1^{st} Topic

Partial Differential Equations

Formation of partial differential equations

Prepared by:
Prof. Sunil
Department of Mathematics & Scientific Computing
NIT Hamirpur (HP)

PARTIAL DIFFERENTIAL EQUATIONS

INTRODUCTION:

Real world problems in general, involve functions of several (independent) variables giving rise to partial differential equations more frequently than ordinary differential equations. Thus, most problems in engineering and science reproduce with first and second order linear non-homogeneous partial differential equations. Thus, before discussing the methods of obtaining solutions, we will discuss some definitions, how we can formulate partial differential equations and other related concepts.

PARTIAL DIFFERENTIAL EQUATION

A differential equation, which involves partial derivatives, is called a **partial differential equation**.

For example, $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z$, (i) $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$, (ii) and $\frac{\partial^2 u}{\partial x \partial y} = \left(\frac{\partial u}{\partial z}\right)^3$ (iii)

are the partial differential equations.

Order of the differential equation:

The order of the partial differential equation is the **order** of the **highest** ordered derivative appearing in the partial differential equation.

Degree of a partial differential equation:

The degree of a partial differential equation is the **degree** of the **highest order** partial derivative occurring in the equation.

Thus, equation (i) is of first order, equations (ii) and (iii) are of second order. The degree of all above equations is one.

STANDARD NOTATION

If z is function of two independent variables x and y, then we shall use the following standard notation for the partial derivatives of z:

$$\frac{\partial z}{\partial x}=p,\ \frac{\partial z}{\partial y}=q,\ \frac{\partial^2 z}{\partial x^2}=r,\ \frac{\partial^2 z}{\partial x\partial y}=s,\ \frac{\partial^2 z}{\partial y^2}=t.$$

FORMATION OF PARTIAL DIFFERENTIAL EQUATION:

Unlike the case of ordinary differential equations which arise from the elimination of arbitrary constants; the partial differential equation can be formed either by the **elimination of arbitrary constants** or by the **elimination of arbitrary functions** from a relation involving three or more variables.

By elimination of arbitrary constants:

Let $f(x,y,z,a,b)=0$, (i)

be an equation involving two arbitrary constants a and b.

Differentiating this equation partially w.r.t. x and y, we get

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial z}\frac{\partial z}{\partial x}=0, \tag{ii}$$

$$\frac{\partial f}{\partial y}+\frac{\partial f}{\partial z}\frac{\partial z}{\partial y}=0. \tag{iii}$$

By eliminating a, b from (i), (ii) and (iii), we get an equation of the form

$$f(x,y,z,p,q)=0, \tag{iv}$$

which is a partial differential equation of the first order.

By elimination of arbitrary functions:

(a) One arbitrary function:

Consider $z=f(u)$, (i)

where f(u) is an arbitrary function of u and u is a given (known) function of x, y, z.

i.e. $u=u(x,y,z)$.

Differentiating (i) partially w.r.t. x and y by chain rule, we get

$$\frac{\partial z}{\partial x}=\frac{\partial f}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial f}{\partial u}\frac{\partial u}{\partial z}\frac{\partial z}{\partial x}, \qquad \text{(ii)}$$

$$\frac{\partial z}{\partial y}=\frac{\partial f}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial f}{\partial u}\frac{\partial u}{\partial z}\frac{\partial z}{\partial y}. \qquad \text{(iii)}$$

By eliminating the arbitrary function f from (i), (ii) and (iii), we get an equation a partial differential equation of the first order.

(b) Two arbitrary functions:

Differentiating twice or more, the elimination process results in a partial differential equation of second or higher order.

Always remember:

- If the number of arbitrary constants to be eliminated is **equal** to the number of independent variables, then the **partial differential equations** that arise are of the **first order**.
- If the number of arbitrary constants to be eliminated is **more than** the number of independent variables, then the **partial differential equations** obtained are of **second or higher order**.
- If the partial differential equation is obtained by elimination of arbitrary functions, then the order of the partial differential equation is, in general, equal to the number of arbitrary functions eliminated.
- When n is number of arbitrary functions, one may get several partial differential equations. But generally the one with the least order is chosen.

The method is best illustrated by the following problems:

Q.No.1.: Form partial differential equations from the following equations by **eliminating** the **arbitrary constants**:

(i) $z=ax+by+ab$, **(ii)** $z=ax+a^2y^2+b$,

(iii) $z=(x^2+a)(y^2+b)$, **(iv)** $2z=\frac{x^2}{a^2}+\frac{y^2}{b^2}$, **(v)** $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

Sol.: (i). Given $z=ax+by+ab$.

Here the number of arbitrary constants is **equal** to the number of independent variables.

Differentiating z partially w. r. t. x and y, we get

$$p = \frac{\partial z}{\partial x} = a, \; q = \frac{\partial z}{\partial y} = b.$$

Substituting for a and b in the given equation, we get

$$z = px + qy + pq,$$

which is the partial differential equation of the first order.

(ii). Given $z = ax + a^2y^2 + b$**.**

Here the number of arbitrary constants is **equal** to the number of independent variables.

Differentiating z partially w. r. t. x and y, we get

$$p = \frac{\partial z}{\partial x} = a, \; q = \frac{\partial z}{\partial y} = 2a^2y.$$

Eliminating a between these result, we get

$$q = 2p^2y,$$

which is the partial differential equation of the first order.

(iii). Given $z = (x^2 + a)(y^2 + b)$**.**

Here the number of arbitrary constants is **equal** to the number of independent variables.

Differentiating z partially w. r. t. x and y, we get

$$p = \frac{\partial z}{\partial x} = 2x(y^2 + b), \qquad \text{(i)} \qquad q = \frac{\partial z}{\partial y} = 2y(x^2 + a). \qquad \text{(ii)}$$

Multiplying (i) and (ii), we get

$$pq = 4xy(x^2 + a)(y^2 + b) \Rightarrow pq = 4xyz,$$

which is the partial differential equation of the first order.

(iv). Given $2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$. (i)

Here the number of arbitrary constants is **equal** to the number of independent variables.

Differentiating partially w. r. t. x and y, we get

$$2\frac{\partial z}{\partial x} = \frac{2x}{a^2} \Rightarrow \frac{1}{a^2} = \frac{1}{x}\frac{\partial z}{\partial x} = \frac{p}{x},$$

and $$2\frac{\partial z}{\partial y} = \frac{2y}{b^2} \Rightarrow \frac{1}{b^2} = \frac{1}{y}\frac{\partial z}{\partial y} = \frac{q}{y}.$$

Substituting these values in (i), we get $2z = xp + yq$,

which is the desired partial differential equation of the first order.

(v). Given $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Here the number of arbitrary constants (a, b, c) is greater than the number of independent variables (x, y).

Differentiating partially w. r. t. x and y, we get

$$\frac{2x}{a^2} + \frac{2z}{c^2}.\frac{\partial z}{\partial x} = 0 \Rightarrow c^2x + a^2z\frac{\partial z}{\partial x} = 0, \tag{i}$$

$$\frac{2y}{b^2} + \frac{2z}{c^2}.\frac{\partial z}{\partial y} = 0 \Rightarrow c^2y + b^2z\frac{\partial z}{\partial y} = 0. \tag{ii}$$

Again differentiating (i) partially w. r. t. x, we get

$$c^2 + a^2\left(\frac{\partial z}{\partial x}\right)^2 + a^2z\frac{\partial^2 z}{\partial x^2} = 0 \Rightarrow \frac{c^2}{a^2} + \left(\frac{\partial z}{\partial x}\right)^2 + z\frac{\partial^2 z}{\partial x^2} = 0.$$

Substituting $\frac{c^2}{a^2} = -\frac{z}{x}\frac{\partial z}{\partial x}$ (from (i)), we have

$$-\frac{z}{x}\frac{\partial z}{\partial x} + \left(\frac{\partial z}{\partial x}\right)^2 + \frac{\partial^2 z}{\partial x^2} = 0 \Rightarrow xz\frac{\partial^2 z}{\partial x^2} + x\left(\frac{\partial z}{\partial x}\right)^2 - z\frac{\partial z}{\partial x} = 0,$$

which is the partial differential equation of the second order.

Always remember:

If the partial differential equation is obtained by elimination of arbitrary functions, then the **order** of the partial differential equation is, in general, equal to the number of arbitrary functions eliminated.

Q.No.2.: Form the partial differential equations (by **eliminating** the arbitrary **functions**)

(i) $z = f\left(x^2 - y^2\right)$, **(ii)** $f\left(x^2 + y^2, z - xy\right) = 0$, **(iii)** $z = f(x + at) + g(x - at)$,

(iv) $z = f(x + it) + g(x - it)$, **(v)** $z = xf_1(x + t) + f_2(x + t)$.

Sol.: (i). Given $z = f\left(x^2 - y^2\right)$.

Here we have one arbitrary function.

Differentiating z partially w. r. t. x and y, we get

$$p=\frac{\partial z}{\partial x}=f'(x^2-y^2)2x, \quad q=\frac{\partial z}{\partial y}=f'(x^2-y^2)(-2y).$$

Division gives $\frac{p}{q}=-\frac{x}{y}\Rightarrow yp+xq=0$,

which is the required partial differential equations of first order.

(ii). Given $f(x^2+y^2,\ z-xy)=0$.

Here we have one arbitrary function.

Let $x^2+y^2=u$ and $z-xy=v$, so that f(u, v) = 0.

Differentiating partially w. r. t. x and y, we get

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}p\right)+\frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}p\right)=0\Rightarrow\frac{\partial f}{\partial u}(2x)+\frac{\partial f}{\partial v}(-y+p)=0 \qquad \text{(i)}$$

and $$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}q\right)+\frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}q\right)=0\Rightarrow\frac{\partial f}{\partial u}(2y)+\frac{\partial f}{\partial v}(-x+q)=0. \qquad \text{(ii)}$$

Eliminating $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ from (i) and (ii), we get

$$\begin{vmatrix}2x & -y+p\\ 2y & -x+q\end{vmatrix}=0\Rightarrow xq-yp=x^2-y^2.$$

which is the desired partial differential equation of the first order.

(iii). Given $z=f(x+at)+g(x-at)$. (i)

Here we have two arbitrary functions.

Differentiating z partially w. r. t. x and t, we get

$$\frac{\partial z}{\partial x}=f'(x+at)+g'(x-at),\quad \frac{\partial^2 z}{\partial x^2}=f''(x+at)+g''(x-at) \qquad \text{(ii)}$$

$$\frac{\partial z}{\partial t}=af'(x+at)-ag'(x-at),\quad \frac{\partial^2 z}{\partial t^2}=a^2f''(x+at)+a^2g''(x-at)=a^2\frac{\partial^2 z}{\partial x^2}. \qquad \text{(iii)}$$

From (ii) and (iii), we get $\frac{\partial^2 z}{\partial t^2}=a^2\frac{\partial^2 z}{\partial x^2}$,

which is the required partial differential equation of the second order.

(iv). Given $z=f(x+it)+g(x-it)$.

Here we have two arbitrary functions.

Differentiating z twice partially w. r. t. x and t, we get

$$\frac{\partial z}{\partial x}=f'(x+it)+g'(x-it)\Rightarrow\frac{\partial^2 z}{\partial x^2}=f''(x+it)+g''(x-it) \qquad \text{(i)}$$

$$\frac{\partial z}{\partial t}=if'(x+it)-ig'(x-it)\Rightarrow\frac{\partial^2 z}{\partial t^2}=i^2f''(x+it)+i^2g''(x-it)$$

$$\Rightarrow\frac{\partial^2 z}{\partial t^2}=-f''(x+it)-g''(x-it). \qquad \text{(ii)}$$

Adding (i) and (ii), we get

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial t^2}=0,$$

which is the desired partial differential equation of the second order.

(v). Given $z=xf_1(x+t)+f_2(x+t)$**.**

Here we have two arbitrary functions.

Differentiating z twice partially w. r. t. x and t, we get

$$\frac{\partial z}{\partial x}=f_1(x+t)+xf_1'(x+t)+f_2'(x+t)$$

$$\frac{\partial^2 z}{\partial x^2}=f_1'(x+t)+f_1'(x+t)+xf_1''(x+t)+f_2''(x+t)$$

$$\frac{\partial^2 z}{\partial x^2}=2f_1'(x+t)+xf_1''(x+t)+f_2''(x+t) \qquad \text{(i)}$$

$$\frac{\partial z}{\partial t}=xf_1'(x+t)+f_2'(x+t)$$

$$\frac{\partial^2 z}{\partial t^2}=xf_1''(x+t)+f_2''(x+t). \qquad \text{(ii)}$$

Subtracting (ii) from (i), we get

$$\frac{\partial^2 z}{\partial x^2}-\frac{\partial^2 z}{\partial t^2}=2f_1'(x+t)\ . \qquad \text{(iii)}$$

Also $\dfrac{\partial^2 z}{\partial x\partial t}=f_1'(x+t)+xf_1''(x+t)+f_2''(x+t)=\dfrac{1}{2}\left[\dfrac{\partial^2 z}{\partial x^2}-\dfrac{\partial^2 z}{\partial t^2}\right]+\dfrac{\partial^2 z}{\partial t^2}$

$$\Rightarrow \frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial t} + \frac{\partial^2 z}{\partial t^2} = 0,$$

which is the desired partial differential equation of the second order.

Solve some more problems

Q.No.3.: Form the partial differential equations (by eliminating the arbitrary functions)

$$z = y^2 + 2f\left(\frac{1}{x} + \log y\right).$$

Sol.: Given $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$.

Here we have one arbitrary function.

Differentiating z partially w. r. t. x and y, we get

$$p = \frac{\partial z}{\partial x} = 2f'\left(\frac{1}{x} + \log y\right).\left(-\frac{1}{x^2}\right) \Rightarrow -px^2 = 2f'\left(\frac{1}{x} + \log y\right) \quad \text{(i)}$$

$$q = \frac{\partial z}{\partial y} = 2y + 2f'\left(\frac{1}{x} + \log y\right).\left(\frac{1}{y}\right) \Rightarrow qy - 2y^2 = 2f'\left(\frac{1}{x} + \log y\right). \quad \text{(ii)}$$

From (i) and (ii), we get

$$-px^2 = qy - 2y^2 \Rightarrow x^2 p + yq = 2y^2,$$

which is the required partial differential equation of the first order.

Q.No.4.: Form the partial differential equations (by eliminating the arbitrary functions)

$$f\left(x + y + z, x^2 + y^2 + z^2\right) = 0.$$

Sol.: Given $f\left(x + y + z, x^2 + y^2 + z^2\right) = 0$.

Here we have one arbitrary function.

Let $x + y + z = u$ and $x^2 + y^2 + z^2 = v$, then $f(u, v) = 0$

Differentiating partially w. r. t. x and y, we get

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}p\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}p\right) = 0 \Rightarrow \frac{\partial f}{\partial u}(1 + p) + \frac{\partial f}{\partial v}(2x + 2zp) = 0 \quad \text{(i)}$$

and $$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}q\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}q\right) = 0 \Rightarrow \frac{\partial f}{\partial u}(1 + q) + \frac{\partial f}{\partial v}(2y + 2zq) = 0. \quad \text{(ii)}$$

Eliminating $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ from (i) and (ii), we get

$(1+p)(2y+2zq) = (1+q)(2x+2zp) \Rightarrow (y-z)p + (z-x)q = x-y$,

which is the required partial differential equation of the first order.

Q.No.5.: Find the differential equation of all planes which are at a constant distance 'a' from the origin.

Sol.: Equation of the plane in normal form is $\ell x + my + nz = a$. (i)

where ℓ, m, n are the d. c.'s of the normal from the origin to the plane.

Then, $\ell^2 + m^2 + n^2 = 1 \Rightarrow n = \sqrt{\left(1-\ell^2+m^2\right)}$.

$\therefore$ (i) becomes $\ell x + my + \sqrt{\left(1-\ell^2-m^2\right)}z = a$. (ii)

Differentiating partially w. r. t. x , we get

$\ell + \sqrt{\left(1-\ell^2-m^2\right)}.p = 0$. (iii)

Differentiating partially w. r. t. y, we get

$m + \sqrt{\left(1-\ell^2-m^2\right)}.q = 0$. (iv)

Now we have to eliminate ℓ, m from (ii), (iii) and (iv), we get

From (iii), $\ell = -\sqrt{\left(1-\ell^2-m^2\right)}.p$ and $m = -\sqrt{\left(1-\ell^2-m^2\right)}.q$.

Squaring and adding, we get $\ell^2 + m^2 = \left(1-\ell^2-m^2\right)\left(p^2+q^2\right)$.

$$\Rightarrow \left(\ell^2+m^2\right)\left(1+p^2+q^2\right) = p^2+q^2 \Rightarrow 1-\ell^2-m^2 = 1-\frac{p^2+q^2}{1+p^2+q^2} = \frac{1}{1+p^2+q^2}.$$

Also $\ell = -\frac{p}{\sqrt{\left(1+p^2+q^2\right)}}$ and $m = -\frac{q}{\sqrt{\left(1+p^2+q^2\right)}}$.

Substitute the value of ℓ, m and $1-\ell^2-m^2$ in (ii), we obtain

$$\frac{-px}{\sqrt{\left(1+p^2+q^2\right)}} - \frac{qy}{\sqrt{\left(1+p^2+q^2\right)}} + \frac{1}{\sqrt{\left(1+p^2+q^2\right)}}z = a$$

$$\Rightarrow z = px + qy + a\sqrt{\left(1+p^2+q^2\right)},$$

which is the required partial differential equation.

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Home Assignments

Form the partial differential equations (by eliminating the arbitrary constants)

Q.No.1.: Form the partial differential equation (by eliminating the arbitrary constants) from: $z = ax + by + a^2 + b^2$.

Ans.: $z = px + qy + p^2 + q^2$.

Q.No.2.: Form the partial differential equation (by eliminating the arbitrary constants) from: $(x-a)^2 + (y-b)^2 + z^2 = c^2$.

Ans.: $z^2\left(p^2 + q^2 + 1\right) = c^2$.

Q.No.3.: Form the partial differential equation (by eliminating the arbitrary constants) from: $z = xy + y\sqrt{x^2 - a^2} + b$.

Ans.: $px + qy = pq$.

Q.No.4.: Form the partial differential equation (by eliminating the arbitrary constants) from: $z = ax^2 + bxy + cy^2$.

Ans.: $x^2 r + 2xys + y^2 t = 2z$.

Q.No.5.: Form the partial differential equation (by eliminating the arbitrary constants) from: $z = axe^y + \frac{1}{2}a^2 e^{2y} + b$.

Ans.: $q = px + p^2$.

Q.No.6.: Form the partial differential equation (by eliminating the arbitrary constants) from: $z = a(x+y) + b(x-y) + abt + c$.

Ans.:

Q.No.7.: Form the partial differential equation (by eliminating the arbitrary constants) from: $z = \left(x - a^2\right) + (y-b)^2$.

Ans.: $p^2 + q^2 = 4z$.

Q.No.8.: Form the partial differential equation (by eliminating the arbitrary constants)

from:. $z = a\log\left\{\frac{b(y-1)}{1+x}\right\}$.

Ans.: $p + q = px + qy$.

Form the partial differential equations (by eliminating the arbitrary functions)

Q.No.9.: Form the partial differential equation (by eliminating the arbitrary functions)

from: $xyz = \phi(x+y+z)$.

Ans.: $x(y-z)p + y(z-x)q = z(x-y)$.

Q.No.10.: Form the partial differential equation (by eliminating the arbitrary functions)

from: $z = f(x+4t) + g(x-4t)$.

Ans.: $16\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial t^2} = 0$

Q.No.11.: Form the partial differential equation (by eliminating the arbitrary functions)

from: $z = f\left(\frac{xy}{z}\right)$.

Ans.: $px = qy$

Q.No.12.: Form the partial differential equation (by eliminating the arbitrary functions)

from: $z = yf(x) + xg(y)$.

Ans.: $xys = px + qy - z$

Q.No.13.: Form the partial differential equation (by eliminating the arbitrary functions)

from: $z = f(x) + e^y g(x)$.

Ans.: $\frac{\partial^2 z}{\partial y^2} = \frac{\partial z}{\partial y}$

Q.No.14.: Form the partial differential equation (by eliminating the arbitrary functions)

from: $xyz = \phi(x+y+z)$.

Ans.: $x(y-z)p + y(z-x)q = z(x-y)$

Q.No.15.: Form the partial differential equation (by eliminating the arbitrary functions) from: $z = f_1(x)f_2(y)$.

Ans.: $z\dfrac{\partial^2 z}{\partial x\partial y} = \dfrac{\partial z}{\partial x}.\dfrac{\partial z}{\partial y}$.

Q.No.16.: Form the partial differential equation (by eliminating the arbitrary functions) from: $z = f_1(y+2x) + f_2(y-3x)$.

Ans.: $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial x\partial y} - 6\dfrac{\partial^2 z}{\partial y^2} = 0$

Q.No.17.: Form the partial differential equation (by eliminating the arbitrary functions) from: $v = \dfrac{1}{r}[f(r-at) + F(r+at)]$.

Ans.: $\dfrac{\partial^2 v}{\partial t^2} = \dfrac{a^2}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial v}{\partial r}\right)$.

Q.No.18.: Form the partial differential equation (by eliminating the arbitrary functions) from: $F(xy+z^2, x+y+z) = 0$.

Ans.: $p(x-2z) + q(2z-y) = y - x$.

Q.No.19.: Form the partial differential equation (by eliminating the arbitrary functions) from: $z = x^2 f(y) + y^2 g(x)$.

Ans.: $xyr = 2(px + qy - 2z)$

Q.No.20.: Form the partial differential equation (by eliminating the arbitrary functions) from: $z = e^{my}\phi(x-y)$.

Ans.: $p + q = mz$.

Q.No.21.: Form the partial differential equation (by eliminating the arbitrary functions) from: $z = f\left(\dfrac{y}{x}\right)$.

Ans.: $px + qy = 0$.

Some miscellaneous problems

Q.No.22.: Find the differential equation of all spheres of radius 3-units having their centers in the xy-plane.

Ans.: $z^2\left(b^2+q^2+1\right)=9$.

Q.No.23.: Find the differential equation of all spheres whose centers lies on z-axes.

Ans.: $py-qx=0$.

Q.No.24.: If $u=f\left(x^2+2yz,\, y^2+2zx\right)$,

prove that $\left(y^2-zx\right)\dfrac{\partial u}{\partial x}+\left(x^2-yz\right)\dfrac{\partial u}{\partial y}+\left(z^2-xy\right)\dfrac{\partial u}{\partial z}=0$

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Solution of partial differential equation

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Prepared by:
Prof. Sunil
Department of Mathematics & Scientific Computing
NIT Hamirpur (HP)

SOLUTIONS OF A PARTIAL-DIFFERENTIAL EQUATION:

It is clear that a partial differential equation can be obtained by elimination of arbitrary constants or by the elimination of arbitrary functions.

Partial differential equation of first order

The general form of a first order partial differential equation is

$$f(x, y, z, p, q) = 0, \qquad \text{(i)}$$

where x, y are the two independent variables, z is the dependent variable and $\frac{\partial z}{\partial x} = p$, $\frac{\partial z}{\partial y} = q$.

Complete solution:

Any function f(x, y, z, a, b) = 0, (ii)

involving two arbitrary constants a, b and satisfying the partial differential equation (i) is known as **complete solution** or **complete integral** or **primitive.**

General solution or general integral:

Any arbitrary function F of specific (given) functions u, v

$$F(u, v) = 0, \qquad \text{(iii)}$$

satisfying partial differential equation (i) is known as **general solution or general integral.**

Particular solution or particular integral:

A solution obtained from the complete integral by assigning particular values to the arbitrary constants is called a **particular solution or particular integral.**

Singular solution or singular integral:

The envelope of the family of surfaces (ii), with parameters a and b, if it exist, is called a **singular solution or singular integral**.

Remarks: The *singular integral* differs from the *particular integral* in that it is not obtained from the complete integral by giving particular values to the constants.

A solution of a partial differential equation in a region R is a function of the independent variables, whose partial derivatives satisfy the partial differential equation at every point in R. As such, a partial differential equation may have a large number of entirely different solutions.

For example, $u = x^2 - y^2$, $u = \log\left(x^2 - y^2\right)$, $u = \sin kx \cos ky$ are solutions of the Laplace equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$. The unique solution of a partial differential equation corresponding to a physical problem must satisfy certain other conditions at the boundary of the region R. These are known as the **boundary conditions**.

If these conditions are given for the time t = 0, they are known as the **initial conditions**.

Theorem: Show that if u_1 and u_2 are two solutions of the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = A\frac{\partial^2 u}{\partial t^2} + B\frac{\partial u}{\partial t}$$, then $c_1 u_1 + c_2 u_2$ is also a solution.

