L\omega}{R}\right)+c$$

$$\Rightarrow I=\frac{E}{\sqrt{\left(R^2+\omega^2L^2\right)}}\sin\left(\omega t-\varphi\right)+ce^{-Rt/L}\quad\text{where }\tan\phi=\frac{L\omega}{R}.\qquad\text{(i)}$$

Initial value problem:

Initially when t = 0, I = 0. $\quad\therefore 0=\frac{E\sin(-\phi)}{\sqrt{\left(R^2+\omega^2L^2\right)}}+c\Rightarrow c=\frac{E\sin\phi}{\sqrt{\left(R^2+\omega^2L^2\right)}}$

Thus (i) takes the form $I=\frac{E\sin\left(\omega t-\varphi\right)}{\sqrt{\left(R^2+\omega^2L^2\right)}}+\frac{E\sin\varphi}{\sqrt{\left(R^2+\omega^2L^2\right)}}.e^{-Rt/L}$

$\Rightarrow I = \dfrac{E}{\sqrt{\left(R^2+\omega^2L^2\right)}}\left[\sin\omega t - \varphi + \sin\varphi . e^{-Rt/L}\right]$, which gives the current at any time t.

Problems on RC-Circuit

Q.No.4.: The equation of the electromotive force in terms of current I for an electrical circuit having resistance R and a condenser of capacity C, in series, is $E = RI + \int \dfrac{I(t)}{C} dt$. Find the current I at any time t, when $E = E_0 \sin\omega t$.

Sol.: **Mathematical model:**

The current I(t) in the RC-circuit is determined by the first order linear differential equation.

$$\boxed{RI + \int \frac{I(t)}{C} dt = E}.$$

Now here given equation can be written as $RI + \int \dfrac{I(t)}{C} dt = E_0 \sin\omega t$.

Differentiating both sides w. r. t. t, we get

$$R\frac{dI}{dt} + \frac{I}{C} = \omega E_0 \cos\omega t \Rightarrow \frac{dI}{dt} + \frac{I}{RC} = \frac{\omega E_0}{R}\cos\omega t. \qquad \text{(i)}$$

which is Leibnitz's linear equation.

Its I.F. $= e^{\int \frac{1}{RC}dt} = e^{\frac{t}{RC}}$.

Solution: $\therefore$The solution of (i) is

$$I.e^{t/RC} = \int \frac{\omega E_0}{R}\cos\omega t . e^{t/RC} dt + k = \frac{\omega E_0}{R}\int e^{t/RC}\cos\omega t dt + k$$

$$= \frac{\omega E_0}{R} . \frac{e^{t/RC}}{\sqrt{\left(\frac{1}{RC}\right)^2 + \omega^2}}\cos\left(\omega t - \tan^{-1}\frac{\omega}{\frac{1}{RC}}\right) + k$$

$$\left[\because \int e^{ax}\cos bx dx = \frac{e^{ax}}{\sqrt{a^2+b^2}}\cos\left(bx - \tan^{-1}\frac{b}{a}\right)\right]$$

$$= \frac{\omega C E_0}{\sqrt{1+R^2C^2\omega^2}} e^{t/RC}\cos(\omega t - \phi) + k, \quad \text{where } \tan\phi = RC\omega$$

$\Rightarrow I = \dfrac{\omega C E_0}{\sqrt{1+R^2C^2\omega^2}}\cos(\omega t - \varphi) + ke^{-t/RC}$, which gives the current at any time t.

Q.No.5.: A capacitor C = 0.01 F in series with a resistor R = 20 Ohms is charged from a battery E_0 = 10V. Assuming that initially the capacitor is completely uncharged. Determine the charge Q(t), voltage V(t) on the capacitor and the current I(t) in the circuit.

Sol.: **Given:** Capacitor C = 0.01 F, Resistor R = 20 Ohms, E_0 = 10V.

Mathematical model:

The current I(t) in the RC-circuit is determined by the first order linear differential equation.

$$\boxed{RI + \int \frac{I}{C}dt = E}.$$

But $Q = \int I dt$. i.e. $I = \dfrac{dQ}{dt}$.

Thus, the above first order linear differential equation becomes

$$R\frac{dQ}{dt} + \frac{Q}{C} = E \quad \Rightarrow 20\frac{dQ}{dt} + \frac{Q}{0.01} = 10$$

$$\Rightarrow \frac{dQ}{dt} + 5Q = 0.5$$

which is Leibnitz's linear differential equation.

Solution: The general solution is $Q.e^{5t} = 0.5\int e^{5t}dt = 0.5.\dfrac{e^{5t}}{5} + c$

$\Rightarrow Q = 0.1 + ce^{-5t}$.

Initial value problem:

At $t = 0,\ Q = 0$ so $0 = 0.1 + c \Rightarrow c = -0.1$.

Thus the charge $Q(t) = 0.1\left(1 - e^{-5t}\right)$.

Voltage $V(t) = \dfrac{Q(t)}{C} = \dfrac{0.1\left(1-e^{-5t}\right)}{0.01} = 10\left(1 - e^{-t}\right)$.

Current $I(t) = \dfrac{dQ}{dt} = 0.5e^{-5t}$.