Proof: Since u_1 and u_2 are two solutions of the given equation, we have

$$\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2} + \frac{\partial^2 u_1}{\partial z^2} = A\frac{\partial^2 u_1}{\partial t^2} + B\frac{\partial u_1}{\partial t}, \qquad \text{(i)}$$

and $$\frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2} + \frac{\partial^2 u_2}{\partial z^2} = A\frac{\partial^2 u_2}{\partial t^2} + B\frac{\partial u_2}{\partial t}. \qquad \text{(ii)}$$

Now $\frac{\partial^2}{\partial x^2}(c_1u_1 + c_2u_2) + \frac{\partial^2}{\partial y^2}(c_1u_1 + c_2u_2) + \frac{\partial^2}{\partial z^2}(c_1u_1 + c_2u_2)$

$$= c_1\left(\frac{\partial^2 u_1}{\partial x^2}+\frac{\partial^2 u_1}{\partial y^2}+\frac{\partial^2 u_1}{\partial z^2}\right)+c_2\left(\frac{\partial^2 u_2}{\partial x^2}+\frac{\partial^2 u_2}{\partial y^2}+\frac{\partial^2 u_2}{\partial z^2}\right)$$

$$= c_1\left(A\frac{\partial^2 u_1}{\partial t^2}+B\frac{\partial u_1}{\partial t}\right)+c_2\left(A\frac{\partial^2 u_2}{\partial t^2}+B\frac{\partial u_2}{\partial t}\right) \qquad \text{[using (i) and (ii)]}$$

$$= A\frac{\partial^2}{\partial t^2}(c_1u_1+c_2u_2)+B\frac{\partial}{\partial t}(c_1u_1+c_2u_2).$$

$\Rightarrow c_1u_1+c_2u_2$ is also a solution of the given equation.

Generalization: If $u_1, u_2, u_3 \ldots\ldots\ldots u_n$ are n independent solutions,

then $c_1u_1+c_2u_2+c_3u_3+\ldots\ldots\ldots+c_nu_n$ is also a solution.

Problem on verifications of a solution

Q.No.1.: Verify that $e^{-n^2t}\sin nx$ is a solution of the heat equation $\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}$.

Hence, show that $\sum_{n=1}^{N} c_n e^{-n^2t}\sin nx$, where $c_1, c_2, \ldots\ldots\ldots\ldots, c_N$ are arbitrary constants, is a solution of this equation satisfying the boundary conditions $u(0,t)=0$ and $u(\pi,t)=0$.

Sol.: 1st Part: Show that $e^{-n^2t}\sin nx$ is a solution of the heat equation $\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}$.

Here $u=e^{-n^2t}\sin nx$.

Then $\frac{\partial u}{\partial t}=-n^2e^{-n^2t}\sin nx$, $\frac{\partial u}{\partial x}=ne^{-n^2t}\cos nx$, $\frac{\partial^2 u}{\partial x^2}=-n^2e^{-n^2t}\sin nx$.

So that $\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}$.

$\therefore u=e^{-n^2t}\sin nx$ is a solution of the given equation.

2nd Part: Show that $\sum_{n=1}^{N} c_n e^{-n^2t}\sin nx$ is also a solution.

For n = 1, 2, …………, N, we get N different solutions.

Their linear combination

$u = c_1 e^{-1^2 t} \sin 1x + c_2 e^{-2^2 t} \sin 2x + ... + c_N e^{-N^2 t} \sin Nx$ is also a solution.

$\Rightarrow u(x,t) = \sum_{n=1}^{N} c_n e^{-n^2 t} \sin nx$ is also a solution.

Clearly, $u(0,t) = 0$ and $u(\pi,t) = 0$, since $\sin n\pi = 0$, where n is an integer.

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Home Assignments

Q.No.1.: Verify that $z = f(x^2 + y^2)$ is a solution of $y\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} = 0$.

Sol.:

Q.No.2.: Verify that $u = \cos kx \sinh ky$ is a solution of the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Sol.:

Q.No.3.: Verify that $e^{-k^2 t} \sin\left(\frac{kx}{c}\right)$ is a solution of the heat equation $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$.

Hence, show that $\sum_{k=1}^{n} A_k e^{-k^2 t} \sin\left(\frac{kx}{c}\right)$, where A_1, A_2, ………. are arbitrary constants, is a solution satisfying the boundary conditions $u(0,t) = u(c\pi,t) = 0$.

Sol.:

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EQUATIONS SOLVABLE BY DIRECT INTEGRATION:

Those equations, which contain **only one partial derivative**, can be solved by **direct integration**. In place of usual constants of integration, we must, use arbitrary functions of the variable kept constant.

Now let us solve some partial differential equations, which can be solved by direct integration:

Q.No.1.: Solve the following partial differential equation:

$$\frac{\partial^3 z}{\partial x^2 \partial y} + 18xy^2 + \sin(2x - y) = 0.$$

Sol.: Given partial differential equation is $\frac{\partial^3 z}{\partial x^2 \partial y} + 18xy^2 + \sin(2x - y) = 0$.

Integrate twice w.r.t. x (keeping y fixed), we get

$$\frac{\partial^2 z}{\partial x \partial y} + 9x^2y^2 - \frac{1}{2}\cos(2x - y) = f(y),$$

$$\frac{\partial z}{\partial y} + 3x^3y^2 - \frac{1}{4}\sin(2x - y) = xf(y) + g(y).$$

Now integrate w.r.t. y (keeping x fixed), we get

$$z + x^3y^3 - \frac{1}{4}\cos(2x - y) = x\int f(y)dy + \int g(y)dy + w(x).$$

The result may simplified by writing $\int f(y)dy = u(y)$ and $\int g(y)dy = v(y)$.

Thus, $z = \frac{1}{4}\cos(2x - y) - x^3y^3 + xu(y) + v(y) + w(x)$,

where u, v, w are arbitrary functions.

This is the required solution.

Q.No.2.: Solve the following partial differential equation: $\frac{\partial^2 z}{\partial x^2} + z = 0$,

given that when $x = 0$, $z = e^y$ and $\frac{\partial z}{\partial x} = 1$.

Sol.: If z were a function of x alone, the solution would have been $z = A\sin x + B\cos x$, where A and B are arbitrary constants.

But here z is a function of x and y, therefore, A and B can be arbitrary functions of y, the independent variable kept constant.

Hence, the solution of the given equation is $z = f(y)\sin x + \phi(y)\cos x$.

$\frac{\partial z}{\partial x} = f(y)\cos x - \phi(y)\sin x$.

When $x = 0$, $z = e^y$. $\therefore e^y = \phi(y)$.

When $x = 0$, $\frac{\partial z}{\partial x} = 1$. $\therefore 1 = f(y)$.

Hence, the desired solution is $z = \sin x + e^y \cos x$.

Q.No.3.: Solve the following partial differential equation: $\frac{\partial^2 z}{\partial x \partial y} = \sin x \sin y$,

given that $\frac{\partial z}{\partial y} = -2\sin y$, when $x = 0$;

and $z = 0$, when y is an odd multiple of $\frac{\pi}{2}$.

Sol.: Given partial differential equation is $\frac{\partial^2 z}{\partial x \partial y} = \sin x \sin y$.

Integrating w.r.t. x, keeping y as constant, we get

$$\frac{\partial z}{\partial y} = -\cos x \sin y + f(y). \qquad \text{(i)}$$

When $x = 0$, $\frac{\partial z}{\partial y} = -2\sin y$.

$\therefore -2\sin y = -\sin y + f(y) \Rightarrow f(y) = -\sin y$.

From (i), we get $\frac{\partial z}{\partial y} = -\cos x \sin y - \sin y$.

Integrating w.r.t. y, keeping x as constant, we get

$$z = \cos x \cos y + \cos y + \phi(x). \qquad \text{(ii)}$$

When y is an odd multiple of $\frac{\pi}{2}$, $z = 0$.

$\therefore 0 = 0 + 0 + \phi(x)$, since $\cos(2n+1)\frac{\pi}{2} = 0 \Rightarrow \phi(x) = 0$.

$\therefore$ From (ii), we get $z = (1 + \cos x)\cos y$,

which is the required particular solution.

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Home Assignments

Q.No.1.: Solve the following partial differential equation: $\frac{\partial^2 z}{\partial x \partial y} = \frac{x}{y} + a$.

Ans.: $z = \frac{x^2}{2} \log y + axy + \phi(x) + \psi(y)$.

Q.No.2.: Solve the following partial differential equation: $\frac{\partial^2 z}{\partial x^2} = xy$.

Ans.: $z = \frac{1}{6} x^3 y + x f(y) + \phi(y)$.

Q.No.3.: Solve the following partial differential equation: $\frac{\partial^2 u}{\partial x \partial t} = e^{-t} \cos x$.

Ans.: $u = -e^{-t} \sin x + \phi(x) + \psi(t)$.

Q.No.4.: Solve the following partial differential equation: $\frac{\partial^3 z}{\partial x^2 \partial y} = \cos(2x + 3y)$.

Ans.: $z = f(y) + x\phi(y) + \psi(y) - \frac{1}{12} \sin(2x + 3y)$.

Q.No.5.: Solve the following partial differential equation:

$$\frac{\partial^2 z}{\partial y^2} = z, \text{ gives that when } y = 0,\ z = e^x \text{ and } \frac{\partial z}{\partial y} = e^{-x}.$$

Ans.: $z = e^x \cosh y + e^{-x} \sinh y$

Q.No.6.: Solve the following partial differential equation:

$$\frac{\partial^2 z}{\partial x^2} = a^2 z, \text{ gives that when } x = 0,\ \frac{\partial z}{\partial x} = a \sin y \text{ and } \frac{\partial z}{\partial y} = 0.$$

Ans.: $z = A \cosh x + \sin y \sinh ax$.

Q.No.7.: Solve the following partial differential equation: $\frac{\partial^2 z}{\partial x \partial y} = \frac{1}{xy}$.

Sol.: $z = \log x \log y + g(y) + \phi(x)$.

Q.No.8.: Solve the following partial differential equation: $\dfrac{\partial^2 z}{\partial x \partial y} = 2x + 2y$.

Ans.: $z = xy(x + y) + g(y) + \phi(x)$.

Q.No.9.: Solve the following partial differential equation: $\dfrac{\partial^2 z}{\partial y^2} = \sin(xy)$.

Ans.: $z = -\dfrac{1}{x^2}\sin xy + yf(x) + \phi(x)$.

Q.No.10.: Solve the following partial differential equation: $\dfrac{\partial^2 u}{\partial y \partial x} = 4x \sin(3xy)$.

Ans.: $u = -\dfrac{4}{9y}\sin(3xy) + f(x) + \phi(y)$.

Q.No.11.: Solve the following partial differential equation: $\log\left[\dfrac{\partial^2 z}{\partial x \partial y}\right] = x + y$.

Ans.: $z = e^{x+y} + g(y) + \phi(x)$.

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3rd Topic

Partial Differential Equations

Linear partial differential equation of first order

(Lagrange's linear equation)

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3rd Topic

Partial Differential Equations

Linear partial differential equation of first order

(Lagrange's linear equation)

Prepared by:
Prof. Sunil
Department of Mathematics & Scientific Computing
NIT Hamirpur (HP)

DEFINITIONS:

PARTIAL DIFFERENTIAL EQUATION OF THE FIRST ORDER:

A differential equation involving first order partial derivatives p and q only is called a **partial differential equation of the first order**.

LINEAR PARTIAL DIFFERENTIAL EQUATION:

A partial differential equation is said to be **linear** if the dependent variable z and its derivatives are of degree one and products of z and its derivatives do not appear in the equation.

e.g. $x^2 p + y^2 q = z$ is linear partial differential equation in z and of first order.

QUASI-LINEAR PARTIAL DIFFERENTIAL EQUATION:

Partial differential equation is said to be **quasi-linear** if degree of highest ordered derivative is one and no products of partial derivatives of the highest order are present.

e.g. $zz_{xx} + (z_y)^2 = 0$ is quasi-linear partial differential equation in z and of second order.

NON-LINEAR PARTIAL DIFFERENTIAL EQUATION:

A P.D.E., which is not linear, is known as non-linear P.D.E.

e.g. $\left(\frac{\partial^2 u}{\partial x^2}\right)^2 + u^2\left(\frac{\partial u}{\partial y}\right) = f(x, y)$ is non-linear in u and of second order.

HOMOGENEOUS PARTIAL DIFFERENTIAL EQUATION:

If in a P.D.E., each term contains the dependent variable or its derivatives, THE p.d.e. is known as homogeneous p.d.e. Otherwise p.d.e. is non-homogeneous.

LINEAR PARTIAL DIFFERENTIAL EQUATION OF THE FIRST ORDER:

If p and q both occur in the first degree only and are not multiplied together, then it is called a **linear partial differential equation of the first order.**

LAGRANGE'S LINEAR EQUATION:

The general form of a **quasi-linear** partial differential equation of the first order is

$$P(x,y,z)p + Q(x,y,z)q = R(x,y,z). \tag{i}$$

This equation (i) is known as **Lagrange's linear equation.**

If P and Q are independent of z and R is linear in z, then (i) is a **linear equation**.

Such an equation is obtained by eliminating an arbitrary function ϕ from $\phi(u,v)=0$, (ii)

where u, v are specific (known) functions of x, y, z.

Lagrange's auxiliary equations:

Differentiating (ii) partially w. r. t. x and y.

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}p\right)+\frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}p\right)=0 \quad \text{and} \quad \frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}q\right)+\frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}q\right)=0.$$

Eliminating $\frac{\partial \phi}{\partial u}$ and $\frac{\partial \phi}{\partial v}$, we get

$$\begin{vmatrix} \frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}p & \frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}p \\ \frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}q & \frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}q \end{vmatrix}=0$$

$$\Rightarrow \left(\frac{\partial u}{\partial y}.\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}.\frac{\partial v}{\partial y}\right)p+\left(\frac{\partial u}{\partial z}.\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}.\frac{\partial v}{\partial z}\right)q=\frac{\partial u}{\partial x}.\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}.\frac{\partial v}{\partial x}. \tag{iii}$$

This is of the same form as (i) with

$$P=\left(\frac{\partial u}{\partial y}.\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}.\frac{\partial v}{\partial y}\right),\ Q=\left(\frac{\partial u}{\partial z}.\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}.\frac{\partial v}{\partial z}\right) \text{ and } R=\left(\frac{\partial u}{\partial x}.\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}.\frac{\partial v}{\partial x}\right).$$

To determine u, v, from P, Q, R:

Suppose u = a and v = b, where a, b are constants, so that

$$\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+\frac{\partial u}{\partial z}dz=du=0,$$

$$\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy+\frac{\partial v}{\partial z}dz=dv=0.$$

By cross-multiplication, we get

$$\frac{dx}{\frac{\partial u}{\partial y}.\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}.\frac{\partial v}{\partial y}}=\frac{dy}{\frac{\partial u}{\partial z}.\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}.\frac{\partial v}{\partial z}}=\frac{dz}{\frac{\partial u}{\partial x}.\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}.\frac{\partial v}{\partial x}}.$$

$$\Rightarrow \frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}. \qquad \text{[by virtue of (i) and (iii)]} \qquad \text{(iv)}$$

The solution of these equations are u = a and v = b.

Thus, determining u, v from the simultaneous equations $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$,

we have the solution of the partial differential equation.

Thus, $\phi(u,v)=0$ or $u=f(v)$ is the required solution of $Pp+Qq=R$.

Note: Equations $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$ are called **Lagrange's auxiliary equations** or **subsidiary equations**

Method of obtaining General Solution:

(i) Rewrite the equation in the standard form $Pp+Qq=R$.

(ii) Form the Lagrange's auxiliary equations (A.E.) $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$.

(iii) Solve the auxiliary equations by the method of grouping or the method of multiplier or both to get **two independent solutions** u = a and v = b, where a, b are arbitrary constants.

(iv) Then $\phi(u,v)=0$ or $u=f(v)$ is the general solution of the equation $Pp+Qq=R$.

Extended form for more than two independent variables:

In the differential equation $Pp + Qq = R$, there are two independent variables x and y. To solve it, we have to find two independent solutions satisfying the equation $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$. This procedure can be extended to linear partial equations of the last order involving more than two independent variables.

If $u_1 = c_1, u_2 = c_2, \ldots\ldots\ldots u_n = c_n$ are n independent solutions of the equations

$$\frac{dx_1}{P_1} = \frac{dx_2}{P_2} = \ldots\ldots. = \frac{dx_n}{P_n} = \frac{dz}{R}.$$

Then, the general solution of the differential equation

$$P_1 \frac{\partial z}{\partial x_1} + P_2 \frac{\partial z}{\partial x_2} + \ldots\ldots\ldots P_n \frac{\partial z}{\partial x_n} = R.$$

is $\phi(u_1, u_2, \ldots\ldots\ldots u_n) = 0$, where ϕ is an arbitrary function.

Now let us solve some linear partial differential equation of the first order:

Q.No.1.: Solve the following differential equation: $pz - qz = z^2 + (x+y)^2$.

Sol.: Given differential equation is $pz - qz = z^2 + (x+y)^2$.

This is a Lagrange's linear partial differential equation of first order.

Now here the Lagrange's auxiliary equations are $\frac{dx}{z} = \frac{dy}{-z} = \frac{dz}{z^2 + (x+y)^2}$.

Taking the first two members, we get $dx + dy = 0$.

Integrating, we get $x + y = a$. (i)

This is first independent solution.

Again taking first and third members, we get

$$\frac{dx}{z} = \frac{dz}{z^2 + (x+y)^2} \Rightarrow dx = \frac{zdz}{z^2 + a^2}, \text{ since } x + y = a.$$

$$\Rightarrow \frac{2zdz}{z^2 + a^2} = 2dx.$$

Integrating both sides, we get

$$\log(z^2 + a^2) = 2x + b \Rightarrow \log\left[z^2 + (x+y)^2\right] - 2x = b. \quad \text{(ii)}$$

This is second independent solution.

Since we know that if u = a and v = b are two independent solutions, then $\phi(u, v) = 0$ or $u = f(v)$ is the general solution of the equation $Pp + Qq = R$.

Thus, from (i) and (ii), the required general solution is

$\phi\left[x + y, \log\left(x^2 + y^2 + z^2 + 2xy\right) - 2x\right] = 0$. Ans.

Q.No.2.: Solve the following differential equation:

$$(mz - ny)\frac{\partial z}{\partial x} + (nx - \ell z)\frac{\partial z}{\partial y} = \ell y - mx.$$

Sol.: Given differential equation is $(mz - ny)\frac{\partial z}{\partial x} + (nx - \ell z)\frac{\partial z}{\partial y} = \ell y - mx$.

This is a Lagrange's linear partial differential equation of first order.

Now here the Lagrange's auxiliary equations are $\frac{dx}{mz - ny} = \frac{dy}{nx - \ell z} = \frac{dz}{\ell y - mx}$.

Using multipliers x, y, z, we get

each fraction $= \frac{xdx + ydy + zdz}{0} \Rightarrow xdx + ydy + zdz = 0$.

Integrating, we get

$x^2 + y^2 + z^2 = a$. (i)

Again using multipliers ℓ, m and n, we get

each fraction $= \frac{\ell dx + mdy + ndz}{0} \Rightarrow \ell dx + mdy + ndz = 0$.

Integrating, we get

$\ell x + my + nz = b$. (ii)

Since we know that if u = a and v = b are two independent solutions, then $\phi(u, v) = 0$ or $u = f(v)$ is the general solution of the equation $Pp + Qq = R$.

Hence, from (i) and (ii), the required general solution is

$x^2 + y^2 + z^2 = f(\ell x + my + nz)$.

Q.No.3.: Solve the following differential equation: $\left(x^2 - y^2 - z^2\right)p + 2xyq = 2xz$.

Sol.: Given differential equation is $\left(x^2 - y^2 - z^2\right)p + 2xyq = 2xz$.

This is a Lagrange's linear partial differential equation of first order.

Now here the Lagrange's auxiliary equations are $\frac{dx}{x^2-y^2-z^2}=\frac{dy}{2xy}=\frac{dz}{2xz}$.

From the last two fractions, we have $\frac{dy}{y}=\frac{dz}{z}$.

Integrating both sides, we get

$$\log y=\log z+\log a \Rightarrow \frac{y}{z}=a. \qquad \text{(i)}$$

Using multipliers x, y and z, we have

each fraction $=\frac{xdx+ydy+zdz}{x\left(x^2+y^2+z^2\right)} \Rightarrow \frac{xdx+ydy+zdz}{x\left(x^2+y^2+z^2\right)}=\frac{dz}{2xz}$

$$\Rightarrow \frac{2xdx+2ydy+2zdz}{\left(x^2+y^2+z^2\right)}=\frac{dz}{z}.$$

Integrating both sides, we get

$$\log\left(x^2+y^2+z^2\right)=\log z+\log b \Rightarrow \frac{x^2+y^2+z^2}{z}=b. \qquad \text{(ii)}$$

Since we know that if u = a and v = b are two independent solutions, then $\phi(u,v)=0$ or $u=f(v)$ is the general solution of the equation $Pp+Qq=R$.

Hence, from (i) and (ii), required general solution is $x^2+y^2+z^2=zf\left(\frac{y}{z}\right)$.

Q.No.4.: Solve the following differential equation: $\left(x^2-yz\right)p+\left(y^2-zx\right)q=z^2-xy$.

Sol.: Given differential equation is $\left(x^2-yz\right)p+\left(y^2-zx\right)q=z^2-xy$.

This is a Lagrange's linear partial differential equation of first order.

Now here the Lagrange's auxiliary equations are $\frac{dx}{x^2-yz}=\frac{dy}{y^2-zx}=\frac{dz}{z^2-xy}$. (i)

Each of these equations $=\frac{dx-dy}{x^2-y^2-(y-x)z}=\frac{dy-dz}{y^2-z^2-x(z-y)}$.

$$\Rightarrow \frac{d(x-y)}{(x-y)(x+y+z)}=\frac{d(y-z)}{(y-z)(x+y+z)} \Rightarrow \frac{d(x-y)}{x-y}=\frac{d(y-z)}{y-z}.$$

Integrating both sides, we get

$\log(x-y)=\log(y-z)+\log c \Rightarrow \dfrac{x-y}{y-z}=c$. (ii)

Using multipliers x, y and z, each fraction of (i)

$$=\frac{xdx+ydy+zdz}{x^3+y^3+z^3-3xyz}=\frac{xdx+ydy+zdz}{(x+y+z)\left(x^2+y^2+z^2-yz-zx-xy\right)}. \qquad \text{(iii)}$$

Also, each of the Lagrange's auxiliary equations $=\dfrac{dx+dy+dz}{x^2+y^2+z^2-yz-zx-xy}$. (iv)

Equating (iii) and (iv) and canceling the common factor, we get

$$\frac{xdx+ydy+zdz}{x+y+z}=dx+dy+dz \Rightarrow (xdx+ydy+zdz)=(x+y+z)d(x+y+z)$$

Integrating both sides, we get

$$\int (xdx+ydy+zdz)=\int (x+y+z)d(x+y+z)+c$$

$$\Rightarrow x^2+y^2+z^2=(x+y+z)^2+2c$$

$$\Rightarrow xy+yz+zx+c'=0. \qquad \text{(v)}$$

Since we know that if u = a and v = b are two independent solutions, then $\phi(u,v)=0$ or $u=f(v)$ is the general solution of the equation $Pp+Qq=R$.

Hence, from (i) and (ii), required general solution is $\phi\left(\dfrac{x-y}{y-z},\ xy+yz+zx\right)=0$.

Q.No.5.: Solve the following differential equation: $y^2zp+x^2zq=y^2x$.

Sol.: Given differential equation is $y^2zp+x^2zq=y^2x$.

This is a Lagrange's linear partial differential equation of first order.

Now here the Lagrange's auxiliary equations are $\dfrac{dx}{y^2z}=\dfrac{dy}{x^2z}=\dfrac{dz}{xy^2}$.

Taking first two members, we get $x^2dx=y^2dy$.

Integrating, we get

$x^3-y^3=a$. (i)

Again taking first and third members, we get $xdx=zdz$

Integrating, we get

$x^2-z^2=b$. (ii)

Since we know that if u = a and v = b are two independent solutions, then $\phi(u, v) = 0$ or $u = f(v)$ is the general solution of the equation $Pp + Qq = R$.

Hence, from (i) and (ii), required general solution is $\phi\left(x^3 - y^3,\ x^2 - z^2\right) = 0$.

Q.No.6.: Solve the following differential equation: $x^2(y-z)p + y^2(z-x)q = z^2(x-y)$.

Sol.: Given differential equation is $x^2(y-z)p + y^2(z-x)q = z^2(x-y)$.

This is a Lagrange's linear partial differential equation of first order.

Now here the Lagrange's auxiliary equations are $\dfrac{dx}{x^2(y-z)} = \dfrac{dy}{y^2(z-x)} = \dfrac{dz}{z^2(x-y)}$.

Using $\dfrac{1}{x^2}, \dfrac{1}{y^2}, \dfrac{1}{z^2}$ as multipliers, we get

$$\text{Each fraction} = \frac{\dfrac{1}{x^2}dx + \dfrac{1}{y^2}dy + \dfrac{1}{z^2}dz}{0}.$$

$$\therefore \frac{dx}{x^2} + \frac{dy}{y^2} + \frac{dz}{z^2} = 0.$$

Integrating, we get

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = a. \qquad \text{(i)}$$

Again using $\dfrac{1}{x}, \dfrac{1}{y}, \dfrac{1}{z}$ as multipliers, we get

$$\text{Each fraction} = \frac{\dfrac{1}{x}dx + \dfrac{1}{y}dy + \dfrac{1}{z}dz}{0} \Rightarrow \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0.$$

Integrating, we get

$$\log x + \log y + \log z = \log b \Rightarrow \log(xyz) = \log b \Rightarrow xyz = b. \qquad \text{(ii)}$$

Since we know that if u = a and v = b are two independent solutions, then $\phi(u, v) = 0$ or $u = f(v)$ is the general solution of the equation $Pp + Qq = R$.

Hence, from (i) and (ii), required general solution is $\phi\left(\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z}, xyz\right) = 0$.

Q.No.7.: Solve the following differential equation:

$$\left(z^2-2yz-y^2\right)p+(xy+zx)q=xy-zx.$$

Sol.: Given differential equation is $\left(z^2-2yz-y^2\right)p+(xy+zx)q=xy-zx$.

This is a Lagrange's linear partial differential equation of first order.

Now here the auxiliary equations are $\dfrac{dx}{z^2-2yz-y^2}=\dfrac{dy}{xy+zx}=\dfrac{dz}{xy-zx}$.

Taking x, y, z as multipliers, we get

Each fraction $=\dfrac{xdx+ydy+zdz}{0}$

$\therefore xdx+ydy+zdz=0$.

Integrating, we get

$$x^2+y^2+z^2=a. \qquad \text{(i)}$$

Again taking the last two numbers, we get

$$\frac{dy}{xy+zx}=\frac{dz}{xy-zx}\Rightarrow(y-z)dy=(y+z)dz\Rightarrow ydy-(zdy+ydz)-zdz=0$$

$\Rightarrow ydy-d(yz)-zdz=0$.

Integrating, we get

$$y^2-2yz-z^2=b. \qquad \text{(ii)}$$

Since we know that if u = a and v = b are two independent solutions, then $\phi(u,v)=0$ or $u=f(v)$ is the general solution of the equation $Pp+Qq=R$.

Hence, from (i) and (ii), required general solution is $\phi\left(x^2+y^2+z^2,y^2-2yz-z^2\right)=0$.

Q.No.8.: Solve the following differential equation: $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}+z\dfrac{\partial u}{\partial z}=xyz$.

Sol.: Given differential equation is $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}+z\dfrac{\partial u}{\partial z}=xyz$.

This is a Lagrange's linear partial differential equation of first order.

Now here the Lagrange's auxiliary equations are $\dfrac{dx}{x}=\dfrac{dy}{y}=\dfrac{dz}{z}=\dfrac{du}{xyz}$.

Taking first and second members, we get $\dfrac{dx}{x}=\dfrac{dy}{y}$.

Integrating both sides, we get

$$\log x = \log y + \log c_1 \Rightarrow \log\frac{x}{y} = \log c_1 \Rightarrow \frac{x}{y} = c_1. \qquad \text{(i)}$$

Similarly, taking second and third members, we get $\frac{y}{z} = c_2$. (ii)

Also, we have $\frac{yzdx + zxdy + xydz}{3xyz} = \frac{du}{xyz} \Rightarrow d(xyz) = 3du$.

Integrating both sides, we get

$$xyz - 3u = c_3. \qquad \text{(iii)}$$

Since we know that if u = a and v = b are two independent solutions, then $\phi(u, v) = 0$ or $u = f(v)$ is the general solution of the equation $Pp + Qq = R$.

Thus, from (i), (ii) and (iii), the required general solution is

$$\phi\left(\frac{x}{y}, \frac{y}{z}, xyz - 3u\right) = 0. \text{ Ans.}$$

Q.No.9.: Solve the following differential equation: $xp + yq = 3z$.

Sol.: This is a PDE of first order $Pq + Qq = R$ with P = x, Q = y and R = 3z.

The Lagrange's auxiliary equations are $\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{3z}$.

Integrating the first two equations $\frac{dx}{x} = \frac{dy}{y}$, we get

$$\text{In } x = \text{In } y + c_1 \Rightarrow \frac{x}{y} = c.$$

Integrating first and last equations $\frac{dx}{x} = \frac{dz}{3z}$, we get

$$3 \text{ In } x = \text{In } z + c_2 \qquad \therefore x^3 = c_1 z.$$

Thus the required solution is $x^3 = z f\left(\frac{x}{y}\right)$.

The general solution can also be written as $F\left(\frac{x^3}{z}, \frac{x}{y}\right) = 0$,

Note: By integrating 2nd and 3rd equations $\frac{dy}{y} = \frac{dz}{3z}$, we also get $y^3 = c_2 z$ so the general solution is also given by $y^3 = zf\left(\frac{x}{y}\right)$.

Q.No.10.: Solve the following differential equation: $p - q = \log(x + y)$.

Sol.: Given differential equation is $p - q = \log(x + y)$.

This is a Lagrange's linear partial differential equation of first order.

Now here the Lagrange's auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{-1} = \frac{dz}{\text{In}(x+y)}$$

Integrating the first two fractions $dx + dy = 0$ yields

$$x + y = c_1$$

From first and last fractions $\text{In}\,(x + y)dx = dz$

Put $x + y = c_1$, then $\text{In}\,c_1 dx = dz$

Integrating $x\,\text{In}\,c_1 = z + c_2 \Rightarrow x.\,\text{In}(x + y) = z + c_2$.

The general solution is

$$F(x + y, \text{In}(x + y) - z) = 0.$$

Q.No.11.: Solve $yzp - xzq = xy$.

Sol.: Given differential equation is $yzp - xzq = xy$.

This is a Lagrange's linear partial differential equation of first order.

Now here the Lagrange's auxiliary equations are

$$\frac{dx}{yz} = \frac{dy}{-xz} = \frac{dz}{xy}$$

from first and second fractions, we get

$$\frac{dx}{yz} = \frac{dy}{-xz} \Rightarrow \frac{dx}{y} = \frac{dy}{-z} \Rightarrow xdx + ydy = 0$$

Integrating $x^2 + y^2 = c_1$

From first and third fraction

$$\frac{dx}{yz} = \frac{dz}{xy} \Rightarrow \frac{dx}{z} = \frac{dz}{x}$$

Integrating, $x^2 - z^2 = c_2$

Thus the general solution is

$F\left(x^2 + y^2,\ x^2 - z^2\right) = 0$.

Q.No.12.: Solve $z\left(z^2 + xy\right)(px - qy) = x^4$

Sol.: Given differential equation is $z\left(z^2 + xy\right)(px - qy) = x^4$.

This is a Lagrange's linear partial differential equation of first order.

Now here the Lagrange's auxiliary equations are

$$\frac{dx}{xz\left(z^2 + xy\right)} = \frac{dy}{-yz\left(z^2 + xy\right)} = \frac{dz}{x^4}$$

From first and second fractions, we get

$$\frac{dx}{x} = \frac{dy}{-y}$$

on integration $xy = c_1$

From first and third fraction

$$x^3 dx = \left(z^3 + xyz\right)dz$$

using $xy = c_1$, $x^3 dx = \left(z^3 + c_1 z\right)dz$

Integrating $\frac{x^4}{4} = \frac{z^4}{4} + c_1 \frac{z^2}{2} + c_2 \Rightarrow x^4 - z^4 - 2c_1 z^2 = c_2$

Substituting for $c_1 . x^4 - z^4 - 2(xy)z^2 = c_2$

The general solution is

$F\left(xy, x^4 - z^4 - 2xyz^2\right) = 0$.

Q.No.13.: Solve $(z - y)p + (x - z)q = y - x$.

Sol.: Given differential equation is $(z - y)p + (x - z)q = y - x$.

This is a Lagrange's linear partial differential equation of first order.

Now here the Lagrange's auxiliary equations are

$$\frac{dx}{z - y} = \frac{dy}{x - z} = \frac{dz}{y - x}$$

Choosing multipliers as 1, 1, 1

$dx + dy + dz = (z - y) + (x - z) + (y - z) = 0$

Integrating $x + y + z = c_1$

Choosing multipliers as x, y, z

$xdx + ydy + zdz = x(z - y) + y(x - z) + z(y - x) = 0$

Integrating $x^2 + y^2 + z^2 = c_2$

The general solution is

$F(x + y + z, x^2 + y^2 + z^2) = 0$.

Q.No.14.: Solve $(y + zx)p - (x + yz)q = x^2 - y^2$.

Sol.: Given differential equation is $(y + zx)p - (x + yz)q = x^2 - y^2$.

This is a Lagrange's linear partial differential equation of first order.

Now here the Lagrange's auxiliary equations are

$$\frac{dx}{y + zx} = \frac{dy}{-(x + yz)} = \frac{dz}{x^2 - y^2}.$$

Choosing multipliers as $x, y, -z$

$xdx + ydy - zdz = x(y + zx) + y(-1)(x + yz) - z(x^2 - y^2) = 0$

Integrating $x^2 + y^2 - z^2 = c_1$

Choosing multipliers as y, x, 1, we get

$ydx + xdy + dz = y(y + zx) + x(-1)(x + yz) + (x^2 - y^2) = 0$

Integrating $xy + z = c_2$.

The general solution is

$F(x^2 + y^2 - z^2, xy + z) = 0$.

Q.No.15.: Solve $(y^2 + z^2)p - xyq + zx = 0$.

Sol.: Given differential equation is $(y^2 + z^2)p - xyq + zx = 0$.

This is a Lagrange's linear partial differential equation of first order.

Now here the Lagrange's auxiliary equations are

$$\frac{dx}{y^2 + z^2} = \frac{dy}{-xy} = \frac{dz}{-zx}$$

From the second and third fractions

$$\frac{dy}{y}=\frac{dz}{z}\Rightarrow\frac{y}{z}=c_1$$

Choosing multipliers as x, y, z

$$xdx+ydy+zdz=x\left(y^2+z^2\right)+y(-xy)+z(-zx)=0$$

Integrating $x^2+y^2+z^2=c_2$

The general solution is

$$F\left(\frac{y}{z},x^2+y^2+z^2\right)=0.$$

Q.No.16.: Solve $px(x+y)=qy(x+y)-(2x+2y+z)(x-y)$.

Sol.: Given differential equation is $px(x+y)=qy(x+y)-(2x+2y+z)(x-y)$.

This is a Lagrange's linear partial differential equation of first order.

Now here the Lagrange's auxiliary equations are

$$\frac{dx}{x(x+y)}=\frac{dy}{-y(x+y)}=\frac{dz}{-x(x-y)(2x+2y+z)}$$

From first two fractions, canceling $(x+y)$,

We get

$$\frac{dx}{x}=-\frac{dy}{y}\Rightarrow d(\text{In } x)+d(\text{In } y)=c$$

which on integration gives $xy=c_1$

$$\frac{dx+dy}{x(x+y)-y(x+y)}=\frac{dx+dy}{(x+y)(x-y)}=\frac{dz}{-x(x-y)(2x+2y+z)}$$

Canceling the $x-y$ terms, we get

$$(2x+2y+z)(dx+dy)+(x+y)dz=0$$

or $(x+y+z)(dx+dy)+(x+y)(dx+dy)+(x+y)dz=0$

$$(x+y+z)d(x+y)+(x+y)d(x+y+z)=0$$

i.e. $d[(x+y)(x+y+z)]=0$

Integrating $(x+y)(x+y+z)=c_2$.

Thus the general solution is

$F[xy, (x+y)(x+y+z)] = 0$.

Q.No.17.: Solve $(x^2 - y^2 - yz)p + (x^2 - y^2 - zx)q = z(x-y)$.

Sol.: Auxiliary equations are

$$\frac{dx}{x^2 - y^2 - yz} = \frac{dy}{x^2 - y^2 - zx} = \frac{dz}{z(x-y)}$$

$$dx - dy - dz = x^2 - y^2 - yz - (x^2 - y^2 - zx) - z(x-y) = 0$$

Integrating $x - y - z = c_1$

From first and second fraction

$$\frac{xdx - ydy}{x^3 - xy^2 - x^2y - y^3} = \frac{dz}{z(x-y)} \quad \Rightarrow \frac{xdx - ydy}{(x^2 - y^2)(x-y)} = \frac{dz}{z(x-y)}$$

i.e., $\frac{1}{2}d\left[\text{In}\left(x^2 - y^2\right)\right] = d(\text{In } z)$

$$\therefore \frac{\left(x^2 - y^2\right)}{z^2} = c_2$$

$\therefore$ The general solution is

$$f\left(x - y - z, \frac{\left(x^2 - y^2\right)}{z^2}\right) = 0.$$

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Home Assignments

Q.No.1.: Solve the following differential equation: $p\sqrt{x} + q\sqrt{y} = \sqrt{x}$.

Ans.: $\sqrt{x} - \sqrt{y} = f\left(\sqrt{x} - \sqrt{z}\right)$**.**

Q.No.2.: Solve the following differential equation: $p\cos(x+y) + q\sin(x+y) = z$.

Ans.: $[\cos(x+y) + \sin(x+y)]e^{y-z} = \phi\left[z^{\sqrt{2}}\tan\left(\frac{x+y}{2} + \frac{\pi}{8}\right)\right]$**.**

Q.No.3.: Solve the following differential equation: $pyz + qzx = xy$.

Ans.: $x^2 - y^2 = f\left(y^2 - z^2\right)$.

Q.No.4.: Solve the following differential equation: $p\tan x + q\tan y = \tan z$.

Ans.: $\phi\left(\dfrac{\sin x}{\sin y}, \dfrac{\sin y}{\sin z}\right) = 0$

Q.No.5.: Solve the following differential equation: $xp - yq = y^2 - x^2$.

Ans.: $x^2 + y^2 + 2z = [\log(xy)]$.

Q.No.6.: Solve the following differential equation: $(y+z)p - (z+x)q = x - y$.

Ans.: $x^2 + y^2 - z^2 = f(x + y + z)$.

Q.No.7.: Solve the following differential equation: $x(y-z)p + y(z-x)q = z(x-y)$.

Ans.: $x + y + z = f(xyz)$.

Q.No.8.: Solve the following differential equation:

$$x\left(y^2 - z^2\right)p + y\left(z^2 - x^2\right)q - z\left(x^2 - y^2\right) = 0.$$

Ans.: $\phi\left(x^2 + y^2 + z^2, xyz\right) = 0$.

Q.No.9.: Solve the following differential equation: $y^2p - xyq = x(z - 2y)$.

Ans.: $x^2 + y^2 = f\left(y^2 - yz\right)$.

Q.No.10.: Solve the following differential equation: $px\left(z - 2y^2\right) = (z - qy)\left(z - y^2 - 2x^3\right)$.

Ans.: $\phi\left(\dfrac{y}{z}, \dfrac{z}{x} - \dfrac{y^2}{x} + x^2\right) = 0$.

Q.No.11.: Solve the following differential equation: $2p + 3q = 1$.

Ans.: $\phi(3x - 2y, y - 3z) = 0$.

Q.No.12.: Solve the following differential equation: $(y - z)p + (x - y)q = z - x$.

Ans.: $\phi\left(x + y + z, \dfrac{x^2}{2} + yz\right)$.

Q.No.13.: Solve the following differential equation: $(y + z)p + y(z + x)q = x + y$.

Ans.: $\phi\left[\dfrac{x-y}{y-z},(x-y)^2(x+y+z)\right]=0$.

Q.No.14.: Solve the following differential equation: $\left(\dfrac{y-z}{yz}\right)p+\left(\dfrac{z-x}{zx}\right)q=\dfrac{x-y}{xy}$.

Ans.: $\phi(x+y+z, xyz)=0$.

Q.No.15.: Solve the following differential equation: $x^2p+y^2q=(x+y)z$.

Ans.: $\phi\left(\dfrac{x-y}{z},\dfrac{1}{y}-\dfrac{1}{x}\right)=0$.

Q.No.16.: Solve the following differential equation: $z(xp-yq)=y^2-x^2$.

Ans.: $\phi\left(xy, x^2+y^2+z^2\right)=0$.

Q.No.17.: Solve the following differential equation:

$$\left(y^3x-2x^4\right)p+\left(2y^4-x^3y\right)q=9z\left(x^3-y^3\right).$$

Ans.: $\phi\left(\dfrac{y}{x^2}+\dfrac{x}{y^2}, xyz^{1/3}\right)=0$.

Q.No.18.: Solve the following differential equation:

$$x_1p_1+2x_2p_2+3x_3p_3+4x_4p_4=0.$$

Ans.: $z=\phi\left(\dfrac{x_1{}^2}{x_2},\dfrac{x_1{}^3}{x_3},\dfrac{x_1{}^4}{x_4}\right)$.

Q.No.19.: Solve $xp+yq=z$.

Ans.: $F\left(\dfrac{x}{z},\dfrac{y}{z}\right)=0$.

Q.No.20.: Solve $x^2p+y^2q=z^2$.

Ans.: $F\left(\dfrac{1}{x}-\dfrac{1}{y},\dfrac{1}{y}-\dfrac{1}{z}\right)=0$.

Q.No.21.: Solve $p+3q=5z+\tan(y-3x)$.

Ans.: $F\left(y-3x, e^{-5x}\{5z+\tan(y-3x)\}\right)=0$.

Q.No.22.: Solve $\left(x^3+3xy^2\right)p+\left(y^3+3x^2y\right)q=2\left(x^2+y^2\right)z$.

Ans.: $F\left[(x-y)^{-2}-(x+y)^{-2}, \frac{xy}{z^2}\right]=0$.

Q.No.23.: Solve $x^2\left(y^3-z^3\right)p+y^2\left(z^3-x^3\right)q=z^2\left(x^3-y^3\right)$.

Ans.: $F\left(x^2+y^2+z^2, \frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)=0$.

Q.No.24.: Solve $\left(2x^2+y^2+z^2-2yz-zx-xy\right)p+\left(x^2+2y^2+z-yz-2zx-xy\right)q$

$$=\left(x^2+y^2+2z^2-yz-2xy\right).$$

Ans.: $F\left(\frac{x-y}{y-z}, \frac{y-z}{z-x}\right)=0$.

Q.No.25.: Solve $(x+2z)p+(4zx-y)q=2x^2+y$.

Ans.: $F\left(xy-z^2, x^2-y-z\right)=0$.

Q.No.26.: Solve $x(y-z)p+y(z-x)q=z(x-y)$.

Ans.: $F(x+y+z, xyz)=0$.

Q.No.27.: Solve $(y+z)p+(z+x)q=x+y$.

Ans.: $F\left(\frac{x-y}{y-z}, \frac{y-x}{\sqrt{x+y+z}}\right)=0$.

Q.No.28.: Solve $px\left(z-2y^2\right)=(z-qy)\left(z-y^2-2x^3\right)$.

Ans.: $F\left(\frac{y}{z}, \frac{y^2}{x}-\frac{z}{x}-x^2\right)=0$.

Q.No.29.: Solve $x\left(y^2-z^2\right)p+y\left(z^2-x^2\right)q=z\left(x^2-y^2\right)$.

Ans.: $F\left(x^2+y^2+z^2, xyz\right)=0$.

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4th Topic

Partial Differential Equations

Non-linear partial differential equation of first order

Standard forms:

(i) $f(p,q)=0$,

(ii) $f(z,p,q)=0$,

(iii) $f(x,p)=F(y,q)$,

(iv) $z=px+qy+f(p,q)$.

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4^{th} Topic

Partial Differential Equations

Non-linear partial differential equation of first order

Standard forms:

(i) $f(p,q)=0$, **(ii)** $f(z,p,q)=0$,

(iii) $f(x,p)=F(y,q)$, **(iv)** $z=px+qy+f(p,q)$

Prepared by:
Prof. Sunil
Department of Mathematics & Scientific Computing
NIT Hamirpur (HP)

PARTIAL DIFFERENTIAL EQUATION OF THE FIRST ORDER:

A differential equation involving first order partial derivatives p and q only is called a **partial differential equation of the first order**.

NON-LINEAR PARTIAL DIFFERENTIAL EQUATION:

A partial differential equation, which is not linear, is known as non-linear P.D.E.

e.g. $\left(\dfrac{\partial^2 u}{\partial x^2}\right)^2 + u^2\left(\dfrac{\partial u}{\partial y}\right) = f(x,y)$ is non-linear in u and of second order.

NON-LINEAR PARTIAL DIFFERENTIAL EQUATION OF THE FIRST ORDER:

A partial differential equation, which involves **first order** partial derivatives p and q of **degree (power) higher than one** and/or product of terms p and q, is called a **non-linear partial differential equation.** Some special types of non-linear first order partial differential equations are presented.

Form 1. Equation of the form f(p, q) = 0:

i.e. Equation involving only p and q and no x, y, z.

Complete solution:

In this case, the **complete solution** is $z = ax + by + c$, (i)

where a and b are connected by the relation $f(a, b) = 0$. (ii)

$$\left[\text{since } p = \frac{\partial z}{\partial x} = a \text{ and } q = \frac{\partial z}{\partial y} = b\right]$$

From (ii), we can find b in terms of a and let us suppose $b = \phi(a)$.

Putting this value in (i), the **complete solution** is $z = ax + \phi(a)y + c$,

where a and c are arbitrary constants.

Now let us solve some partial differential equations for the present case i.e. when equation of the form f(p, q) = 0:

Q.No.1.: Solve the following partial differential equations:

(i) $\sqrt{p} + \sqrt{q} = 1$, **(ii)** $pq = p + q$.

Sol.: (i). Given partial differential equation is $\sqrt{p} + \sqrt{q} = 1$.

This non-linear partial differential equation is of the form f(p, q) = 0.

$\therefore$ The complete solution is $z = ax + by + c$, (i)

where $\sqrt{a} + \sqrt{b} = 1 \Rightarrow b = \left(1 - \sqrt{a}\right)^2$.

Thus, from (i), the complete solution is $z = ax + \left(1 - \sqrt{a}\right)^2 y + c$. Ans.

(ii). Given partial differential equation is $pq = p + q$.

This non-linear partial differential equation is of the form f(p, q) = 0.

$\therefore$ The complete solution is $z = ax + by + c$, (i)

where $ab = a + b \Rightarrow b = \dfrac{a}{a-1}$.

Thus, from (i), the complete solution is $z = ax + \dfrac{a}{a-1}y + c$. Ans.

Equations reducible to the form f(p, q) = 0:

Q.No.2.: Solve the following partial differential equation: $(y - x)(qy - px) = (p - q)^2$.

Sol.: Given partial differential equation is $(y - x)(qy - px) = (p - q)^2$.

This equation is not of the form f(p, q) = 0, but this can be reduced this form.

Let $x+y=X$ and $xy=Y$,

so that $p=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial X}.\frac{\partial X}{\partial x}+\frac{\partial z}{\partial Y}.\frac{\partial Y}{\partial x}=\frac{\partial z}{\partial X}+y\frac{\partial z}{\partial Y}$

and $q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial X}.\frac{\partial X}{\partial y}+\frac{\partial z}{\partial Y}.\frac{\partial Y}{\partial y}=\frac{\partial z}{\partial X}+x\frac{\partial z}{\partial Y}$.

$\therefore qy-px=(y-x)\frac{\partial z}{\partial X}$ and $p-q=(y-x)\frac{\partial z}{\partial Y}$.

$\therefore$ The given equation can be written as $\frac{\partial z}{\partial X}=\left(\frac{\partial z}{\partial Y}\right)^2$.

$\Rightarrow P=Q^2$, where $P=\frac{\partial z}{\partial X}$ and $Q=\frac{\partial z}{\partial Y}$.

Now this equation is of the form f(P, Q) = 0.

$\therefore$ The complete solution is $z=aX+bY+c$, (i)

where $a=b^2 \Rightarrow b=\sqrt{a}$.

Thus, from (i), the complete solution is $z=aX+\sqrt{a}.Y+c$

$$\Rightarrow z=a(x+y)+\sqrt{a}.xy+c. \text{ Ans.}$$

Form 2. Equation of the form $z=px+qy+f(p,q)$:

Complete solution:

In this case, the **complete solution** is

$z=ax+by+f(a,b)$

obtained by writing a for p and b for q.

Now let us solve some partial differential equations for the present case i.e. when equation of the form $z=px+qy+f(p,q)$:

Q.No.3.: Solve the following partial differential equation: $z=px+qy+\sqrt{1+p^2+q^2}$.

Sol.: Given partial differential equation is $z=px+qy+\sqrt{1+p^2+q^2}$.

This non-linear partial differential equation is of the form $z=px+qy+f(p,q)$.

Thus, its complete solution is $z=ax+by+\sqrt{1+a^2+b^2}$.

Equations reducible to the form $z = px + qy + f(p,q)$.

Q.No.4.: Solve the following partial differential equation: $4xyz = pq + 2px^2y + 2qxy^2$.

Sol.: Given partial differential equation is $4xyz = pq + 2px^2y + 2qxy^2$.