Home Assignments

Q.No.1.: A generator having emf 100volts is connected bin series with a 10 ohm resistor and an inductor of 2 H. If the switch is closed at a time t = 0, determine the current at time $t > 0$.

Ans.: $I = 10(1 - e^{-5t})$.

Q.No.2.: Solve the above example when the generator is replaced by on having an emf of $20\cos 5t$ volts.

Ans.: $I = \cos 5t + \sin 5t - e^{-5t}$.

Q.No.3.: A decaying emf $E = 200e^{-5t}$ is connected in series with a 20 ohm resistor and 0.01 farad capacitor. Find the charge and current at any time assuming Q = 0 at t = 0. Show that the charge reaches a maximum, calculate it and find the time when it reached.

Ans.: $Q(t) = 10te^{-5t}$, $I(t) = 10e^{-5t} - 50te^{-5t}$.

Maximum of $Q = 10.\frac{1}{5}.e^{-1} = \frac{2}{c} \sim 0.74$ coulombs when $t = \frac{1}{5}$ second

Q.No.4.: Find the current I(t) in the RL-circuit with R = 10 Ohms, L = 100 H, E = 40 Volts. when $0 \le t \le 100$ and $E = 0$ when $t > 100$ and (I) $= 4$.

Ans.: I = 4, when $0 \le t \le 100$, $I = 4.e^{10}e^{-t/10}$ when $t > 100$.

Q.No.5.: Find $I(t)$ in an RL-circuit with E = 10 V, R = 5 Ohm, $L = (10 - t)$ H, when $0 \le t \le 10$ sec. and L = 0 when $t > 10$ sec and I(0) = 0.

Ans.: $I = 2 - 2(1 - 0.1t)^5$ when $0 \le t \le 10$, when t > 10, $I = 2$.

Q.No.6.: Solve the (RL-circuit) equation $L\frac{dI}{dt} + RI = E(t)$ when (a) $E(t) = E_0$ and the initial current is I_0. (b). Solve the problem when L = 3 H, R = 15 ohm, emf is the 60 cycle sine wave of amplitude 110 volts and I(t = 0) = 0.

Ans.: (a). $f(t) = \frac{E_0}{R}\left(1 - e^{-Rt/L}\right) + I_0 e^{-Rt/L}$

(b). $I(t)=\frac{22}{3}.\frac{\sin 120\pi t-24\pi\cos 120\pi t+24\pi e^{-5t}}{1+576\pi^2}$

Q.No.7.: Solve the RC-circuit) equation $R\frac{dQ}{dt}+\frac{Q}{c}=E$ with R = 10 Ohm, $c=10^{-3}$ farad and $E(t)=100\sin 120\pi t$ assuming Q(t = 0) = 0. Find I(t) given I(t = 0) =5.

Ans.: $Q(t)=\frac{\sin(120\pi t-\phi)}{2\sqrt{(25+36\pi^2)}}+\frac{3\pi e^{-100t}}{25+36\pi^2}$.

Q.No.8.: Determine the current at time t > 0 in a series RL-circuit having an emf given by E(t) = $100\sin 40tV$, a resistor of 10Ω and an inductor of 0.5 H given that initial current is zero. Find the period and the phase angle.

Ans.: $I(t)=2(\sin 40t-2\cos 40t)+4e^{-20t,}$.

Period = $\frac{\pi}{20}$, Phase angle = $\phi\approx -1.11$

Q.No.9.: Find the current in RC-circuit with R = 10, c = 0.1, $E(t)=110\sin 314t$, I(0) = 0

Ans.: $I(t)=0.035(\cos 314t+314\sin t-e^{-t})$.

Q.No.10.: Determine the charge and current at time t > 0 in a RC-circuit with R = 10, $c=2\times10^{-4}$, E = 100 V given that Q(t = 0) = 0.

Ans.: $Q(t)=\frac{(1-e^{-500t})}{50}$, $I(t)=10e^{-500t}$.

Q.No.11.: When a switch is closed in a circuit containing a battery E, a resistance R and an inductance L, the current i builds up at a rate given by $L\frac{di}{dt}+Ri=E$. Find i as a function of t. How long will it be, before the current has reached one-half its final value if E = 6 volts, R = 100 ohms and L = 0.1 henry ?

Ans.:

Q.No.12.: When a resistance R ohms is connected in series with an inductance L henries with an e. m. f. of E volts, the current i amperes at time t is given by

$L\frac{di}{dt}+Ri=E$. If $E=10\sin t$ volts and i = 0 when t = 0, find I as a function of t.

Ans.:

Q.No.13.: A resistance R in series with inductance L is shunted by an equal resistance R with capacity C. An alternating e. m. f. $E\sin pt$ produces currents i_1 and i_2 in two branches. If initially there is no current, determine i_1 and i_2 from the equations $L\frac{di_1}{dt}+Ri_1=E\sin pt$ and $\frac{i_2}{C}+R\frac{di_2}{dt}=pE\cos pt$. Verify that if $R^2C=L$, the total current i_1+i_2 will be $\frac{(E\sin pt)}{R}$.

Ans.:

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