This equation is not of the form $z = px + qy + f(p,q)$, but this can be reduced this form.

Let $x^2 = X$ and $y^2 = Y$

so that $p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}.\frac{\partial X}{\partial x} = 2x\frac{\partial z}{\partial X}$ and $q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y}.\frac{\partial Y}{\partial y} = 2y\frac{\partial z}{\partial Y}$.

$\therefore$ The given equation becomes

$$4xyz = 4xy\frac{\partial z}{\partial X}.\frac{\partial z}{\partial Y} + 4x^3y\frac{\partial z}{\partial X} + 4xy^3\frac{\partial z}{\partial Y}$$

$$\Rightarrow z = x^2\frac{\partial z}{\partial X} + y^2\frac{\partial z}{\partial Y} + \frac{\partial z}{\partial X}.\frac{\partial z}{\partial Y} = X\frac{\partial z}{\partial X} + Y\frac{\partial z}{\partial Y} + \frac{\partial z}{\partial X}.\frac{\partial z}{\partial Y}$$

$$\Rightarrow z = PX + QY + PQ, \text{ where } P = \frac{\partial z}{\partial X} \text{ and } Q = \frac{\partial z}{\partial Y}.$$

Now this equation is of the form $z = PX + QY + f(P,Q)$.

Thus, the complete solution is $z = aX + bY + ab \Rightarrow z = ax^2 + by^2 + ab$. Ans.

Form 3. Equation of the form $f(z,p,q) = 0$:

i.e. Equation not containing x and y.

Let us assume $z = \phi(x + ay) = \phi(u)$,

where $u = x + ay$ as a trial solution of the given equation.

$$\therefore p = \frac{\partial z}{\partial x} = \phi'(x + ay) = \phi'(u) = \frac{\partial z}{\partial u} \text{ and } q = \frac{\partial z}{\partial y} = \phi'(x + ay)a = a\phi'(u) = a\frac{\partial z}{\partial u}.$$

Substituting these values of p and q, we get $f\left(z, \frac{dz}{du}, a\frac{dz}{du}\right) = 0$,

which is the ordinary differential equation of the first order.

Integrated it, we get the complete solution.

Method:

(i) Assume $u = x + ay$, so that $p = \frac{\partial z}{\partial u}$ and $q = a\frac{\partial z}{\partial u}$.

(ii) Substitute these values of p and q in the given equation.

(iii) Solve these resulting ordinary differential equations in z and u.

(iv) Replace u by $x + ay$.

Now let us solve some partial differential equations for the present case

i.e. when equation of the form $f(z, p, q) = 0$:

Q.No.5.: Solve the following partial differential equation: $z^2\left(p^2 + q^2 + 1\right) = a^2$.

Sol.: Given partial differential equation is $z^2\left(p^2 + q^2 + 1\right) = a^2$

This non-linear partial differential equation is of the form $f(z, p, q) = 0$.

Let $u = x + by$, [Note the use of b instead of a, since a is given in the constant]

so that $p = \dfrac{\partial z}{\partial u}$ and $q = b\dfrac{\partial z}{\partial u}$.

Substituting these values of p and q in the given equation, we get

$$z^2\left[\left(\frac{dz}{du}\right)^2 + b^2\left(\frac{dz}{du}\right)^2 + 1\right] = a^2 \Rightarrow z^2\left(1 + b^2\right)\left(\frac{dz}{du}\right)^2 = a^2 - z^2$$

$$\Rightarrow z\sqrt{1 + b^2}\frac{dz}{du} = \sqrt{a^2 - z^2} \Rightarrow \pm\sqrt{1 + b^2}.\frac{z}{\sqrt{a^2 - z^2}}dz = du$$

Integrating on both sides, we get

$$\pm\sqrt{1 + b^2}\sqrt{a^2 - z^2} = u + c \Rightarrow \left(1 + b^2\right)\left(a^2 - z^2\right) = (x + by + c)^2,$$

which is the required complete solution.

Q.No.6.: Solve the following partial differential equation: $z^2\left(p^2x^2 + q^2\right) = 1$.

Sol.: Given partial differential equation is $z^2\left(p^2x^2 + q^2\right) = 1$.

This equation can be written as $z^2\left[\left(x\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2\right] = 1$. (i)

Let $X = \log x$,

so that $\dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial X}.\dfrac{\partial X}{\partial x} = \dfrac{\partial z}{\partial X}.\dfrac{1}{x} \Rightarrow x\dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial X}$.

$\therefore$ The equation (i) reduces to

$$z^2\left[\left(\frac{\partial z}{\partial X}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right]=1 \Rightarrow z^2\left(P^2+q^2\right)=1, \text{ where } P=\frac{\partial z}{\partial X}. \qquad \text{(ii)}$$

Let $u = X + ay$, so that $p=\frac{\partial z}{\partial u}$ and $q = a\frac{\partial z}{\partial u}$.

$\therefore$ From (ii), we get $z^2\left[\left(\frac{\partial z}{\partial u}\right)^2+a^2\left(\frac{\partial z}{\partial u}\right)^2\right]=1 \Rightarrow \left(1+a^2\right)z^2\left(\frac{dz}{du}\right)^2=1$

$\Rightarrow \sqrt{1+a^2}.zdz = \pm du$.

Integrating on both sides, we get

$\sqrt{1+a^2}.\frac{z^2}{2} = \pm u + b \Rightarrow \sqrt{1+a^2}.z^2 = \pm 2(X+ay)+2b$.

$\Rightarrow \sqrt{1+a^2}.z^2 = \pm 2(\log x + ay) + c$,

which is the required complete solution.

Form 4. Equation of the form $f_1(x,p) = f_2(y,q)$:

i. e. Equations in which z is absent and the terms involving x and p can be separated from those involving y and q.

As a trial solution, let us put each side equal to an arbitrary constant a, then

$f_1(x,p) = f_2(y,q) = a$.

Solving the equations for p and q, let $p = F_1(x)$ and $q = F_2(y)$.

Since $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = pdx + qdy$

$\therefore dz = F_1(x)dx + F_2(y)dy$

Integrating on both sides, we get

$z = \int F_1(x)dx + \int F_2(y)dy + b$,

which is the required complete solution.

Now let us solve some partial differential equations for the present case i.e. when of the form $f_1(x,p)=f_2(y,q)$:

Q.No.7.: Solve the following partial differential equation: $yp = 2yx + \log q$.

Sol.: Given partial differential equation is $yp = 2yx + \log q$.

This equation can be written as $p = 2x + \frac{1}{y}\log q \Rightarrow p - 2x = \frac{1}{y}\log q$.

This non-linear partial differential equation is of the form $f_1(x,p)=f_2(y,q)$.

Let $p - 2x = \frac{1}{y}\log q = a$, then $p = 2x + a$ and $\log q = ay \Rightarrow q = e^{ay}$.

Substituting these values of p and q in $dz = pdx + qdy$, we get

$dz = (2x + a)dx + e^{ay}dy$.

Integrating on both sides, we get $z = x^2 + ax + \frac{1}{a}e^{ay} + b$,

which is the required complete solution.

Q.No.8.: Solve the following partial differential equation: $z^2(p^2+q^2) = x^2 + y^2$.

Sol.: Given partial differential equation is $z^2(p^2+q^2) = x^2 + y^2$.

This equation can be written as $\left(z\frac{\partial z}{\partial x}\right)^2 + \left(z\frac{\partial z}{\partial y}\right)^2 = x^2 + y^2$. (i)

Let $zdz = dZ$, so that $Z = \frac{1}{2}z^2$.

Now $\frac{\partial Z}{\partial x} = \frac{\partial Z}{\partial z}.\frac{\partial z}{\partial x} = z\frac{\partial z}{\partial x}$ and $\frac{\partial Z}{\partial y} = \frac{\partial Z}{\partial z}.\frac{\partial z}{\partial y} = z\frac{\partial z}{\partial y}$.

$\therefore$ The equation (i) becomes $\left(\frac{\partial Z}{\partial x}\right)^2 + \left(\frac{\partial Z}{\partial y}\right)^2 = x^2 + y^2$.

$\Rightarrow P^2 + Q^2 = x^2 + y^2$, where $P = \frac{\partial Z}{\partial x}$ and $Q = \frac{\partial Z}{\partial y}$.

$\Rightarrow P^2 - x^2 = y^2 - Q^2$,

This partial differential equation is the form of $f_1(x,P) = f_2(y,Q)$.

Let $P^2 - x^2 = y^2 - Q^2 = a$, then $P = \sqrt{x^2 + a}$ and $Q = \sqrt{y^2 - a}$.

Substituting these values of P and Q in $dZ = Pdx + Qdy$, we get

$dZ = \sqrt{x^2 + a}dx + \sqrt{y^2 - a}dy$.

Integrated on both sides, we get

$$Z = \frac{1}{2}x\sqrt{x^2 + a} + \frac{a}{2}\log\left(x + \sqrt{x^2 + a}\right) + \frac{1}{2}y\sqrt{y^2 - a} - \frac{a}{2}\log\left(y + \sqrt{x^2 - a}\right) + b$$

$$\Rightarrow z^2 = x\sqrt{x^2 + a} + y\sqrt{y^2 - a} + a\log\frac{x + \sqrt{x^2 + a}}{y + \sqrt{y^2 - a}} + c, \text{ (where } c = 2b),$$

which is the required complete solution.

Q.No.9.: Solve the following partial differential equation: $p^2 - q^2 = x - y$.

Sol.: Given partial differential equation is $p^2 - x = q^2 - y$.

This non-linear partial differential equation is of the form $f_1(x, p) = f_2(y, q)$.

Let $p^2 - x = q^2 - y = a$.

Then $p^2 = x + a$ and $q^2 = y + a$

i.e. $p = \sqrt{x + a}$ and $q = \sqrt{y + a}$.

Substituting these values of p and q in $dz = pdx + qdy$, we get

$dz = \sqrt{x + a}dx + \sqrt{y + a}dy$.

Integrating on both sides, we get $z = \frac{2}{3}(x + a)^{3/2} + \frac{2}{3}(y + a)^{3/2} + b$,

which is the required complete solution.

Now let us solve some more partial differential equations:

Q.No.10.: Solve the following partial differential equation: $x^2p^2 + y^2q^2 = z^2$

Sol.: Given partial differential equation is $x^2p^2 + y^2q^2 = z^2$.

This equation is not of the form f(p, q) = 0, but this can be reduced this form.

Given equation can be reduced to the above form by writing it as

$$\left(\frac{x}{z}.\frac{\partial z}{\partial x}\right)^2+\left(\frac{y}{z}.\frac{\partial z}{\partial y}\right)^2=1. \qquad \text{(i)}$$

and setting $\frac{dx}{x}=du, \frac{dy}{y}=dv, \frac{dz}{z}=dw$ so that $u=\log x, v=\log y, w=\log z$.

Then (i) becomes $\left(\frac{\partial w}{\partial u}\right)^2+\left(\frac{\partial w}{\partial v}\right)^2=1$.

i.e. $P^2+Q^2=1$ where $P=\frac{\partial w}{\partial u}$ and $P=\frac{\partial w}{\partial v}$.

This equation is of the form f(p, q) = 0.

$\therefore$ The complete solution is $w=au+bv+c$. (ii)

where $a^2+b^2=1 \Rightarrow b=\sqrt{\left(1-a^2\right)}$.

$\therefore$ (ii) becomes $w=au+\sqrt{\left(1-a^2\right)}v+c$.

$\Rightarrow \log z=a\log x+\sqrt{\left(1-a^2\right)}\log y+c$,

which is the required solution.

Q.No.11.: Solve the following partial differential equation: $p(1+q)=qz$.

Sol.: Given partial differential equation is $p(1+q)=qz$.

This non-linear partial differential equation is of the form $f(z,p,q)=0$.

Let $u=x+ay$, so that $p=\frac{dz}{du}$ and $q=a\frac{dz}{du}$.

Substituting values of p and q in the given equation, we have

$$\frac{dz}{du}\left(1+a\frac{dz}{du}\right)=az\frac{dz}{du} \Rightarrow a\frac{dz}{du}=az-1 \Rightarrow \int\frac{adz}{az-1}=\int du+b$$

$\Rightarrow \log(az-1)=u+b \Rightarrow \log(az-1)=x+ay+b$,

which is the required complete solution.

Q.No.12.: Solve the following partial differential equation: $q^2=z^2p^2\left(1-p^2\right)$.

Sol.: Given partial differential equation is $q^2=z^2p^2\left(1-p^2\right)$.

This non-linear partial differential equation is of the form $f(z,p,q)=0$.

Setting $u = y + ax$ and $z = f(u)$, we get

$$p = \frac{\partial z}{\partial x} = \frac{dz}{du}.\frac{\partial u}{\partial x} = a\frac{dz}{du} \quad \text{and} \quad q = \frac{dz}{du}.\frac{\partial u}{\partial y} = \frac{dz}{du}.$$

$\therefore$ The given equation becomes $\left(\frac{dz}{du}\right)^2 = a^2z^2\left(\frac{dz}{du}\right)^2\left\{1 - a^2\left(\frac{dz}{du}\right)^2\right\}$. (i)

$$\Rightarrow a^4z^2\left(\frac{dz}{du}\right)^2 = a^2z^2 - 1 \Rightarrow \frac{dz}{du} = \frac{\sqrt{\left(a^2z^2 - 1\right)}}{a^2z}.$$

Integrating on both sides, we get

$$\int \frac{a^2z}{\sqrt{\left(a^2z^2 - 1\right)}}dz = \int du + c \Rightarrow \left(a^2z^2 - 1\right)^{1/2} = u + c$$

$$\Rightarrow Z = aX + bY + c \quad [\because u = y + ax].$$

The second function in (i) is $\frac{dz}{du} = 0 \Rightarrow z = c'$,

which is the required complete solution.

Q.No.13.: Solve the following partial differential equation: $p^2 + q^2 = x + y$.

Sol.: Given partial differential equation is $p^2 + q^2 = x + y$.

This partial differential equation can be written as $p^2 - x = y - q^2$

This non-linear partial differential equation is of the form $f_1(x, p) = f_2(y, q)$.

Let $p^2 - x = y - q^2 = a$, say

$\therefore p^2 - x = a$ gives $p = \sqrt{(a + x)}$

and $y - q^2 = a$ gives $q = \sqrt{(y - a)}$.

Substituting these values of p and q in $dz = pdx + qdy$, we get

$$dz = \sqrt{(a + x)}dx + \sqrt{(y - a)}dy.$$

Integrating on both sides, we get $z = \frac{2}{3}(a + x)^{3/2} + \frac{2}{3}(y - a)^{3/2} + b$,

which is the required complete solution.

Q.No.14.: Solve the following partial differential equation:

$(x+y)(p+q)^2+(x-y)(p-q)^2=1$.

Sol.: Given partial differential equation is $(x+y)(p+q)^2+(x-y)(p-q)^2=1$.

This equation can be reduced to the form $f_1(x,p)=f_2(y,q)$.

By putting $u=x+y$, $v=x-y$ and taking z = z (u, v).

Then $p=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}.\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}.\frac{\partial v}{\partial x}=P+Q$,

and $q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}.\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}.\frac{\partial v}{\partial y}=P-Q$, where $P=\frac{\partial z}{\partial u}$, $Q=\frac{\partial z}{\partial v}$.

Substituting these, the given equation reduces to

$u(2P)^2+v(2Q)^2=1 \Rightarrow 4P^2u=1-4Q^2v$.

This equation is of the form $f_1(u,P)=f_2(v,Q)$.

Let $4P^2u=1-4Q^2v=a$ (say).

$\therefore P=\pm\frac{1}{2}\sqrt{\frac{a}{u}}$, $Q=\pm\frac{1}{2}\sqrt{\frac{1-a}{v}}$.

$\therefore dz=\frac{\partial z}{\partial u}du+\frac{\partial z}{\partial v}dv=Pdu+Qdv=\pm\frac{\sqrt{a}}{2}\frac{du}{\sqrt{u}}\pm\frac{\sqrt{1-a}}{2}\frac{dv}{\sqrt{v}}$.

Integrating on both sides, we get

$z=\pm\sqrt{a}\sqrt{u}\pm\sqrt{1-a}\sqrt{v}+b \Rightarrow z=\pm\sqrt{a(x+y)}\pm\sqrt{(1-a)(x-y)}+b$,

which is the required complete solution.

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Home Assignments

Q.No.1.: Solve the following partial differential equation: $pq+p+q=0$.

Ans.: $z=ax-\frac{ay}{(1+a)}+b$.

Q.No.2.: Solve the following partial differential equation: $p^3 - q^3 = 0$.

Ans.:

Q.No.3.: Solve the following partial differential equation: $z = p^2 + q^2$.

Ans.: $4z\left(1 + a^2\right) = (x + ay + b)^2$.

Q.No.4.: Solve the following partial differential equation: $z^2 = 1 + p^2 + q^2$.

Ans.: $z = \cosh\left\{\dfrac{x + ay + b}{\sqrt{\left(1 + a^2\right)}}\right\}$.

Q.No.5.: Solve the following partial differential equation: $yp + xq + pq = 0$.

Ans.: $2z = ay^2 - \left[\dfrac{a}{(a+1)}\right]x^2 + b$.

Q.No.6.: Solve the following partial differential equation: $p + q = \sin x + \sin y$.

Ans.: $z = a(x - y) - (\cos x + \cos y) + b$.

Q.No.7.: Solve the following partial differential equation: $\sqrt{p} + \sqrt{q} = x + y$.

Ans.: $3z = (x + a)^3 + (y - a)^3 + b$.

Q.No.8.: Solve the following partial differential equation: $\left(p^2 - q^2\right)z = x - y$.

Ans.:

Q.No.9.: Solve the following partial differential equation: $z = px + qy - 2\sqrt{(pq)}$

Ans.: $z = ax + by - 2\sqrt{(ab)}$.

Q.No.10.: Solve the following partial differential equation: $p^2 + q^2 = 1$.

Ans.: $z = ax + \sqrt{\left(1 - a^2\right)}y + c$.

Q.No.11.: Solve the following partial differential equation: $p^2 + q^2 = x^2 + y^2$.

Ans.: $z = \dfrac{a^2}{2}\sinh^{-1}\dfrac{x}{a} + \dfrac{x_1\sqrt{\left(x^2 + a^2\right)}}{2} + \dfrac{y\sqrt{\left(y^2 - a^2\right)}}{2} - \dfrac{a^2}{2}\cosh^{-1}\dfrac{y}{a} + b$.

Q.No.12.: Solve the following partial differential equation: $\dfrac{d^3y}{dx^3} + y = 3 + 5e^x$.

Ans.: $y = c_1 e^{-x} + e^{\frac{1}{2}x}\left(c_2 \cos\frac{\sqrt{3}}{2}x + c_3 \sin\frac{\sqrt{3}}{2}x\right) + 3 + \frac{5}{2}e^{x}$.

Q.No.13.: Solve the following partial differential equation: $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = -2\cosh x$.

Ans.: $y = e^{-2x}(c_1 \cos x + c_2 \sin x) - \frac{1}{10}e^{x} - \frac{1}{2}e^{-x}$.

Q.No.14.: Solve the following partial differential equation: $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = \sin 3x$.

Ans.: $y = e^{x}(c_1 \cos 2x + c_2 \sin 2x) + \frac{1}{26}(3\cos 3x - 2\sin 3x)$.

Q.No.15.: Solve the following partial differential equation:

$$\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} - \frac{dy}{dx} + y = \sin 2x .$$

Ans.: $y = c_1 e^{-x} + c_2 \cos x + c_3 \sin x + \frac{1}{15}(2\cos 2x - \sin 2x)$.

Q.No.16.: Solve the following partial differential equation: $\frac{d^3y}{dx^3} + y = \sin 3x - \cos^2\frac{x}{2}$.

Ans.:

$$y = c_1 e^{-x} + e^{\frac{1}{2}x}\left(c_2 \cos\frac{\sqrt{3}}{2}x + c_3 \sin\frac{\sqrt{3}}{2}x\right) + \frac{1}{730}(\sin 3x + 27\cos 3x) - \frac{1}{2} - \frac{1}{4}(\cos x - \sin x)$$

Q.No.17.: Solve the following partial differential equation:

$$\left(D^2 - 4D + 3\right)y = \sin 3x \cos 2x .$$

Ans.: $y = c_1 e^{-x} + c_2 e^{3x} + \frac{1}{884}(10\cos 5x + 11\sin x) + \frac{1}{20}(\sin x + 2\cos x)$.

Q.No.18.: Solve the following partial differential equation:

$$\left(D^2 - 3D + 2\right)y = 6e^{-3x} + \sin 2x .$$

Ans.: $y = c_1 e^{x} + c_2 e^{2x} + \frac{3}{10}e^{-3x} + \frac{1}{20}(3\cos 2x - \sin 2x)$.

Q.No.19.: Solve the following partial differential equation: $\frac{d^2y}{dx^2} + 4y = e^{x} + \sin 2x$.

Ans.: $y = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{5}e^x - \frac{x}{4}\cos 2x$.

Q.No.20.: Solve the following partial differential equation:

$$\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} + 4\frac{dy}{dx} = e^{2x} + \sin 2x.$$

Ans.: $y = c_1 + e^x\left(c_2 \cos\sqrt{3x} + c_3 \sin\sqrt{3x}\right) + \frac{1}{8}\left(e^{2x} + \sin 2x\right)$.

Q.No.21`.: Solve the following partial differential equation: $\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} - 6\frac{dy}{dx} = 1 + x^2$.

Ans.: $y = c_1 + c_2e^{3x} + c_3e^{-2x} - \frac{1}{18}\left(x^3 - \frac{x^2}{2} + \frac{25}{6}x\right)$.

Q.No.22.: Solve the following partial differential equation:

$$\frac{d^2y}{dx^2} + y = e^{2x} + \cosh 2x + x^3.$$

Ans.: $y = c_1 \cos x + c_2 \sin x + \frac{1}{5}e^{2x} + \frac{1}{5}\cosh 2x + x^3 - 6x$.

Q.No.23.: Solve the following partial differential equation:

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = xe^{3x} + \sin 2x.$$

Ans.: $y = c_1e^x + c_2e^{2x} + \frac{1}{4}e^{3x}(2x-3) + \frac{1}{20}(3\cos 2x - \sin 2x)$.

Q.No.24.: Solve the following partial differential equation: $\left(D^2 - 2D\right)y = e^x \sin x$.

Ans.: $y = c_1 + c_2e^{2x} - \frac{1}{2}e^x \sin x$.

QNo.25.: Solve the following partial differential equation:

$$\frac{d^2y}{dx^2} + 2y = x^2e^{3x} + e^x \cos 2x.$$

Ans.: $y = c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x + \frac{e^{3x}}{11}\left(x^2 - \frac{12}{11}x + \frac{50}{121}\right) + \frac{e^x}{17}(4\sin 2x - \cos 2x)$.

QNo.26.: Solve the following partial differential equation:

$$\left(D^3 + 2D^2 + D\right)y = x^2e^{2x} + \sin^2 x .$$

Ans.: $y = c_1 + (c_2 + c_3x)e^x + \dfrac{x}{2} + \dfrac{e^{2x}}{18}\left(x^2 - \dfrac{7}{3}x + \dfrac{11}{6}\right) + \dfrac{1}{100}(3\sin 2x + 4\cos 2x)$.

QNo.27.: Solve the following partial differential equation:

$$\left(D^2 - 4D + 4\right)y = 8x^2e^{2x}\sin 2x .$$

Ans.: $y = (c_1 + c_2x)e^{2x} - e^{2x}\left[4\cos 2x + \left(2x^2 - 3\right)\sin 2x\right]$.

QNo.28.: Solve the following partial differential equation:

$$(D-1)^2(D+1)^2 y = \sin^2\frac{x}{2} + e^x + x .$$

Ans.: $y = (c_1 + c_2x)e^x + (c_3 + c_4x)e^{-x} + \dfrac{1}{2} - \dfrac{1}{8}\cos x + \dfrac{x^2}{8}e^x + x$.

QNo.29.: Solve the following partial differential equation: $\dfrac{d^2y}{dx^2} + a^2y = \sec ax$.

Ans.: $y = c_1\cos ax + c_2\sin ax + \dfrac{1}{a}\left(x\sin ax + \cos ax\dfrac{\log\cos ax}{a}\right)$.

5th Topic

Partial Differential Equations

Method for finding the complete integral

of a non-linear partial differential equation

(Charpit's Method)

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5th Topic

Partial Differential Equations

Method for finding the complete integral of a non-linear partial differential equation

(Charpit's Method)

Prepared by:
Prof. Sunil
Department of Mathematics & Scientific Computing
NIT Hamirpur (HP)

CHARPIT'S METHOD:

Charpit's method is a general method for finding the complete solution of non-linear partial differential equation of the first order of the form

$$f(x, y, z, p, q) = 0. \tag{i}$$

Since we know that $dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy = pdx + qdy$. (ii)

Integrating (ii), we get the **complete solution** of (i).

Note: In order to integrate (ii), we must know p and q in terms of x, y, z.

For this purpose, introduce another non-linear partial differential equation of the first order of the form

$$F(x, y, z, p, q) = 0, \tag{iii}$$

Solving (i) and (iii), we get

$$p = p(x, y, z, a),\ q = q(x, y, z, b). \tag{iv}$$

Let (iii) be the relation such that when the values of p and q derived from it and the given equation (i) are substituted in (ii), it becomes integrable.

On substitution of (iv) in (ii), equation (ii) becomes integrable, resulting in the complete solution of (i) in the form

$$\phi(x, y, z, a, b) = 0, \tag{v}$$

containing two arbitrary constants a and b.

To determine F: We differentiate (i) and (iii) partially w. r. t. x and y. Thus

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}.p + \frac{\partial f}{\partial p}.\frac{\partial p}{\partial x} + \frac{\partial f}{\partial q}.\frac{\partial q}{\partial x} = 0, \tag{vi}$$

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}.p + \frac{\partial F}{\partial p}.\frac{\partial p}{\partial x} + \frac{\partial F}{\partial q}.\frac{\partial q}{\partial x} = 0, \tag{vii}$$

$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}.q + \frac{\partial f}{\partial p}.\frac{\partial p}{\partial y} + \frac{\partial f}{\partial q}.\frac{\partial q}{\partial y} = 0, \tag{viii}$$

$$\frac{\partial F}{\partial y} + \frac{\partial F}{\partial z}.q + \frac{\partial F}{\partial p}.\frac{\partial p}{\partial y} + \frac{\partial F}{\partial q}.\frac{\partial q}{\partial y} = 0. \tag{ix}$$

Eliminating $\frac{\partial p}{\partial x}$ between (vi) and (vii), we get

$$\left(\frac{\partial f}{\partial x}.\frac{\partial F}{\partial p} - \frac{\partial F}{\partial x}.\frac{\partial f}{\partial p}\right) + \left(\frac{\partial f}{\partial z}.\frac{\partial F}{\partial p} - \frac{\partial F}{\partial z}.\frac{\partial f}{\partial p}\right).p + \left(\frac{\partial f}{\partial q}.\frac{\partial F}{\partial p} - \frac{\partial F}{\partial q}.\frac{\partial f}{\partial p}\right)\frac{\partial q}{\partial x} = 0. \tag{x}$$

Eliminating $\frac{\partial q}{\partial y}$ between (viii) and (ix), we get

$$\left(\frac{\partial f}{\partial y}.\frac{\partial F}{\partial q} - \frac{\partial F}{\partial y}.\frac{\partial f}{\partial q}\right) + \left(\frac{\partial f}{\partial z}.\frac{\partial F}{\partial q} - \frac{\partial F}{\partial z}.\frac{\partial f}{\partial q}\right).q + \left(\frac{\partial f}{\partial p}.\frac{\partial F}{\partial q} - \frac{\partial F}{\partial p}.\frac{\partial f}{\partial q}\right)\frac{\partial p}{\partial y} = 0. \tag{xi}$$

Since $\frac{\partial q}{\partial x} = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial p}{\partial y}$ and the last term in (x) and (xi) differ in sign only, then adding (x) and (xi), we get

$$\left(\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}\right)\frac{\partial F}{\partial p} + \left(\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}\right)\frac{\partial F}{\partial q} + \left(-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}\right)\frac{\partial F}{\partial z} + \left(-\frac{\partial f}{\partial p}\right)\frac{\partial F}{\partial x} + \left(-\frac{\partial f}{\partial q}\right)\frac{\partial F}{\partial y} = 0, \tag{xii}$$

which is the **linear** partial differential equation (Lagrange's linear equation) of the first order with x, y, z, p, q as independent variables and F as the dependent variable.

$\therefore$ The auxiliary equations of (xii) are

$$\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}}=\frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}}=\frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}}=\frac{dx}{-\frac{\partial f}{\partial p}}=\frac{dy}{-\frac{\partial f}{\partial q}}=\frac{dF}{0}. \qquad \text{(xiii)}$$

These equations (xiii) are known as **Charpit's auxiliary equations**.

Any integral of (xiii) will satisfy (xii). Take the simplest relation involving at least one of p and q for F = 0. From f = 0 and F = 0, find the values of p and q and substitute in

$$dz = pdx + qdy$$

which on integration gives the solution.

Solving (xiii), we get relations (iv) of p and q, using which, the equation (ii) is integrated resulting in the complete solution (v).

Note: All the equations of Charpit's equations (xiii) need NOT be used. Choose the simplest of (xiii), so that p and q are easily obtained.

Now let us solve complete solution of non-linear partial differential equation of the first order by Charpit's method:

Q.No.1.: Solve the following non-linear partial differential equation by Charpit's method: $2zx - px^2 - 2qxy + pq = 0$.

Sol.: Given non-linear partial differential equation is $f = 2zx - px^2 - 2qxy + pq = 0$. (i)

$$\therefore \frac{\partial f}{\partial x}=2z-2px-2qy,\ \frac{\partial f}{\partial y}=-2qx,\ \frac{\partial f}{\partial z}=2x,\ \frac{\partial f}{\partial p}=-x^2+q,\ \frac{\partial f}{\partial q}=-2xy+p.$$

Charpit's auxiliary equations are

$$\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}}=\frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}}=\frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}}=\frac{dx}{-\frac{\partial f}{\partial p}}=\frac{dy}{-\frac{\partial f}{\partial q}}.$$

$$\Rightarrow \frac{dp}{2z-2qy}=\frac{dq}{0}=\frac{dz}{px^2-2pq+2qxy}=\frac{dx}{x^2-q}=\frac{dy}{2xy-p}.$$

From second member, we get q = a.

Putting $q = a$ in (i), we get $p = \frac{2x(z-ay)}{x^2 - a}$.

Since we know that $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = pdx + qdy$.

$\therefore dz = pdx + qdy = \frac{2x(z-ay)}{x^2-a}dx + ady \Rightarrow \frac{dz - ady}{z - ay} = \frac{2x}{x^2 - a}dx$.

Integrating on both sides, we get

$\log(z - ay) = \log(x^2 - a) + \log b \Rightarrow z - ay = b(x^2 - a)$.

$\Rightarrow z = ay + b(x^2 - a)$,

which is the required **complete solution** involving two arbitrary constants a and b.

Q.No.2.: Solve the following non-linear partial differential equation by Charpit's method: $(p^2 + q^2)y = qz$ or $qz - p^2y - q^2y = 0$.

Sol.: Given non-linear partial differential equation is $f = (p^2 + q^2)y - qz = 0$. (i)

$\therefore \frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = p^2 + q^2$, $\frac{\partial f}{\partial z} = -q$, $\frac{\partial f}{\partial p} = 2py$, $\frac{\partial f}{\partial q} = 2qy - z$.

Charpit's auxiliary equations are

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}}.$$

$$\Rightarrow \frac{dp}{-pq} = \frac{dq}{p^2} = \frac{dz}{-qz} = \frac{dx}{-2py} = \frac{dy}{-2qy + z}.$$

From the first two members, we get $pdp + qdq = 0$.

Integrating, we get $p^2 + q^2 = a^2 \Rightarrow p = \sqrt{a^2 - q^2}$. (ii)

Putting $p^2 + q^2 = a^2$ in (i), we get $q = \frac{a^2y}{z}$.

$\therefore$ From (ii), we get $p = \sqrt{a^2 - q^2} = \sqrt{a^2 - \frac{a^4y^2}{z^2}} = \frac{a}{z}\sqrt{z^2 - a^2y^2}$.

Since we know that $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = pdx + qdy$.

$\therefore dz = pdx + qdy = \frac{a}{z}\sqrt{z^2 - a^2y^2}dx + \frac{a^2y}{z}dy$.

$\Rightarrow zdz - a^2ydy = a\sqrt{z^2 - a^2y^2}dx \Rightarrow \dfrac{\frac{1}{2}d\left(z^2 - a^2y^2\right)}{\sqrt{z^2 - a^2y^2}} = adx$.

Integrating on both sides, we get

$\sqrt{z^2 - a^2y^2} = ax + b \Rightarrow z^2 = (ax + b)^2 + a^2y^2$,

which is the required **complete solution** involving two arbitrary constants a and b.

Q.No.3.: Solve the following non-linear partial differential equation by Charpit's

method: $2z + p^2 + qy + 2y^2 = 0$.

Sol.: Given non-linear partial differential equation is $f = 2z + p^2 + qy + 2y^2 = 0$. (i)

Charpit's auxiliary equations are

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}}.$$

$$\Rightarrow \frac{dp}{2p} = \frac{dq}{4y + 3q} = \frac{dz}{-\left(2p^2 + qy\right)} = \frac{dx}{-2p} = \frac{dy}{-y}.$$

From first and fourth ratios, we get $dp = -dx \Rightarrow p = -x + a$.

Substituting $p = a - x$ in (i), we get

$q = \frac{1}{y}\left[-2z - 2y^2 - (a - x)^2\right]$.

Since we know that $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = pdx + qdy$.

$\therefore dz = pdx + qdy = (a - x)dx - \frac{1}{y}\left[2z + 2y^2 + (a - x)^2\right]dy$.

Multiplying both sides by $2y^2$, we get

$2y^2dz + 4yzdy = 2y^2(a - x)dx - 4y^3dy - 2y(a - x)^2dy$

Integrating on both sides, we get

$2zy^2 = -\left[y^2(a-x)^2 + y^4\right] + b \Rightarrow y^2\left[(x-a)^2 + 2z + y^2\right] = b$,

which is the required **complete solution** involving two arbitrary constants a and b.

Q.No.4.: Solve the following non-linear partial differential equation by Charpit's method: $z = p^2x + q^2y$.

Sol.: Given non-linear partial differential equation is $f = p^2x + q^2y - z = 0$. (i)

Charpit's auxiliary equations are

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}}.$$

$$\Rightarrow \frac{dp}{-p+p^2} = \frac{dq}{-q+q^2} = \frac{dz}{-2\left(p^2x+q^2y\right)} = \frac{dx}{-2px} = \frac{dy}{-2pq}.$$

From which, we have $\dfrac{p^2dx + 2pxdp}{p^2x} = \dfrac{q^2dy + 2qydq}{q^2y}$.

Integrating on both sides, we get $\log\left(p^2x\right) = \log\left(q^2y\right) + \log a \Rightarrow p^2x = aq^2y$. (ii)

From (i) and (ii), we have $aq^2y + q^2y = z \Rightarrow q = \left[\dfrac{z}{(1+a)y}\right]^{1/2}$.

From (ii), we have $p = \left[\dfrac{az}{(1+a)x}\right]^{1/2}$.

Since we know that $dz = \dfrac{\partial z}{\partial x}dx + \dfrac{\partial z}{\partial y}dy = pdx + qdy$.

$$\therefore dz = pdx + qdy = \left[\frac{az}{(1+a)x}\right]^{1/2}dx + \left[\frac{z}{(1+a)y}\right]^{1/2}dy \Rightarrow \sqrt{(1+a)}\frac{dz}{\sqrt{z}} = \sqrt{a}\frac{dx}{\sqrt{x}} + \frac{dy}{\sqrt{y}}.$$

Integrating on both sides, we get

$$\sqrt{\{(1+a)z\}} = \sqrt{(ax)} + \sqrt{y} + b$$

$$\Rightarrow z = \frac{\left[\sqrt{ax} + \sqrt{y} + b\right]^2}{(1+a)}, \text{ Ans.}$$

which is the required **complete solution** involving two arbitrary constants a and b.

Q.No.5.: Solve the following non-linear partial differential equations by Charpit's method: $pxy + pq + qy = yz$.

Sol. Given non-linear partial differential equation is $f = pxy + pq + qy - yz = 0$. (i)

Charpit's auxiliary equations are

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}}.$$

$$\Rightarrow \frac{dp}{py + p(-y)} = \frac{dq}{(px+q)+qp} = \frac{dz}{-p(xy+q) - q(p+y)} = \frac{dx}{-(xy+q)} = \frac{dy}{-(p+y)}.$$

$$\Rightarrow \frac{dp}{0} = \frac{dq}{(px+q)+qp} = \frac{dz}{-p(xy+q) - q(p+y)} = \frac{dx}{-(xy+q)} = \frac{dy}{-(p+y)}.$$

From first member, we get $dp = 0 \Rightarrow p = a$.

Putting $p = a$ in (i), we get

$$axy + aq + qy = yz \Rightarrow q(a+y) = y(z - ax) \Rightarrow q = \frac{y(z-ax)}{a+y}.$$

Since we know that $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = pdx + qdy$.

$$\therefore dz = pdx + qdy = adx + \frac{y(z-ax)}{a+y}dy \Rightarrow \frac{dz - adx}{z - ax} = \frac{ydy}{a+y} \Rightarrow \frac{dz - adx}{z-ax} = \left(1 - \frac{a}{a+y}\right)dy.$$

Integrating on both sides, we get

$\log(z - ax) = y - a\log(a+y) + b$, Ans.

which is the required **complete solution** involving two arbitrary constants a and b.

Q.No.6.: Solve the following non-linear partial differential equations by Charpit's method: $z^2 = pqxy$.

Sol.: Given non-linear partial differential equation is $f = z^2 - pqxy = 0$. (i)

Charpit's auxiliary equations are

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}}$$

$$\Rightarrow \frac{dp}{-(f_x + pf_z)} = \frac{dq}{-(f_y + qf_z)} = \frac{dz}{pf_p + qf_q} = \frac{dx}{df_p} = \frac{dy}{df_q}$$

$$\Rightarrow \frac{dp}{-(-pqy + 2pz)} = \frac{dq}{-(-pqx + 2qz)} = \frac{dz}{-2pqxy} = \frac{dx}{-qxy} = \frac{dy}{-pxy}$$

Using the multipliers p, q, o, x, y, we have

$$\frac{pdx + xdp}{-pqxy + xpqy - 2pxz} = \frac{qdy + ydq}{-qpxy + ypqx - 2yqz}$$

$$\Rightarrow \frac{pdx + xdp}{-2xpz} = \frac{qdy + ydq}{-2yqz} \Rightarrow \frac{d(xp)}{(xp)} = \frac{d(yq)}{(yq)}.$$

Integrating on both sides, we get $xp = a\, yq \Rightarrow q = \frac{xp}{ay}$.

Substituting $q = \frac{xp}{ay}$ in (i) i.e. $z^2 = pqxy$, we get

$$z^2 = p.\left(\frac{xp}{ay}\right)xy = \frac{p^2x^2}{a} \Rightarrow p = \sqrt{a}.\frac{z}{x}.$$

Then $q = \frac{xp}{ay} = \frac{x}{ay}.\sqrt{a}.\frac{z}{x} = \frac{z}{\sqrt{a}y}$.

Since we know that $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = pdx + qdy$.

$$\therefore dz = pdx + qdy = \sqrt{a}\frac{z}{x}dx + \frac{1}{\sqrt{a}}\frac{z}{y}dy$$

$$\Rightarrow \frac{dz}{z} = \sqrt{a}\frac{dx}{x} + \frac{1}{\sqrt{a}}\frac{dy}{y}.$$

Integrating on both sides, we get

$z = ax^b y^{1/b}$, Ans.

which is the required **complete solution** involving two arbitrary constants a and b.

Q.No.7.: Solve the following non-linear partial differential equations by Charpit's method: $2(z + px + qy) = yp^2$.

Ans.: Given non-linear partial differential equation is $f = 2(z + px + qy) - yp^2 = 0$. (i)

Charpit's auxiliary equations are

$$\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}}=\frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}}=\frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}}=\frac{dx}{-\frac{\partial f}{\partial p}}=\frac{dy}{-\frac{\partial f}{\partial q}}.$$

$$\Rightarrow \frac{dp}{-(2p+2p)}=\frac{dq}{-\left(2q-p^2+2q\right)}=\frac{dz}{2xp-2yp^2+2qy}=\frac{dx}{2x-2yp}=\frac{dy}{2y}$$

$$\Rightarrow \frac{dp}{-2p}=\frac{dq}{-\left(2q-\frac{p^2}{2}\right)}=\frac{dz}{xp-yp^2+yq}=\frac{dx}{x-yp}=\frac{dy}{y}$$

Using first and fifth members, we have

$$\frac{dy}{y}=\frac{dp}{-2p}\Rightarrow p=ay^{-2}=\frac{a}{y^2}.$$

Substituting the value of p (i) i.e. $2(z+px+qy)=yp^2$, we obtain

$$2yq=y\left(\frac{a}{y^2}\right)^2-2z-2x\left(\frac{a}{y^2}\right)\Rightarrow q=\frac{a^2}{2y^4}-\frac{z}{y}-\frac{ax}{y^3}.$$

Since we know that $dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy=pdx+qdy$.

$$\therefore dz=pdx+qdy=\frac{a}{y^2}dx+\left(\frac{a^2}{2y^4}-\frac{z}{y}-\frac{ax}{y^3}\right)dy.$$

Regrouping the terms, we get

$$\left(\frac{ydz+zdy}{y}\right)=\left(\frac{aydx-axdy}{y^3}\right)+\frac{a^2}{2y^4}dy.$$

Multiplying throughout by y, we obtain

$$d(yz)=ad\left(\frac{x}{y}\right)+\frac{a^2}{2}\frac{dy}{y^3}.$$

Integrating on both sides, we get $yz=a\frac{x}{y}+\frac{a^2}{2}.\left(\frac{1}{-2y^2}\right)+b$.

$$\Rightarrow z=\frac{ax}{y^2}-\frac{a^2}{4y^3}+\frac{b}{y}, \text{ Ans.}$$

which is the required **complete solution** involving two arbitrary constants a and b.

Q.No.8.: Solve the following non-linear partial differential equations by Charpit's method: $px + qy = pq$.

Ans.: Given non-linear partial differential equation is $f \equiv px + qy - pq = 0$. (i)

Charpit's auxiliary equations are

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}}$$

$$\Rightarrow \frac{dp}{p} = \frac{dq}{q} = \frac{dz}{-p(x-q) - q(y-p)} = \frac{dx}{-(x-q)} = \frac{dy}{-(y-p)}.$$

Taking first two members, we have

$$\frac{dp}{p} = \frac{dq}{q}.$$

Integrating on both sides, $\log p = \log q + \log a \Rightarrow p = aq$. (ii)

Putting $p = aq$ in (i), we have

$$aqx + qy = aq^2 \Rightarrow q = \frac{y + ax}{a}.$$

From (ii), we obtain $p = aq = y + ax$

Since we know that $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = pdx + qdy$.

$$\therefore dz = (y + ax)dx + \frac{(y + ax)}{a}dy \Rightarrow adz = (y + ax)(dy + adx)$$

Integrating on both sides, we get

$$az = \frac{1}{2}(y + ax)^2 + b,$$

which is the required **complete solution** involving two arbitrary constants a and b.

General Integral: Writing $b = \phi(a)$, we have

$$az = \frac{1}{2}(y + ax)^2 + \phi(a) \qquad \text{(iii)}$$

Differentiating (iii) partially w.r.t. a, we have

$$z = x(y + ax) + \phi'(a). \qquad \text{(iv)}$$

General integral is obtained by eliminating a from (iii) and (iv).

Singular Integral: Differentiating the complete integral partially w.r.t. a and b, we have

$z = x(y + ax)$ and $0 = 1$. Hence there is no singular integral.

Q.No.9.: Solve the following non-linear partial differential equations by Charpit's method: $2(xy - px - qy) + p^2 + q^2 = 0$.

Sol.: Given non-linear partial differential equation is $2(xy - px - qy) + p^2 + q^2 = 0$.

Here $f \equiv p^2 + q^2 - 2px - 2qy + 2xy = 0$ (i)

Charpit's auxiliary equations are

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}}$$

$$\Rightarrow \frac{dp}{-2p + 2y} = \frac{dq}{-2q + 2x} = \frac{dx}{2x - 2p} = \frac{dy}{2y - 2q}$$

$$\Rightarrow \frac{dp}{-p + y} = \frac{dq}{-q + x} = \frac{dx}{x - p} = \frac{dy}{y - q}$$

$$\Rightarrow \frac{dp + dq}{x + y - p - q} = \frac{dx + dy}{x + y - p - q}$$

$\Rightarrow dp + dq = dx + dy$

$\therefore (p - x) + (q - y) = a$ (ii)

$dp + dq = dx + dy$, $(p - x) + (q - y) = 0$

Equation (i) can be written as

$(p - x)^2 + (q - y)^2 = (x - y)^2$ (iii)

Putting the values of $(q - y)$ from (ii) in (iii), we have

$$(p - x)^2 + [a - (p - x)]^2 = (x - y)^2 \Rightarrow 2(p - x)^2 - 2a(p - x) + \{a^2 - (x - y)^2\} = 0$$

$$p - x = 2a + \frac{\sqrt{[4a^2 - 8\{a^2 - (x - y)^2\}]}}{4}, \text{ (Taking only +ve sign)}$$

$$\Rightarrow p = x + \frac{1}{2}\left[a + \sqrt{\{2(x - y)^2 - a^2\}}\right]$$

$$\therefore \text{ From (ii), } q - y = a - \frac{1}{2}\left[a + \sqrt{\{2(x - y)^2 - a^2\}}\right]$$

$q = y + \frac{1}{2}\left[a - \sqrt{\left\{2(x-y)^2 - a^2\right\}}\right]$.

Since we know that $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = pdx + qdy$.

$$\therefore dz = xdx + ydy + \frac{a}{2}(dx + dy) + \frac{1}{2}\sqrt{\left\{2(x-y)^2 - a^2\right\}}(dx - dy)$$

$$= xdx + ydy + \frac{a}{2}(dx + dy) + \frac{1}{\sqrt{2}}\sqrt{\left\{(x-y)^2 - \frac{a^2}{2}\right\}}(dx - dy)$$

Integrating on both sides, we have

$$z = \frac{x^2}{2} + \frac{y^2}{2} + \frac{a}{2}(x+y) + \frac{1}{\sqrt{2}}\left[\frac{x-y}{2}\sqrt{\left\{(x-y)^2 - \frac{a^2}{2}\right\}} - \frac{a^2}{4}\log\left\{(x-y) + \sqrt{\left((x-y)^2 - \frac{a^2}{2}\right)}\right\}\right] + b$$

which is the required **complete solution** involving two arbitrary constants a and b.

Q.No.10.: Solve the following non-linear partial differential equations by Charpit's method: $z = px + qy + p^2 + q^2$

Sol.: Given non-linear partial differential equation is $f \equiv z - px - qy - p^2 - q^2 = 0$. (i)

Charpit's auxiliary equations are

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}}$$

$$\Rightarrow \frac{dp}{-p+p} = -\frac{dq}{-q+q} = \frac{dz}{-p(-x-2p) - q(-y-2q)} = \frac{dx}{-(-x-2p)} = \frac{dy}{(-y-2q)}$$

$$\Rightarrow \frac{dp}{0} = -\frac{dq}{0} = \frac{dz}{p(x+2p) + q(y+2q)} = \frac{dx}{(x+2p)} = \frac{dy}{-(y+2q)}$$

From first two members, we get $dp = 0$ and $dq = 0$.

Integrating, we obtain p = a and q = b.

Putting in (i), we get

$$z = ax + by + a^2 + b^2,$$

which is the required **complete solution** involving two arbitrary constants a and b.

Q.No.11.: Solve the following non-linear partial differential equations by Charpit's method: $z^2\left(p^2z^2 + q^2\right) = 1$

Sol.: Given non-linear partial differential equation is $z^2\left(p^2z^2+q^2\right)=1$.

Here $f \equiv p^2z^4+q^2z^2-1=0$.

Charpit's auxiliary equations are

$$\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}}=\frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}}=\frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}}=\frac{dx}{-\frac{\partial f}{\partial p}}=\frac{dy}{-\frac{\partial f}{\partial q}}$$

$$\Rightarrow \frac{dp}{0+p.\left(4p^2z^3+2q^2z\right)}=\frac{dq}{q\left(4p^2z^3+2q^2z\right)}=\frac{dz}{-p\left(2pz^4\right)-q\left(2qz^2\right)}=\frac{dx}{-2pz^4}=\frac{dy}{-2qz^2}.$$

Taking first two members, we have $\frac{dp}{p}=\frac{dq}{q}$.

Integrating on both sides, we obtain $\log p=\log q+\log a \Rightarrow p=aq$.

Putting $p=aq$ in $z^2\left(p^2z^2+q^2\right)=1$, we get

$$q^2=\frac{1}{z^2\left(a^2z^2+1\right)}\Rightarrow q=\frac{1}{z\sqrt{\left(a^2z^2+1\right)}}$$

$$\therefore p=aq \Rightarrow p=\frac{a}{z\sqrt{\left(a^2z^2+1\right)}}.$$

Since we know that $dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy=pdx+qdy$.

$$\therefore dz=\frac{a}{z\sqrt{\left(a^2z^2+1\right)}}dx+\frac{1}{z\sqrt{\left(a^2z^2+1\right)}}dy \Rightarrow z\sqrt{\left(a^2z^2+1\right)}dz=adx+dy$$

Integrating on both sides, we get $\frac{1}{3a^2}\left(a^2z^2+1\right)^{3/2}=ax+y+b$

$\Rightarrow \left(a^2z^2+1\right)^3=9a^4(ax+y+b)^2$, Ans.

which is the required **complete solution** involving two arbitrary constants a and b.

Q.No.12.: Solve the following non-linear partial differential equations by Charpit's method: $p^2+q^2-2px-2qy+1=0$.

Sol.: Given non-linear partial differential equation is $f \equiv p^2+q^2-2px-2qy+1=0$. (i)

Charpit's auxiliary equations are

$$\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}}=\frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}}=\frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}}=\frac{dx}{-\frac{\partial f}{\partial p}}=\frac{dy}{-\frac{\partial f}{\partial q}}$$

$$\Rightarrow \frac{dp}{-2p}=\frac{dq}{-2q}=\frac{dz}{-p(2p-2x)-q(2q-2y)}=\frac{dx}{-(2p-2x)}=\frac{dy}{-(2q-2y)}$$

Taking the first two members, we have

$$\frac{dp}{p}=\frac{dq}{q}\Rightarrow \log p=\log q+\log a\Rightarrow p=aq.$$

Putting in (i), we get

$$a^2q^2+q^2-2aqx-2qy+1=0\Rightarrow \left(a^2+1\right)q^2-2(ax+y)q+1=0$$

$$\Rightarrow q=\frac{2(ax+y)+\sqrt{\left\{4(ax+y)^2-4\left(a^2+1\right)\right\}}}{2\left(a^2+1\right)}$$

$$\Rightarrow q=\frac{(ax+y)+\sqrt{\left\{(ax+y)^2-\left(a^2+1\right)\right\}}}{\left(a^2+1\right)}.\ \text{(Taking the positive sign only)}$$

and $p=aq=\dfrac{a\left[(ax+y)+\sqrt{\left\{(ax+y)^2-\left(a^2+1\right)\right\}}\right]}{a^2+1}$

Since we know that $dz=\dfrac{\partial z}{\partial x}dx+\dfrac{\partial z}{\partial y}dy=pdx+qdy$.

$$\therefore dz=\frac{(ax+y)+\sqrt{\left\{(ax+y)^2-\left(a^2+1\right)\right\}}}{\left(a^2+1\right)}(adx+dy)$$

Putting $ax+y=t$, so that $adx+dy=dt$, we have, $\left(a^2+1\right)dz=\left[t+\sqrt{\left\{t^2-\left(a^2+1\right)\right\}}\right]dt$.

Integrating on both sides, we get

$$\left(a^2+1\right)z=\frac{t^2}{2}+\frac{t}{2}\sqrt{\left\{t^2-\left(a^2+1\right)\right\}}-\frac{a^2+1}{2}\log\left[t+\sqrt{\left\{t^2-\left(a^2+1\right)\right\}}\right]+b,\ \text{where } t=ax+y,$$

which is the required **complete solution** involving two arbitrary constants a and b.

Q.No.13.: Solve the following non-linear partial differential equations by Charpit's method: $p=(qy+z)^2$.

Sol.: Given non-linear partial differential equation is $f\equiv -p+(qy+z)^2=0$. (i)

Charpit's auxiliary equations are

$$\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}}=\frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}}=\frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}}=\frac{dx}{-\frac{\partial f}{\partial p}}=\frac{dy}{-\frac{\partial f}{\partial q}}$$

$$\Rightarrow \frac{dp}{2p(qy+z)}=\frac{dq}{4q(qy+z)}=\frac{dz}{(-p)(-1)-q.2(qy+z).y}=\frac{dx}{-(-1)}=\frac{dy}{-2y(qy+z)}.$$

Taking first and fifth members, we have

$$\frac{dp}{p}+\frac{dy}{y}=0 \Rightarrow \log p+\log y=\log a \Rightarrow p=\frac{a}{y}.$$

$\therefore$ From (i), we obtain $\frac{a}{y}=(qy+z)^2 \Rightarrow \sqrt{\left(\frac{a}{y}\right)}=qy+z \Rightarrow q=\frac{\sqrt{a}}{y^{3/2}}-\frac{z}{y}$.

Since we know that $dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy=pdx+qdy$.

$$\therefore dz=\frac{a}{y}dx+\left(\frac{\sqrt{a}}{y^{3/2}}-\frac{z}{y}\right)dy \Rightarrow ydz+zdy=adx+\frac{\sqrt{a}}{\sqrt{y}}dy.$$

Integrating on both sides, we get

$$yz=ax+2\sqrt{(ay)}+b,$$

which is the required **complete solution** involving two arbitrary constants a and b.

Q.No.14.: Solve the following non-linear partial differential equations by Charpit's method: $px+qy=z(1+pq)^{1/2}$.

Sol.: Given non-linear partial differential equation is $f \equiv px+qy-z(1+pq)^{1/2}=0$. (i)

Charpit's auxiliary equations are

$$\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}}=\frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}}=\frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}}=\frac{dx}{-\frac{\partial f}{\partial p}}=\frac{dy}{-\frac{\partial f}{\partial q}}$$

$$\Rightarrow \frac{dp}{p-p(1+pq)^{1/2}}=\frac{dq}{q-q(1+pq)^{1/2}}=\ldots\ldots$$

Taking the first two members, we have

$$\frac{dp}{p}=\frac{dq}{q} \Rightarrow \log p=\log q+\log a \Rightarrow p=aq.$$

Putting in (i), we have

$$aqx + qy = z\left(1 + aq^2\right)^{1/2} \Rightarrow q^2(ax + y)^2 = z^2\left(1 + aq^2\right)$$

$$q^2 = \frac{z^2}{(ax + y)^2 - az^2} \Rightarrow q = \frac{z}{\sqrt{\left\{(ax + y)^2 - az^2\right\}}} \Rightarrow p = \frac{az}{\sqrt{\left\{(ax + y)^2 - az^2\right\}}}.$$

Since we know that $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = pdx + qdy$.

$$\therefore dz = \frac{(azdx + zdy)}{\sqrt{\left\{(ax + y)^2 - az^2\right\}}} \Rightarrow \frac{dz}{z} = \frac{adx + dy}{\sqrt{\left\{(ax + y)^2 - az^2\right\}}}$$

Putting $ax + y = \sqrt{a}u$, we get

$$\frac{dz}{z} = \frac{\sqrt{a}du}{\sqrt{\left\{\left(au^2 - az^2\right)\right\}}} \Rightarrow \frac{du}{dz} = \frac{1}{z}\sqrt{\left(u^2 - z^2\right)}.$$

Again put, $u = vz$,

so that $v + z\frac{dv}{dz} = \frac{1}{z}\sqrt{\left(v^2z^2 - z^2\right)} \Rightarrow v + z\frac{dv}{dz} = \sqrt{\left(v^2 - 1\right)}$

$$\Rightarrow z\frac{dv}{dz} = \sqrt{\left(v^2 - 1\right)} - v \Rightarrow \frac{dz}{z} = \frac{dv}{\sqrt{\left(v^2 - 1\right)} - v}$$

$$\Rightarrow \frac{dz}{z} = -\left\{\sqrt{\left(v^2 - 1\right)} + v\right\}dv.$$

Integrating on both sides, we get

$$\log z = -\left[\frac{v}{2}\sqrt{\left(v^2 - 1\right)} - \frac{1}{2}\log\left\{v + \sqrt{\left(v^2 - 1\right)}\right\}\right] - \frac{v^2}{2} + b$$

$$\Rightarrow \log z + \frac{v^2}{2} + \frac{v}{2}\sqrt{\left(v^2 - 1\right)} - \frac{1}{2}\log\left\{v + \sqrt{\left(v^2 - 1\right)}\right\} = b, \text{ where } v = \frac{u}{z} = \frac{ax + y}{z\sqrt{a}}.$$

which is the required **complete solution** involving two arbitrary constants a and b.

Q.No.15.: Solve the following non-linear partial differential equations by Charpit's method: $\left(x^2 - y^2\right)pq - xy\left(p^2 - q^2\right) - 1 = 0$.

Sol.: Here $f \equiv \left(x^2 - y^2\right)pq - xy\left(p^2 - q^2\right) - 1 = 0$. (i)

Charpit's auxiliary equations are

$$\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}}=\frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}}=\frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}}=\frac{dx}{-\frac{\partial f}{\partial p}}=\frac{dy}{-\frac{\partial f}{\partial q}}$$

$$\Rightarrow \frac{dp}{2pqx-y\left(p^2-q^2\right)}=\frac{dq}{-2ypq-x\left(p^2-q^2\right)}=\frac{dx}{-\left(x^2-y^2\right)q+2pxy}$$

$$=\frac{dy}{-\left(x^2-y^2\right)p-2qxy}=\ldots\ldots\ldots$$

Using x, y, p, q as multipliers, we have

Each fraction $=\dfrac{xdp+ydq+pdx+qdy}{0}$ $\quad\therefore (xdp+pdx)+(qdy+ydq)=0$.

Integrating, we get $px+qy=a \Rightarrow p=\dfrac{a-qy}{x}$.

$\therefore$ From (i), we have

$$\left(x^2-y^2\right)\left(\frac{a-qy}{x}\right)q-xy\left[\frac{(a-qy)^2}{x^2}-q^2\right]-1=0$$

$$\Rightarrow \frac{a-qy}{x}\left\{\left(x^2-y^2\right)q-(a-qy)y\right\}+xyq^2-1=0$$

$$\Rightarrow \frac{a-qy}{x}\left(x^2q-ay\right)+xyq^2-1=0$$

$$\Rightarrow (a-qy)\left(x^2q-ay\right)+x^2yq^2-x=0$$

$$\Rightarrow ax^2q-a^2y-x^2yq^2+ay^2q+x^2yq^2-x=0$$

$$\Rightarrow qa\left(x^2+y^2\right)=a^2y+x$$

$$\Rightarrow q=\frac{a^2y+x}{a\left(x^2+y^2\right)}$$

$$\therefore p=\frac{1}{x}\left[a-\frac{\left(a^2y+x\right)y}{a\left(x^2+y^2\right)}\right]=\frac{a^2x-y}{a\left(x^2+y^2\right)}.$$

Since we know that $dz=\dfrac{\partial z}{\partial x}dx+\dfrac{\partial z}{\partial y}dy=pdx+qdy$.

$$dz=\frac{\left(a^2x-y\right)dx+\left(a^2y+x\right)dy}{a\left(x^2+y^2\right)} \Rightarrow dz=a\frac{(xdx+ydy)}{x^2+y^2}+\frac{xdy-ydx}{a\left(x^2+y^2\right)}$$

Integrating on both sides, we have

$$z=\frac{a}{2}\log\left(x^2+y^2\right)+\frac{1}{a}\tan^{-1}\frac{y}{x}+b,$$

which is the required **complete solution** involving two arbitrary constants a and b.

Q.No.16.: Solve the following non-linear partial differential equation by Charpit's

method: $yz-p(xy+q)-qy=0$ or $pxy+pq+qy=yz$.

Sol.: Here $yz-p(xy+q)-qy=0$. (i)

Charpit's auxiliary equations are

$$\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}}=\frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}}=\frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}}=\frac{dx}{-\frac{\partial f}{\partial p}}=\frac{dy}{-\frac{\partial f}{\partial q}}.$$

Here $\frac{\partial f}{\partial x}=py$, $\frac{\partial f}{\partial p}=xy+q$, $\frac{\partial f}{\partial y}=px+q-z$, $\frac{\partial f}{\partial q}=p+q$, $\frac{\partial f}{\partial z}=-y$.

$$\Rightarrow\frac{dp}{py+p(-y)}=\frac{dq}{px+q-z+q(-y)}=\frac{dz}{-[p(xy+q)+q(p+y)]}=\frac{dx}{-(xy+q)}=\frac{dy}{-(p+q)}.$$

From 1^{st} and 2^{nd} member, we obtain

$$\frac{dp}{0}=\frac{dq}{px+q-z-qy}\Rightarrow dp=0\Rightarrow p=a.$$

Now from (i), $pxy+pq+qy-yz=0$

$$\Rightarrow axy+(a+y)q-yz=0\Rightarrow q=\frac{yz-axy}{a+y}=\frac{y(z-ax)}{a+y}\Rightarrow q=\frac{y(z-ax)}{a+y}.$$

Now consider $dz=pdx+qdy=adx+\frac{y(z-ax)}{a+y}dy$

$$\Rightarrow dz-adx=\frac{y(z-ax)a}{a+y}dy \tag{ii}$$

Put $z-ax=t$, $dz-adx=dtd$

$\therefore$ (ii) reduces to $dt=\frac{yt}{a+y}dy=\frac{a+y-a}{a+y}dy=\left(1-\frac{a}{a+y}\right)dy$

Integrating both sides, we get

$$\int\frac{1}{t}dt=\int dy-a\int\frac{1}{a+y}dy+\text{constant }(=\log c)$$

$\Rightarrow \log t = y - a\log(a+y) + \log c \qquad [t = z - ax]$

$\Rightarrow \log(z-ax) = y - a\log(a+y) + \log c$, is the required solution.

This solution can also be written as $(z-ax)(y+a^2) = be^y$.

Q.No.17.: Solve the following non-linear partial differential equation by Charpit's method: $q + xp = p^2$.

Sol.: Here $q + xp = p^2$. (i)

Charpit's auxiliary equations are

Let $f(x,y,z,p,q) = q + xp - p^2$

Here $\frac{\partial f}{\partial x} = p$, $\frac{\partial f}{\partial p} = x - 2p$ $\frac{\partial f}{\partial y} = 0$, $\frac{\partial f}{\partial q} = 1$, $\frac{\partial f}{\partial z} = 0$.

Charpit's auxiliary equations are

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}}$$

$$\Rightarrow \frac{dp}{p+0} = \frac{dq}{0 + q\times 0} = \frac{dz}{-[p(x-2p)+q]} = \frac{dx}{-x+2p} = \frac{dy}{-1}$$

From 1st and 5th members

$\frac{dp}{p} = -dy \Rightarrow \log p = -y + \log a \Rightarrow \log\frac{p}{a} = -y \Rightarrow \frac{p}{a} = e^{-y} \Rightarrow p = ae^{-y}$

Also from (i), $q + xp = p^2 \Rightarrow q + axe^{-y} = a^2xe^{-2y} \Rightarrow q = a^2e^{-2y} - axe^{-y}$.

Now consider $dz = pdx + qdy = ae^{-y}dx + (a^2e^{-2y} - axe^{-y})dy$

$dz = a(e^{-y}dx - xdy) + a^2e^{-2y}dy$

Integrating on both sides, we obtain $z = a\int d(e^{-y}x) + a^2\int e^{-2y}dy + c$

$\Rightarrow z = axe^{-y} - \frac{a^2}{2}e^{-2y} + c$, is the required solution.

Q.No.18.: Solve the following non-linear partial differential equation by Charpit's method: $p(p^2+1) + (b-z)q = 0$.

Sol.: Let $f(x,y,z,p,q) = p^3 + p + bq - zq = 0$

Here $\frac{\partial f}{\partial x}=0$, $\frac{\partial f}{\partial p}=3p^2+1$, $\frac{\partial f}{\partial y}=0$, $\frac{\partial f}{\partial q}=b-z$, $\frac{\partial f}{\partial z}=-q$.

Consider Charpit's Auxiliary equations

$$\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}}=\frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}}=\frac{dz}{-\left(p\frac{\partial f}{\partial p}+q\frac{\partial f}{\partial q}\right)}=\frac{dx}{-\frac{\partial f}{\partial p}}=\frac{dy}{-\frac{\partial f}{\partial q}}$$

$$\Rightarrow \frac{dp}{-pq}=\frac{dq}{-q^2}=.....\text{etc}$$

From 1st and 2nd members, $\frac{dp}{p}=\frac{dq}{q}$.

Integrating on both sides, we get

$\log p=\log q+\log c \Rightarrow p=cq$.

Now from (i) $p^3+p+(b-z)q=0 \Rightarrow c^3q^3+cq+(b-z)q=0$

$$\Rightarrow c^3q^2+c+b-z=0 \Rightarrow q^2=\frac{z-c-b}{c^3} \Rightarrow q=\frac{\sqrt{z-c-b}}{c.\sqrt{c}}.$$

Now consider $dz=pdx+qdy=\sqrt{z-c-b}.\frac{1}{\sqrt{c}}dx+\frac{1}{c^{3/2}}.\sqrt{z-c-b}dy$

$$\Rightarrow (z-c-b)^{-1/2}dz=\frac{1}{\sqrt{c}}+\frac{y}{c^{3/2}}dy.$$

Integrating on both sides, we get

$$\int (a-c-b)^{-1/2}dz=\frac{1}{\sqrt{c}}\int dx+\frac{1}{c^{3/2}}\int dy+a$$

$$\Rightarrow 2\sqrt{z-c-b}=\frac{x}{\sqrt{c}}+\frac{y}{c^{3/2}}+a,$$ is the required solution.

Q.No.19.: Solve the following non-linear partial differential equation by Charpit's method: $1+p^2=qz$.

Sol.: Let $f(x,y,z,p,q)=1+p^2-qz=0$

Here $\frac{\partial f}{\partial x}=0$, $\frac{\partial f}{\partial p}=2p$, $\frac{\partial f}{\partial y}=0$, $\frac{\partial f}{\partial q}=-z$, $\frac{\partial f}{\partial z}=-q$.

Consider Charpit's Auxiliary equations

$$\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}}=\frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}}=\frac{dz}{-\left(p\frac{\partial f}{\partial p}+q\frac{\partial f}{\partial q}\right)}=\frac{dx}{-\frac{\partial f}{\partial p}}=\frac{dy}{-\frac{\partial f}{\partial q}}$$

$$\Rightarrow \frac{dp}{-pq}=\frac{dq}{-q^2}=.....\text{etc}$$

From 1^{st} and 2^{nd} members, $\frac{dp}{p}=\frac{dq}{q}$.

Integrating on both sides, we get

$\log p=\log q+\log c \;\Rightarrow p=cq$.

Now from (i) $p^2+1-qz=0 \;\Rightarrow c^2q^2-qz=0 \Rightarrow q=\frac{z\pm\sqrt{z^2-4c^2}}{2c^2}$.

Now consider

$$dz=pdx+qdy=\frac{1}{2c}\left(z\pm\sqrt{z^2-4c^2}\right)dx+\frac{1}{2c^2}\left(z\pm\sqrt{z^2-4c^2}\right)dy$$

$$\Rightarrow \frac{dz}{z\pm\sqrt{z^2-4c^2}}=\frac{1}{2c}dx+\frac{1}{2c^2dy}$$

$$\Rightarrow \frac{1}{z\pm\sqrt{z^2-4c^2}}.\frac{z\mp\sqrt{z^2-4c^2}}{z\mp\sqrt{z^2-4c^2}}dz=\frac{1}{2c}dx+\frac{1}{2c^2}dy.$$

Integrating on both sides, we get

$$\int\frac{\left(z\mp\sqrt{z^2-4c^2}\right)dz}{4c^2}=\int\frac{1}{2c}dx+\int\frac{1}{2c^2}dy+b \qquad \left[(A+B)(A-B)=A^2-B^2\right]$$

$$\Rightarrow \frac{z^2}{2}\pm\left[\frac{z}{2}\sqrt{z^2-4c^2}-2c^2\log\left(z+\sqrt{z^2-4c^2}\right)\right]=4c^2\left(\frac{1}{2c}x+\frac{1}{2c^2}y+b\right)=2cx+2y+d,$$

where $d=4c^2b$,

$$\left[\because \int\sqrt{x^2-a^2}\,dx=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\log\sqrt{x^2-a^2}\right]$$

$$\Rightarrow \frac{z^2}{2}\pm\left[\frac{z}{2}\sqrt{z^2-4c^2}-2c^2\log\left(z+\sqrt{z^2-4c^2}\right)\right]=2cx+2y+d,$$ is the required solution.

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Home Assignments

Q.No.1.: Solve the following non-linear partial differential equation by Charpit's method: $16p^2z^2 + 9q^2z^2 + 4z^2 - 4 = 0$.

Hint:

$$\frac{dp}{32p^3z + 18pq^2z + 8pz} = \frac{dq}{32p^2qz + 18q^3z + 8qz} = \frac{-dx}{32pz^2} = \frac{-dy}{+18qz^2} = \frac{-dz}{32p^2z^2 + 18q^2z^2}$$

$4zdp + 0.dq + 1.dx + 0.dy + 4pdz = 0$, $x + 4pz = a$, $p = -\dfrac{x-a}{4z}$,

$q = \dfrac{2}{3z}\sqrt{1 - z^2 - \dfrac{1}{4}(x-a)^2}$.

Ans.: $\dfrac{(x-a)^2}{4} + \dfrac{(y-b)^2}{\frac{9}{4}} + z^2 = 1$.

Q.No.2.: Solve the following non-linear partial differential equation by Charpit's method: $p(1+q^2) + (b-z)q = 0$.

Hint: $\dfrac{dp}{pq} = \dfrac{dq}{q^2} = \dfrac{dz}{3pq^2 + p + (b-z)q} = \dfrac{dx}{q^2+1} = \dfrac{dy}{-z+b+2pq}$,

(i) (ii)$q = pc$, Sub $q = \sqrt{cz - b} - 1$.

Ans.: $2\sqrt{[c(z-b)-1]} = x + cy + a$; a, c are arbitrary constants.

Q.No.3.: Solve the following non-linear partial differential equation by Charpit's method: $q - px - q^2 = 0$.

Hint: $q = a$, $p = \dfrac{1}{2}\left[\mp x \pm \sqrt{x^2 + 4a}\right]$.

Ans.: $z = -\dfrac{x^2}{4} \pm \dfrac{1}{2}\left[\dfrac{x}{2}\sqrt{x^2+4a} + 2a\log\left\{x + \sqrt{x^2+4a}\right\}\right] + ay + b$.

Q.No.4.: Solve the following non-linear partial differential equation by Charpit's method: $yzp^2 - q = 0$.

Ans.: $z^2 = 2ax + a^2y^2 + b$.

Q.No.5.: Solve the following non-linear partial differential equation by Charpit's method: $2(pq + py + qx) + x^2 + y^2 = 0$.

Ans.: $2z = ax - x^2 + ay - y^2 + \frac{1}{2}(x - y)\sqrt{\{(x-y)^2 + a^2\}}$.

$$+\frac{a^2}{2^{3/2}}\log\left[\sqrt{\{2(x-y)\}} + \sqrt{\{2(x-y)^2 + a^2\}} + b\right]$$

Q.No.6.: Solve the following non-linear partial differential equation by Charpit's method: $q = 3p^2$.

Ans.: $z = ax + 3x^2y + b$.

Q.No.7.: Solve the following non-linear partial differential equation by Charpit's method: $z = pq$.

Ans.: $2\sqrt{z} = \sqrt{ax} + \frac{1}{\sqrt{a}}y + b$.

Q.No.8.: Solve the following non-linear partial differential equation by Charpit's method: $zpq = p + q$.

Ans.: $z^2 = 2(a+1)\left(x + \frac{y}{a}\right) + b$.

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6^{th} Topic

Partial Differential Equations

Homogeneous linear equations with constant co-efficients

(Complementary function and Particular Integral)

Prepared by:
Prof. Sunil
Department of Mathematics & Scientific Computing
NIT Hamirpur (HP)

Homogeneous linear equations with constant co-efficients:

Consider a partial differential equation of the form

$$\left(a_0\frac{\partial^n z}{\partial x^n}+a_1\frac{\partial^n z}{\partial x^{n-1}\partial y}+a_2\frac{\partial^n z}{\partial x^{n-2}\partial y^2}+..........+a_n\frac{\partial^n z}{\partial y^n}\right)+$$

$$\left(b_0\frac{\partial^{n-1} z}{\partial x^{n-1}}+b_1\frac{\partial^{n-1} z}{\partial x^{n-2}\partial y}+b_2\frac{\partial^{n-1} z}{\partial x^{n-3}\partial y^2}+..........+b_{n-1}\frac{\partial^{n-1} z}{\partial y^{n-1}}\right)+$$

$$.............+\left(m_0\frac{\partial z}{\partial x}+m_n\frac{\partial z}{\partial y}\right)+n_0 z=0.$$

i.e. $\phi(D,D')=0$

Here $a_0,a_1,a_2,.............,a_n$, $b_0,b_1,b_2,.............,b_{n-1}$, m_0,m_1,n_0 are all constants. In this equation the dependent variable z and its derivatives are linear, since each term in the LHS contains z or its derivatives.

This equation is called a ***linear partial differential equation of the nth order with constant co-efficients***.

Linear Homogeneous Partial Differential Equation

A linear **homogeneous** partial differential equation is of the form where all the derivatives are of the same order. Thus, an equation of the form

$$a_0 \frac{\partial^n z}{\partial x^n} + a_1 \frac{\partial^n z}{\partial x^{n-1}\partial y} + a_2 \frac{\partial^n z}{\partial x^{n-2}\partial y^2} + \dots\dots\dots + a_n \frac{\partial^n z}{\partial y^n} = F(x, y), \qquad \text{(i)}$$

where $a_0, a_1, a_2, \dots\dots\dots\dots, a_n$ are constants, is called a ***homogeneous linear partial differential equation of the nth order with constant co-efficients***.

Here, all the partial derivatives are of the nth order.

Writing D for $\frac{\partial}{\partial x}$ and D' for $\frac{\partial}{\partial y}$, (i) can be written as

$$\left(D^n + a_1 D^{n-1} D' + a_2 D^{n-2} D'^2 + \dots\dots\dots + a_n D'^n\right) z = F(x, y) \Rightarrow \phi(D, D') z = F(x, y). \qquad \text{(ii)}$$

As in the case of ordinary linear differential equations with constant co-efficients the complete solution of (ii) contains of two parts:

(i) The **complimentary function** (C.F.), which is the complete solution of the equation $\phi(D, D')z = 0$. It must contain n arbitrary functions where n is the order of the differential equation.

(ii) The **particular integral** (P.I.), which is a particular solution (free from arbitrary constants) of $\phi(D, D')z = F(x, y)$.

The complete solution of (ii) is $z = \text{C.F.} + \text{P.I.}$

Rules for finding Complementary function (C.F.):

We explain the procedure for finding the complimentary function of a differential equation of the second order. It can be easily extended to differential equation of the higher order.

Consider the equation $\frac{\partial^2 z}{\partial x^2} + a_1 \frac{\partial^2 z}{\partial x \partial y} + a_2 \frac{\partial^2 z}{\partial y^2} = 0$

which in symbolic form is $\left(D^2 + a_1 DD' + a_2 D'^2\right) z = 0$. (i)

Its auxiliary equation (A.E.) is $m^2 + a_1 m + a_2 = 0$. (ii)

Here we put $m = \frac{D}{D'}$. Let its roots be m_1, m_2.

Case I.: When the A.E. has distinct roots i. e. $m_1 \neq m_2$ then (ii) can be written as

$(D - m_1 D')(D - m_2 D')z = 0$. (iii)

Now the solution of $(D - m_2 D')z = 0$ will be the solution of (iii)

But $(D - m_2 D')z = 0 \Rightarrow p - m_2 q = 0$,

which is of Langrange's form and the auxiliary equations are $\frac{dx}{1} = \frac{dy}{-m_2} = \frac{dz}{0}$.

The first two members give $dy + m_2 dx = 0 \Rightarrow y + m_2 x = a$

Also $dz = 0 \Rightarrow z = b$

$\therefore z = f_2(y + m_2 x)$ is the solution of $(D - m_2 D')z = 0$

Similarly, (iii) will also satisfied by the solution of

$(D - m_1 D')z = 0$ i. e. by $z = f_1(y + m_1 x)$

Hence the complete solution of (ii) is $f_1(y + m_1 x) + f_2(y + m_2 x)$.

Case II.: When the A. E. has equal roots, each = m, then (ii) can be written as

$(D - mD')(D - mD')z = 0$. (iv)

Let $(D - mD')z = u$, then (iv) becomes $(D - mD')u = 0$

Its solution is $u = f(y + mx)$, as proved in case I.

$\therefore (D - mD')z = u$ takes the form $(D - mD')z = f(y + mx)$

$\Rightarrow p - mq = f(y + mx)$ [Langrange' form]

The auxiliary equations are $\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{f(y + mx)}$.

Now $\frac{dx}{1} = \frac{dy}{-m}$ gives $dy + mdx = 0 \Rightarrow y + mx = a$.

Also $\frac{dx}{1} = \frac{dz}{f(a)}$ gives $dz = f(a)dx \Rightarrow z = f(a)x + b$ i. e. $z - xf(y + mx) = b$.

$\therefore$ The complete solution of (ii) is

$z - x\phi(y + mx) = \phi(y + mx) \Rightarrow z = f(y + mx) + xf(y + mx)$.

Note1. We know that for the differential equation $(D^2 + a_1 DD' + a_2 D'^2)z = 0$,

the A.E. is $m^2 + a_1 m + a_2 = 0$. (i)

Now the roots of (i), considered as a quadratic in D/D', are the same as those of

$m^2 + a_1 m + a_2$. (ii)

$\therefore$ This quadratic in m can also be called the A.E.

Hence, we shall write the A.E. in terms of m.

Thus, the A.E. is obtained by putting $D = m$ and $D' = 1$ in $\phi(D, D') = 0$.

Hence, the A.E. of differential equation $\phi(D, D')z = F(x, y)$ is $\phi(m, 1) = 0$.

Note 2. Generalizing the results of case I and case II, we have

(i) if the roots of A.E. are $m_1, m_2, m_3, ...$ (all are distinct roots), then

$$C.F. = f_1(y + m_1 x) + f_2(y + m_2 x) + f_3(y + m_3 x) + ...$$

(ii) if the roots of A.E. are $m_1, m_2, m_3, ...$ (two equal roots), then

$$C.F. = f_1(y + m_1 x) + x f_2(y + m_1 x) + f_3(y + m_2 x) + ...$$

(iii) if the roots of A.E. are $m_1, m_2, m_3, ...$ (three equal roots), then

$$C.F. = f_1(y + m_1 x) + x f_2(y + m_1 x) + x^2 f_3(y + m_1 x) + ...$$

Q.No.1.: Solve the following partial differential equation: $2\frac{\partial^2 z}{\partial x^2} + 5\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = 0$.

Sol.: The given equation in symbolic form is $\left(2D^2 + 5DD' + 2D'^2\right)z = 0$.

Its auxiliary equation is $2m^2 + 5m + 2 = 0 \Rightarrow m = -2, -\frac{1}{2}$.

Here, the complete solution is $f_1(y - 2x) + f_2\left(y - \frac{1}{2}x\right)$.

This complete solution can also be written as $\phi_1(y - 2x) + \phi_2(2y - x)$.

Q.No.2.: Solve the following partial differential equation: $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = 0$.

Sol.: The given equation in symbolic form is $\left(D^2 - DD' - 6D'^2\right)z = 0$

Its auxiliary equation is $m^2 - m - 6 = 0 \Rightarrow (m - 3)(m + 2) = 0 \Rightarrow m = 3, -2$.

Hence, its complete solution is $z = f_1(y + 3x) + f_2(y - 2x)$.

Q.No.3.: Solve the following partial differential equation: $(D + 2D')(D - 3D')^2 z = 0$.

Sol.: The given equation is $(D+2D')(D-3D')^2 z = 0$.

Its auxiliary equation is $(m+2)(m-3)^2 = 0 \Rightarrow m = -2, 3, 3$.

Hence, the complete solution is $z = f_1(y-2x) + f_2(y+3x) + xf_3(y+3x)$.

Q.No.4.: Solve the following partial differential equation: $4r - 12s + 9t = 0$.

Sol.: The given equation is $4r - 12s + 9t = 0$.

Its symbolic form is $\left(4D^2 - 12DD' + 9D'^2\right)z = 0$.

Since $r = \dfrac{\partial^2 z}{\partial x^2} = D^2 z$, $s = \dfrac{\partial^2 z}{\partial x \partial y} = DD'z$, $t = \dfrac{\partial^2 z}{\partial y^2} = D'^2 z$.

Its auxiliary equation is $4m^2 - 12m + 9 = 0 \Rightarrow (2m-3)^2 = 0 \ \Rightarrow m = \dfrac{3}{2}, \dfrac{3}{2}$.

Hence, the complete solution is $z = \phi_1\left(y + \dfrac{3}{2}x\right) + x\phi_2\left(y + \dfrac{3}{2}x\right)$,

This complete solution can also be written as $z = f_1(2y+3x) + xf_2(2y+3x)$.

Q.No.5.: Solve the following partial differential equation: $r + 6s + 9t = 0$.

Sol.: The given equation is $r + 6s + 9t = 0$.

Its symbolic form is $\left(D^2 + 6DD' + 9D'^2\right)z = 0$.

Since $r = \dfrac{\partial^2 z}{\partial x^2} = D^2 z$, $s = \dfrac{\partial^2 z}{\partial x \partial y} = DD'z$, $t = \dfrac{\partial^2 z}{\partial y^2} = D'^2 z$.

Its auxiliary equation is $m^2 + 6m + 9 = 0 \Rightarrow m = -3, \ -3$.

Hence, the complete solution is $z = f_1(y-3x) + xf_2(y-3x)$.

Rules for finding the P.I.:

I. Short methods to find P.I.:

(i). When $F(x,y) = e^{ax+by}$.

$$\text{P.I.} = \frac{1}{\phi(D,D')}e^{ax+by} = \frac{1}{\phi(a,b)}e^{ax+by}$$ (i. e. put $D = a$ and $D' = b$) provided $\phi(a,b) \neq 0$

If $\phi(a,b) = 0$, then it is called a case of failure.

(ii). When $F(x,y) = \sin(ax+by)$.

P. I. $=\dfrac{1}{\phi\left(D^2, DD', D'^2\right)}\sin(ax+by)=\dfrac{1}{\phi\left(-a^2,-ab,-b^2\right)}\sin(ax+by)$,

(i. e. put $D^2=-a^2$, $DD'=-ab$, $D'^2=-b^2$) provided $\phi\left(-a^2,-ab,-b^2\right)\neq 0$.

If $\phi\left(-a^2,-ab,-b^2\right)=0$, then it is a case of failure.

Note: A similar rule holds when $F(x,y)=\cos(ax+by)$.

(iii). When $F(x,y)=x^p y^q$, where p, q are positive integers.

P. I. $=\dfrac{1}{\phi(D,D')}x^p y^q=[\phi(D,D')]^{-1}x^p y^q$.

If $p<q$, expand $[\phi(D,D')]^{-1}$ in powers of $\dfrac{D}{D'}$

If $q<p$, expand $[\phi(D,D')]^{-1}$ in powers of $\dfrac{D'}{D}$

Also, we have $\dfrac{1}{D}F(x,y)=\displaystyle\int_{y\ \text{constant}} F(x,y)dx$ and $\dfrac{1}{D'}F(x,y)=\displaystyle\int_{x\ \text{constant}} F(x,y)dy$.

II. General method to find the P.I.:

F(x, y) is not always of the forms given above. Moreover, there are cases of failure in the short methods. The general method is applicable to all cases where F(x, y) is not of the forms given above or the short methods fail.

Now, $\phi(D,D')$ can be factorized, in general, into n linear factors.

$$\therefore \text{P.I.}=\frac{1}{\phi(D,D')}F(x,y)=\frac{1}{(D-m_1D')(D-m_2D')\ldots\ldots\ldots(D-m_nD')}F(x,y)$$

$$=\frac{1}{D-m_1D'}\cdot\frac{1}{D-m_2D'}\cdots\cdots\cdots\frac{1}{D-m_nD'}F(x,y)$$

Let us evaluate $\dfrac{1}{D-mD'}F(x,y)$.

Consider the equation, $(D-mD')z=F(x,y)\Rightarrow p-mq=F(x,y)$.

The auxiliary equations are $\dfrac{dx}{1}=\dfrac{dy}{-m}=\dfrac{dz}{F(x,y)}$.

From the first two members $dy+mdx=0\Rightarrow y+mx=c$.

From the first and last members, we have

$dz = F(x, y)dx = F(x, c - mx)dx$

$$\therefore z = \int F(x, c - mx)dx \Rightarrow \frac{1}{D - mD'}F(x, y) = \int F(x, c - mx)dx\ ,$$

where c is replaced by $y + mx$ after integration.

By repeated application of above rules, the P. I. can be evaluated.

Working procedure to solve the equation:

$$\frac{\partial^n z}{\partial x^n} + k_1 \frac{\partial^n z}{\partial x^{n-1}\partial y} + \dots\dots\dots + k^n \frac{\partial^n z}{\partial y^n} = F(x, y)$$

Its symbolic form is $\left(D^n + k_1 D^{n-1} D' + \dots\dots\dots + k_n D'^n\right)z = F(x, y)$

or briefly $f(D, D')z = F(x, y)$.

Step1. To find the C.F.:

(i) Write the auxiliary equation (A.E.)

i. e. $m^n + k_1 m^{n-1} + \dots\dots\dots\dots\dots + k_n = 0$ and solve it for m.

(ii) Write the C.F. as follows

Roots of A. E.	C.F
1. $m_1, m_2, m_3 \dots\dots\dots$ (distinct roots)	$f_1(y + m_1 x) + f_2(y + m_2 x) + f_3(y + m_3 x) + \dots$
2. $m_1, m_2, m_3 \dots\dots\dots$ (two equal roots)	$f_1(y + m_1 x) + x f_2(y + m_1 x) + f_3(y + m_3 x) + \dots\dots$
3. $m_1, m_2, m_3 \dots\dots\dots$ (three equal roots)	$f_1(y + m_1 x) + x f_2(y + m_1 x) + x^2 f_3(y + m_1 x) + \dots\dots$

Step2. To find the P.I.:

From the symbolic form, P. I. $= \dfrac{1}{f(D, D')}F(x, y)$.

(i) When $F(x, y) = e^{ax+by}$, P. I. $= \dfrac{1}{f(D, D')}e^{ax+by}$ [put $D = a$ and $D' = b$]

(ii) When $F(x, y) = \sin(mx + ny)$ or $\cos(mx + ny)$

P. I. $= \dfrac{1}{f\left(D^2, DD', D'^2\right)}\sin$ or $\cos(mx + ny)$. [put $D^2 = -m^2$, $DD' = -mn$, $D'^2 = -n^2$]

(iii) When $F(x,y)=x^m y^n$, P.I. $=\dfrac{1}{f(D,D')}x^m y^n=[f(D,D')]^{-1}x^m y^n$.

Expand $[f(D,D')]^{-1}$ in ascending powers of D or D' and operate on $x^m y^n$ term by term.

(iv) When F(x, y) is any function of x and y, P. I. $=\dfrac{1}{f(D,D')}F(x,y)$.

Resolve $\dfrac{1}{f(D,D')}$ into partial fractions considering $f(D,D')$ as a function of D alone and operate each partial fraction on F(x, y) remembering that

$$\frac{1}{D-mD'}F(x,y)=\int F(x,c-mx)dx,$$

where c is replaced by $y+mx$ after integration.

Now let us solve some homogeneous linear partial differential equation with constant co-efficients.

Q.No.1.: Solve the following partial differential equation: $\dfrac{\partial^2 z}{\partial x^2}-\dfrac{\partial^2 z}{\partial x\partial y}=\cos x\cos 2y$.

Sol.: The given equation in symbolic form is $(D^2-DD')z=\cos x\cos 2y$.

Step 1. To find the complementary function

Its auxiliary equation (A.E.) is $m^2-m=0\Rightarrow m(m-1)=0\Rightarrow m=0,\ 1$.

Thus, C.F. $=f_1(y)+f_2(y+x)$.

Step 2. To find the particular integral

$$\text{Now P.I. }=\frac{1}{D^2-DD'}\cos x\cos 2y=\frac{1}{2}.\frac{1}{D^2-DD'}[\cos(x+2y)+\cos(x-2y)]$$

$$=\frac{1}{2}.\left[\frac{1}{D^2-DD'}.\cos(x+2y)+\frac{1}{D^2-DD'}\cos(x-2y)\right]$$

$$=\frac{1}{2}\cos(x+2y)-\frac{1}{6}\cos(x-2y).$$

Step 3. To find the complete solution

Now, since the complete solution is z = C.F. + P.I.

$\therefore\ z = f_1(y) + f_2(y+x) + \frac{1}{2}\cos(x+2y) - \frac{1}{6}\cos(x-2y)$. Ans.

Q.No.2.: Solve the following partial differential equation: $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} = \sin x \cos 2y$.

Sol.: The given equation in symbolic form is $(D^2 - DD')z = \sin x \cos 2y$.

Step 1. To find the complementary function

Its auxiliary equation (A.E.) is $m^2 - m = 0 \Rightarrow m(m-1) = 0 \Rightarrow m = 0,\ 1$.

Thus, C.F. $= f_1(y) + f_2(y+x)$.

Step 2. To find the particular integral

Now P.I. $= \frac{1}{D^2 - DD'}\sin x \cos 2y = \frac{1}{D^2 - DD'}\left[\frac{1}{2}(2\sin x \cos 2y)\right]$

$$= \frac{1}{2}.\frac{1}{D^2 - DD'}[\sin(x+2y) + \sin(x-2y)]$$

$$= \frac{1}{2}.\left[\frac{1}{D^2 - DD'}.\sin(x+2y) + \frac{1}{D^2 - DD'}\sin(x-2y)\right]$$

$$= \frac{1}{2}\sin(x+2y) - \frac{1}{6}\sin(x-2y).$$

Step 3. To find the complete solution

Now, since the complete solution is z = C.F. + P.I.

$\therefore\quad z = f_1(y) + f_2(y+x) + \frac{1}{2}\sin(x+2y) - \frac{1}{6}\sin(x-2y)$. Ans.

Q.No.3.: Solve the following partial differential equation: $\frac{\partial^3 z}{\partial x^3} - 2\frac{\partial^3 z}{\partial x^2 \partial y} = 2e^{2x} + 3x^2 y$.

Sol.: The given equation in symbolic form is $(D^3 - 2D^2D')z = 2e^{2x} + 3x^2 y$.

Step 1. To find the complementary function

Its auxiliary equation (A.E.) is $m^3 - 2m^2 = 0 \Rightarrow m = 0, 0, 2$.

Thus, C.F. $= f_1(y) + xf_2(y) + f_3(y+2x)$.

Step 2. To find the particular integral

Now P.I. $=\dfrac{1}{D^3-2D^2D'}\left(2e^{2x}+3x^2y\right)=2\dfrac{1}{D^3-2D^2D'}e^{2x}+3\dfrac{1}{D^3\left(1-\dfrac{2D'}{D}\right)}x^2y$

$$=2\frac{1}{2^3-2.2^2(0)}e^{2x}+\frac{3}{D^3}\left(1-\frac{2D'}{D}\right)^{-1}x^2y=\frac{1}{4}e^{2x}+\frac{3}{D^3}\left(1+\frac{2D'}{D}+\frac{4D'^2}{D^2}+.....\right)x^2y$$

$$=\frac{1}{4}e^{2x}+\frac{3}{D^3}\left(x^2y+\frac{2}{D}x^2.1\right)=\frac{1}{4}e^{2x}+\frac{3}{D^3}\left(x^2y+\frac{2}{3}x^3\right)\quad\left[\because\frac{1}{D}f(x)=\int f(x)dx\right]$$

$$=\frac{1}{4}e^{2x}+3y\frac{x^5}{3.4.5}+2.\frac{x^6}{4.5.6}\qquad\left[\because\frac{1}{D^3}f(x)=\int\left[\int\left(\int f(x)dx\right)dx\right]dx\right]$$

$$=\frac{e^{2x}}{4}+\frac{x^5y}{20}+\frac{x^6}{60}.$$

Step 3. To find the complete solution

Now, since the complete solution is z = C.F. + P.I.

$\therefore f_1(y)+xf_2(y)+f_3(y+2x)+\dfrac{1}{60}\left(15e^{2x}+3x^5y+x^6\right)$. Ans.

Q.No.4.: Solve the following partial differential equation:

$$\left(D^3-7DD'^2-6D'^3\right)=\sin(x+2y)+e^{2x+y}.$$

Sol.: The given equation is $\left(D^3-7DD'^2-6D'^3\right)=\sin(x+2y)+e^{2x+y}$.

Step 1. To find the complementary function

The auxiliary equation (A.E.) is $m^3-7m-6=0\Rightarrow(m+1)(m-3)(m+2)=0$

$\Rightarrow m=-1,3,-2$.

Thus, C.F. $=f_1(y-x)+f_2(y+3x)+f_3(y-2x)$.

Step 2. To find the particular integral

Now P.I. $=\dfrac{1}{D^3-7DD'^2-6D'^3}\left[\sin(x+2y)+e^{2x+y}\right]$

$$=\frac{1}{D^3-7DD'^2-6D'^3}\sin(x+2y)+\frac{1}{D^3-7DD'^2-6D'^3}e^{2x+y}$$

$$=\frac{1}{D\left(D^2\right)-7D\left(D'^2\right)-6D'\left(D'^2\right)}\sin(x+2y)+\frac{1}{2^3-7(2)(1)^2-6.1^3}e^{2x+y}$$

$$=\frac{1}{D\left(-1^2\right)-7D\left(-2^2\right)-6D'\left(-2^2\right)}\sin(x+2y)-\frac{1}{12}e^{2x+y}$$

$$=\frac{1}{27D+24D'}\sin(x+2y)-\frac{1}{12}e^{2x+y}=\frac{D}{27D^2+24DD'}\sin(x+2y)-\frac{1}{12}e^{2x+y}$$

$$=\frac{D}{27\left(-1^2\right)+24(-1.2)}\sin(x+2y)-\frac{1}{12}e^{2x+y}=-\frac{1}{75}\cos(x+2y)-\frac{1}{12}e^{2x+y}$$

Step 3. To find the complete solution

Now, since the complete solution is z = C.F. + P.I.

$$\therefore\quad f_1(y-x)+f_2(y+3x)+f_3(y-2x)-\frac{1}{75}\cos(x+2y)-\frac{1}{12}e^{2x+y}.$$

Q.No.5.: Solve the following partial differential equation: $\frac{\partial^2 z}{\partial x^2}+3\frac{\partial^2 z}{\partial x\partial y}+2\frac{\partial^2 z}{\partial y^2}=x+y$.

Sol.: The given equation in symbolic form is $\left(D^2+3DD'+2D'^2\right)z=x+y$.

Step 1. To find the complementary function

Its auxiliary equation (A.E.) is $m^2+3m+2=0\Rightarrow m=-1,\ -2$.

Thus, C.F. $=f_1(y-x)+f_2(y-2x)$.

Step 2. To find the particular integral

Now P.I. $=\frac{1}{D^2+3DD'+2D'^2}(x+y)=\frac{1}{D^2\left(1+3\frac{D'}{D}+2\frac{D'^2}{D^2}\right)}(x+y)$

$$=\frac{1}{D^2}\left[1+\left(\frac{3D'}{D}+\frac{2D'^2}{D^2}\right)\right]^{-1}(x+y)=\frac{1}{D^2}\left(1-\frac{3D'}{D}+.........\right)(x+y)$$

$$=\frac{1}{D^2}\left[x+y-\frac{3}{D}(1)\right]=\frac{1}{D^2}[x+y-3x]=\frac{1}{D^2}(y-2x)$$

$$=\frac{1}{D}\left(yx-x^2\right)=\frac{yx^2}{2}-\frac{x^3}{3}.$$

Step 3. To find the complete solution

Now, since the complete solution is z = C.F. + P.I.

$$\therefore\quad f_1(y-x)+f_2(y-2x)+\frac{yx^2}{2}-\frac{x^3}{3}.$$

Q.No.6.: Solve the following partial differential equation: $r - 4s + 4t = e^{2x+y}$.

Sol.: The equation can be written as $\frac{\partial^2 z}{\partial x^2} - 4\frac{\partial^2 z}{\partial x \partial y} + 4\frac{\partial^2 z}{\partial y^2} = e^{2x+y}$.

The given equation in symbolic form is $\left(D^2 - 4DD' + 4D'^2\right)z = e^{2x+y}$.

Step 1. To find the complementary function

Its auxiliary equation (A.E.) is $m^2 - 4m + 4 = 0 \Rightarrow (m-2)^2 = 0 \Rightarrow m = 2,\ 2$.

Thus, C.F. $= f_1(y+2x) + xf_2(y+2x)$.

Step 2. To find the particular integral

Now P.I. $= \frac{1}{(D-2D')^2}e^{2x+y}$, putting $D = 2$ and $D' = 1$, the denominator vanishes.

$\therefore$ It is a case of failure.

Since $\frac{1}{D - mD'}F(x,y) = \int F(x, c - mx)dx$, where c is to be replaced by $(y + mx)$ after integration.

$$\therefore \text{P.I.} = \frac{1}{D-2D'}.\frac{1}{D-2D'}e^{2x+y} = \frac{1}{D-2D'}\int e^{2x+(c-2x)}dx = \frac{1}{D-2D'}\int e^c dx$$

$$= \frac{1}{D-2D'}xe^{y+2x} = \int xe^{(c-2x)+2x}dx = e^c\int xdx = \frac{1}{2}x^2.e^c = \frac{1}{2}x^2e^{y+2x}.$$

Step 3. To find the complete solution

Now, since the complete solution is z = C.F. + P.I.

$$\therefore \quad f_1(y+2x) + xf_2(y+2x) + \frac{1}{2}x^2e^{y+2x}.$$

Q.No.7.: Solve the following partial differential equation:

$$\left(2D^2 - 5DD' + 2D'^2\right)z = 5\sin(2x+y).$$

Sol.: The given equation is $\left(2D^2 - 5DD' + 2D'^2\right)z = 5\sin(2x+y)$.

Step 1. To find the complementary function

Its auxiliary equation (A.E.) is $2m^2 - 5m + 2 = 0 \Rightarrow (2m-1)(m-2) = 0 \Rightarrow m = \frac{1}{2},\ 2$.

Thus, C.F. $= f_1(2y+x) + f_2(y+2x)$.

Step 2. To find the particular integral

Now $\text{P.I.} = 5.\dfrac{1}{2D^2 - 5DD' + 2D'^2}\sin(2x+y)$.

Putting $D^2 = -2^2$, $DD' = -2.1$, $D'^2 = -1^2$.

Denominator $= 2(-4) - 5(-2) + 2(-1) = 0$.

$\therefore$ It is case of failure.

$$\text{Now P.I.} = 5.\frac{1}{(2D-D')(D-2D')}\sin(2x+y) = 5.\frac{1}{2D-D'}\int \sin[2x+(c-2x)]dx$$

$$= \frac{5}{2}.\frac{1}{D-\frac{1}{2}D'}\int \sin c\, dx = \frac{5}{2}.\frac{1}{D-\frac{1}{2}D'} x\sin c$$

$$= \frac{5}{2}.\frac{1}{D-\frac{1}{2}D'} x\sin(y+2x) = \frac{5}{2}\int x\sin\left[\left(c-\frac{1}{2}x\right)+2x\right]dx$$

$$= \frac{5}{2}\int x\sin\left(c+\frac{3}{2}x\right)dx = \frac{5}{2}\left[x.\frac{-\cos\left(c+\frac{3}{2}x\right)}{\frac{3}{2}} - \int 1.\frac{-\cos\left(c+\frac{3}{2}x\right)}{\frac{3}{2}}dx\right]$$

$$= -\frac{5}{3}x\cos\left(c+\frac{3}{2}x\right) + \frac{10}{9}\sin\left(c+\frac{3}{2}x\right)$$

$$= \frac{-5}{3}x\cos\left[\left(y+\frac{1}{2}x\right)+\frac{3}{2}x\right] + \frac{10}{9}\sin\left[\left(y+\frac{1}{2}x\right)+\frac{3}{2}x\right]$$

$$= -\frac{5}{3}x\cos(y+2x) + \frac{10}{9}\sin(y+2x).$$

Step 3. To find the complete solution

Now, since the complete solution is z = C.F. + P.I.

$$\therefore \quad z = f_1(2y+x) + f_2(y+2x) - \frac{5}{3}x\cos(y+2x) + \frac{10}{9}\sin(y+2x)$$

Let $f_2(y+2x) + \dfrac{10}{9}\sin(y+2x) = f_2(y+2x) + \phi(y+2x) = F_2(y+2x)$.

$$\therefore z = f_1(2y+x) + F_2(y+2x) - \frac{5}{3}x\cos(y+2x).$$

Q.No.8.: Solve the following partial differential equation: $\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial x\partial y}-6\frac{\partial^2 z}{\partial y^2}=y\cos x$.

Sol.: The given equation in symbolic form is $\left(D^2+DD'-6D'^2\right)z=y\cos x$.

Step 1. To find the complementary function

Its auxiliary equation (A.E.) is $m^2+m-6=0 \Rightarrow m=-3,\ 2$.

Thus, C.F. $=f_1(y-3x)+f_2(y+2x)$.

Step 2. To find the particular integral

Now P.I. $=\frac{1}{D^2+DD'-6D'^2}y\cos x=\frac{1}{(D+3D')(D-2D')}y\cos x$

$$=\frac{1}{D+3D'}\cdot\frac{1}{D-2D'}y\cos x=\frac{1}{D+3D'}\int (c-2x)\cos x dx,$$

where c is to replaced by $y+mx=y+2x$ after integration

$$=\frac{1}{D+3D'}\left[(c-2x)\sin x-\int -2\sin x dx\right]=\frac{1}{D+3D'}\left[(y+2x-2x)\sin x-2\cos x\right]$$

$$=\frac{1}{D+3D'}(y\sin x-2\cos x)=\int \left[(c+3x)\sin x-2\cos x\right]dx$$

where c is to replaced by $y+mx=y-3x$ after integration

$$=(c+3x)(-\cos x)-\int 3(-\cos x)dx-2\sin x=(y-3x+3x)(-\cos x)+3\sin x-2\sin x$$

$$=-y\cos x+\sin x.$$

Step 3. To find the complete solution

Now, since the complete solution is z = C.F. + P.I.

$\therefore z=f_1(y-3x)+f_2(y+2x)-y\cos x+\sin x$. Ans.

Q.No.9.: Solve the following partial differential equation:

$$\frac{\partial^3 z}{\partial x^3}-3\frac{\partial^3 z}{\partial^2 x\partial y}+4\frac{\partial^3 z}{\partial y^3}=e^{x+2y}.$$

Sol.: The given equation in symbolic form is $\left(D^3-3D^2D'+4D'^3\right)z=e^{x+2y}$

Step 1. To find the complementary function

Its auxiliary equation (A.E.) is $m^3-3m^2+4=0 \Rightarrow (m+1)\left(m^2-4m+4\right)=0$

$$\Rightarrow (m+1)(m-2)^2=0 \Rightarrow m=-1, 2, 2.$$

Thus, C.F. $= f_1(y-x) + f_2(y+2x) + xf_3(y+2x)$.

Step 2. To find the particular integral

Now P.I. $= \dfrac{1}{D^3 - 3D^2D' + 4D'^3} e^{x+2y} = \dfrac{1}{1^3 - 3.1^2.2 + 4.2^3} e^{x+2y} = \dfrac{1}{27} e^{x+2y}$.

(putting $D = 1,\ D' = 2$)

Step 3. To find the complete solution

Now, since the complete solution is z = C.F. + P.I.

$\therefore f_1(y-x) + f_2(y+2x) + xf_3(y+2x) + \dfrac{1}{27} e^{x+2y}$. Ans.

Q.No.10.: Solve the following partial differential equation:

$$\frac{\partial^3 z}{\partial x^3} - 4\frac{\partial^3 z}{\partial x^2 \partial y} + 4\frac{\partial^3 z}{\partial x \partial y^2} = 0.$$

Sol.: The given equation in symbolic form is $\left(D^3 - 4D^2D' + 4DD'^2\right)z = 0$.

Step 1. To find the complementary function

Here auxiliary equation (A.E.) is $m^3 - 4m^2 + 4m = 0$, where $m = \dfrac{D}{D'}$.

$\Rightarrow m(m-2)^2 = 0 \quad \therefore m = 0,\ 2,\ 2$.

Step 2. To find the particular integral

Now P.I. = 0

Step 3. To find the complete solution

Now, since the complete solution is z = C.F. + P.I.

$\therefore z = f_1(y) + f_2(y+2x) + xf_3(y+2x)$. Ans.

Q.No.11.: Solve the following partial differential equation: $4r + 12s + 9t = e^{3x-2y}$.

Sol.: The given equation can be written as $4\dfrac{\partial^2 z}{\partial x^2} + 12\dfrac{\partial^2 z}{\partial x \partial y} + 9\dfrac{\partial^2 z}{\partial y^2} = e^{3x-2y}$.

The given equation in symbolic form is $\left(4D^2 + 12DD' + 9D'^2\right)z = e^{3x-2y}$.

Step 1. To find the complementary function

Here auxiliary equation (A.E.) is $4m^2 + 12m + 9 = 0$, where $m = \dfrac{D}{D'}$.

$\Rightarrow (2m+3)^2 = 0 \Rightarrow m = -\frac{3}{2}, -\frac{3}{2}$.

Thus, C.F. $= f_1\left(y - \frac{3}{2}x\right) + xf_2\left(y - \frac{3}{2}x\right)$.

Step 2. To find the particular integral

Now P.I. $= \frac{1}{4D^2 + 12DD' + 9D'^2} e^{3x-2y}$.

Here the usual rule fails

$\because 4D^2 + 12DD' + 9D'^2 = 0$ for $D = 3,\ D' = -2$

$$= \frac{1}{(2D+3D')^2} e^{3x-2y} = \frac{1}{(2D+3D)} \cdot \frac{1}{(2D+3D')} e^{3x-2y}$$

$$= \frac{1}{(2D+3D')} \int e^{3x-2\left(c+\frac{3}{2}x\right)} dx \qquad \left[y - \frac{3}{2}x = 0,\quad y = c + \frac{3}{2}x\right]$$

$$= \frac{1}{(2D+3D')} \int e^{-2c} dx = \frac{1}{(2D+3D')} xe^{-2c} = \frac{1}{(2D+3D')} xe^{-2\left[y-\frac{3}{2}x\right]}$$

$$= \int xe^{3x-2\left[c+\frac{3}{2}x\right]} dx = \int xe^{-2c} dx = \frac{1}{2}x^2 e^{-2c} = \frac{1}{2}x^2 e^{-(2y-3x)}$$

$$= \frac{1}{2}x^2 e^{3x-2y}.$$

Step 3. To find the complete solution

Now, since the complete solution is z = C.F. + P.I.

$$\therefore z = f_1\left(y - \frac{3}{2}x\right) + xf_2\left(y - \frac{3}{2}x\right) + \frac{1}{2}x^2 e^{3x-2y}$$

$$\Rightarrow z = f_1(2y-3x) + xf_2(2y-3x) + \frac{1}{2}x^2 e^{3x-2y}.$$ Ans.

Q.No.12.: Solve the following partial differential equation: $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = \sin x$.

Sol.: The given equation in symbolic form is $\left(D^2 - 2DD' + D'^2\right)z = \sin x$.

Step 1. To find the complementary function

Here auxiliary equation (A.E.) is $m^2-2m+1=0 \Rightarrow m=1,\ 1$, where $m=\dfrac{D}{D'}$.

Thus, C.F. $=f_1(y+x)+xf_2(y+x)$.

Step 2. To find the particular integral

Now P.I. $=\dfrac{1}{D^2-2DD'+D'^2}\sin x=\dfrac{1}{-1^2-0+0}\sin x=-\sin x$.

Step 3. To find the complete solution

Now, since the complete solution is z = C.F. + P.I.

$\therefore z=f_1(y+x)+xf_2(y+x)-\sin x$. Ans.

Q.No.13.: Solve the following partial differential equation: $\dfrac{\partial^2 y}{\partial t^2}-a^2\dfrac{\partial^2 y}{\partial x^2}=E\sin pt$.

Sol.: The given equation in symbolic form is $\left(D^2-a^2D'^2\right)y=E\sin pt$.

Here $D=\dfrac{\partial}{\partial t}$, $D'=\dfrac{\partial}{\partial x}$, $m=\dfrac{D}{D'}$.

Step 1. To find the complementary function

Here auxiliary equation (A.E.) is $m^2-a^2=0 \Rightarrow m=+a,-a$.

Thus, C.F. $=f_1(x-at)+f_2(x+at)$.

Step 2. To find the particular integral

Now P.I. $=\dfrac{1}{D^2-a^2D'^2}E\sin pt=\dfrac{1}{-p^2-a^2.0}E\sin pt=-\dfrac{E}{p^2}\sin pt$.

Step 3. To find the complete solution

Now, since the complete solution is y = C.F. + P.I.

$\therefore y=f_1(x-at)+f_2(x+at)-\dfrac{E}{p^2}\sin pt$. Ans.

Q.No.14.: Solve the following partial differential equation: $\dfrac{\partial^2 z}{\partial x^2}-\dfrac{\partial^2 z}{\partial y^2}=\cos 2x\cos 3y$.

Sol.: The given equation in symbolic form is $\left[D^2-D'^2\right]z=\cos 2x\cos 3y$.

Step 1. To find the complementary function

Here auxiliary equation (A.E.) is $m^2-1=0 \Rightarrow m=-1,\ 1$.

Thus, $\text{C.F.} = f_1(y+x) + f_2(y-x)$

Step 2. To find the particular integral

$$\text{Now P.I.} = \frac{1}{D^2 - D'^2}\cos 2x \cos 3y = \frac{1}{2}\frac{1}{D^2 - D'^2}[\cos(2x+3y) + \cos(2x-3y)]$$

$$= \frac{1}{2}\left[\frac{1}{-4+9}\cos(2x+3y) + \frac{1}{-4+9}\cos(2x-3y)\right]$$

$$= \frac{1}{5}\cos 2x \cos 3y.$$

Step 3. To find the complete solution

Now, since the complete solution is z = C.F. + P.I.

$$\therefore z = f_1(y+x) + f_2(y-x) + \frac{1!}{5}\cos 2x \cos 3y. \text{ Ans.}$$

Q.No.15.: Solve the following partial differential equation: $(D^2 + 3DD' + 2D'^2)z = 24xy$.

Sol.: The given equation is $(D^2 + 3DD' + 2D'^2)z = 24xy$.

Step 1. To find the complementary function

Here auxiliary equation (A.E.) is $(m^2 + 3m + 2) = 0$

$\Rightarrow (m+1)(m+2) = 0 \Rightarrow m = -1, \ -2$.

Thus, $\text{C. F.} = f_1(y-x) + f_2(y-2x)$

Step 2. To find the particular integral

$$\text{Now P.I.} = \frac{1}{D^2 + 3DD' + 2D'^2}24xy = 24.\frac{1}{D^2}\left[1 + \frac{3D'}{D} + \frac{2D'^2}{D^2}\right]^{-1} xy$$

$$= \frac{24}{D^2}\left[1 - \frac{3D'}{D} - \frac{2D'^2}{D^2} - \dots\dots\dots\right]xy$$

$$= \frac{24}{D^2}\left[xy - \frac{3}{D}.x.1\right] = \frac{24}{D^2}\left[xy - \frac{3x^2}{2}\right] = 24.y\frac{x^3}{2.3} - 24.\frac{3}{2}.\frac{x^4}{3.4}$$

$$= 4x^3y - 3x^4.$$

Step 3. To find the complete solution

Now, since the complete solution is z = C.F. + P.I.

$\therefore z = f_1(y-x) + f_2(y-2x) + 4x^3y - 3x^4$. Ans.

Q.No.16.: Solve the following partial differential equation:

$$\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = x^2 + xy + y^2.$$

Sol.: The given equation in symbolic form is $\left(D^2 + 2DD' + D'^2\right)z = x^2 + xy + y^2$.

Step 1. To find the complementary function

Here auxiliary equation (A.E.) is $\left(m^2 + 2m + 1\right) = 0 \Rightarrow (m+1)^2 = 0 \Rightarrow m = -1, \ -1$.

Thus, C.F. $= f_1(y-x) + xf_2(y-x)$.

Step 2. To find the particular integral

$$\text{Now P.I.} = \frac{1}{D^2 + 2DD' + D'^2}\left(x^2 + xy + y^2\right) = \frac{1}{D^2}\left[1 + 2\frac{D'}{D} + \frac{D'^2}{D^2}\right]^{-1}\left(x^2 + xy + y^2\right)$$

$$= \frac{1}{D^2}\left[1 - \left(2\frac{D'}{D} + \frac{D'^2}{D^2}\right) + \left(2\frac{D'}{D} + \frac{D'^2}{D^2}\right)^2 - .....\right]^{-1}\left(x^2 + xy + y^2\right)$$

$$= \frac{1}{D^2}\left[x^2 + xy + y^2 - 2\left(\frac{x^2}{2} + 2yx\right) - \frac{2x^2}{2} + 4x^2\right]$$

$$= \frac{1}{D^2}\left[3x^2 - 3xy + y^2\right] = 3\frac{x^4}{4.3} - 3y\frac{x^3}{2.3} + \frac{y^2x^2}{2}$$

$$= \frac{1}{4}\left[x^4 + 2x^2y^2 - 2x^3y\right].$$

Step 3. To find the complete solution

Now, since the complete solution is z = C.F. + P.I.

$$\therefore z = f_1(y-x) + xf_2(y-x) + \frac{1}{4}\left[3\frac{x^4}{3} + 2x^2y^2 - 2x^3y\right].$$

Q.No.17.: Solve the following partial differential equation:

$$\left(D^3 + D^2D' - DD'^2 - D'^3\right)z = e^x \cos 2y.$$

Sol.: The given equation is $\left(D^3 + D^2D' - DD'^2 - D'^3\right)z = e^x \cos 2y$.

Step 1. To find the complementary function

Here auxiliary equation (A.E.) is $\left(m^3+m^2-m-1\right)=0$

$\Rightarrow (m+1)(m+1)(m-1)=0 \Rightarrow m=-1,\ -1,\ 1$.

Thus, C.F. $=f_1(y-x)+xf_2(y-x)+f_3(y+x)$.

Step 2. To find the particular integral

Now P.I. $=\dfrac{1}{D^3+D^2D'-DD'^2-D'^3}e^x\cos 2y$.

Take as particular integral

$z=Ae^x\cos 2y+Be^x\sin 2y$.

Then $D^3z=Ae^x\cos 2y+Be^x\sin 2y$

$DD'^2z=-4Ae^x\cos 2y-4Be^x\sin 2y$

$D^2D'z=-2Ae^x\sin 2y+2Be^x\cos 2y$

$D'^3z=8Ae^x\sin 2y-8Be^x\cos 2y$.

Substituting in the given equation, we get

$(5A+10B)e^x\cos 2y+(5B-10A)e^x\sin 2y=e^x\cos 2y$.

So that $5A+10B=1$ and $5B-10A=0$

$\therefore A=\dfrac{1}{25}$ and $B=\dfrac{2}{25}$.

$\therefore$ P.I. is $z=\dfrac{1}{25}e^x\cos 2y+\dfrac{2}{25}e^x\sin 2y$.

Step 3. To find the complete solution

Now, since the complete solution is z = C.F. + P.I.

$\therefore z=f_1(y-x)+xf_2(y-x)+f_3(y+x)+\dfrac{e^x}{25}[\cos 2y+2\sin 2y]$. Ans.

Q.No.18.: Solve the following partial differential equation:

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial x\partial y}-6\frac{\partial^2 z}{\partial y^2}=\cos(2x+y).$$

Sol.: The given equation in symbolic form is $\left(D^2+DD'-6D'^2\right)z=\cos(2x+y)$.

Step 1. To find the complementary function

Here auxiliary equation (A.E.) is $m^2 + m - 6 = 0 \Rightarrow m = -3,\ 2$.

Thus, C.F. $= f_1(y-3x) + f_2(y+2x)$.

Step 2. To find the particular integral

Since $D^2 + DD' - 6D'^2 = -2^2 - (2)(1) - 6(-1)^2 = 0$.

$\therefore$ It is a case of failure and we have to apply the general method.

$$\text{P.I.} = \frac{1}{D^2 + DD' - 6D'^2}\cos(2x+y) = \frac{1}{(D+3D')(D-2D')}\cos(2x+y)$$

$$= \frac{1}{D+3D'}\int \cos\left(2x + \overline{c-2x}\right)dx = \frac{1}{D+3D'}\int \cos c\,dx$$

$$= \frac{1}{D+3D'} x\cos c = \frac{1}{D+3D'} x\cos(y+2x)$$

$$= \int x\cos\left(\overline{c+3x} + 2x\right)dx = \int x\cos(5x+c)dx$$

$$= \frac{x\sin(5x+c)}{5} + \frac{\cos(5x+c)}{25} \qquad \text{[Integrating by parts]}$$

$$= \frac{x}{5}\sin\left(5x + \overline{y-3x}\right) + \frac{1}{25}\cos\left(5x + \overline{y-3x}\right)$$

$$= \frac{x}{5}\sin(2x+y) + \frac{1}{25}\cos(2x+y).$$

Step 3. To find the complete solution

Now, since the complete solution is z = C.F. + P.I.

$\therefore\ z = f_1(y-3x) + f_2(y+2x) + \frac{x}{5}\sin(2x+y) + \frac{1}{25}\cos(2x+y)$. Ans.

Q.No.19.: Solve the following partial differential equation:

$$\frac{\partial^3 z}{\partial x^2 \partial y} - 5\frac{\partial^3 z}{\partial x \partial y^2} + 6\frac{\partial^3 z}{\partial y^3} = e^x.$$

Sol.: The given equation in symbolic form is $\left(D^2D' - 5DD'^2 + 6D'\right)^3 z = e^x$.

Step 1. To find the complementary function

Here auxiliary equation (A.E.) is $6m^3 - 5m^2 + m = 0 \Rightarrow m(3m-1)(2m-1) = 0$

$$\Rightarrow m = 0,\ \frac{1}{3},\ \frac{1}{2}.$$

Thus, C.F. $= f_1(x+m_1y)+f_2(x+m_2y)+f_3(x+m_3y)$

$$= f_1(x+0)+f_2\left(x+\frac{1}{2}y\right)+f_3\left(x+\frac{1}{3}y\right)$$

$$= f_1(x)+f_2(y+2x)+f_3(y+3x).$$

Step 2. To find the particular integral

Now P.I. $= \dfrac{1}{D^2D'-5DD'^2+6D'^3}e^x = \dfrac{1}{D'(3D'-D)(2D'-D)}e^x$

$$= \frac{1}{D'}\frac{1}{(0-1)(0-1)}e^x = \frac{1}{D'}e^x = \int e^x dy = ye^x.$$

Step 3. To find the complete solution

Now, since the complete solution is z = C.F. + P.I.

$\therefore z = f_1(x)+f_2(y+2x)+f_3(y+3x)+ye^x$. Ans.

Q.No.20.: Solve the following partial differential equation:

$$\frac{\partial^2 z}{\partial x^2}+2\frac{\partial^2 z}{\partial x\partial y}+\frac{\partial^2 z}{\partial y^2}=x^2y$$

Sol.: The given equation in symbolic form is $\left(D^2+2DD'+D'^2\right)z = x^2y$.

Step 1. To find the complementary function

Here auxiliary equation (A.E.) is $m^2+2m+1=0$.

$\Rightarrow (m+1)^2 = 0 \Rightarrow m = -1, -1$.

Thus, C.F. $= f_1(y-x)+xf_2(y-x)$.

Step 2. To find the particular integral

Now P.I. $= \dfrac{1}{D^2+2DD'+D'^2}x^2y = \dfrac{1}{D^2}\left[1+\dfrac{2D'}{D}+\dfrac{D'^2}{D^2}\right]^{-1}x^2y$

$$= \frac{1}{D^2}\left[1-\left(\frac{2D'}{D}+\frac{D'^2}{D^2}\right)+\left(\frac{2D'}{D}+\frac{D'^2}{D^2}\right)^2-\ldots\ldots\ldots\right]x^2y$$

$$= \frac{1}{D^2}\left[x^2y-\frac{1}{D}2x^2\right] = \frac{1}{D^2}\left[x^2y-\frac{2}{3}x^3\right] = \frac{x^4y}{3.4}-\frac{2}{3}\frac{x^5}{4.5}$$

$$=\frac{x^4y}{12}-\frac{2}{60}x^5=\frac{x^4}{60}(5y-2x).$$

Step 3. To find the complete solution

Now, since the complete solution is z = C.F. + P.I.

$$\therefore\ z=f_1(y-x)+xf_2(y-x)+\frac{x^4}{60}(5y-2x).\ \text{Ans.}$$

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Home Assignments

Q.No.1.: Solve the following partial differential equation:

$$\frac{\partial^3 z}{\partial x^3}-4\frac{\partial^3 z}{\partial x^2\partial y}+5\frac{\partial^3 z}{\partial x\partial y^2}-2\frac{\partial^3 z}{\partial y^3}=e^{2x+y}.$$

Ans.: $z=f_1(y+x)+xf_2(y+x)+f_3(y+2x)-e^{2x+y}$.

Q.No.2.: Solve the following partial differential equation:

$$\frac{\partial^3 z}{\partial x^3}-4\frac{\partial^3 z}{\partial x^2\partial y}+4\frac{\partial^3 z}{\partial x\partial y^2}=2\sin(3x+2y).$$

Ans.:

Q.No.3.: Solve the following partial differential equation:

$$\frac{\partial^2 z}{\partial x^2}-4\frac{\partial^2 z}{\partial x\partial y}+3\frac{\partial^2 z}{\partial y^2}=\sqrt{x+3y}.$$

Ans.: $z=f_1(y+x)+f_2(y+3x)+\frac{1}{6}(3c-2x)^{3/2}-\frac{1}{24}(3c-8x)^{3/2}$

Q.No.4.: Solve the following partial differential equation:

$$\left(D^2-DD'-2D'^2\right)z=(y-1)e^x.$$

Ans.: $z=f_1(y-x)+f_2(y+2x)+ye^x$

Q.No.5.: Solve the following partial differential equation:

$$\left(D^2 + 2DD' + D'^2\right)z = 2\cos y - x\sin y .$$

Ans.: $z = f_1(y - x) + xf_2(y - x) + x\sin y$

Q.No.6.: Solve the following partial differential equation:

$$\frac{\partial^3 z}{\partial x^3} - 7\frac{\partial^3 z}{\partial x \partial y^2} + 6\frac{\partial^3 z}{\partial y^3} = 0 .$$

Ans.: $z = f_1(y + x) + f_2(y + 2x) + f_3(y - 3x)$.

Q.No.7.: Solve the following partial differential equation:

$$25r - 40s + 16t = 0 .$$

Ans.: $z = f_1(5x + 4y) + xf_2(5y + 4x)$.

Q.No.8.: Solve the following partial differential equation:

$$\frac{\partial^2 z}{\partial x^2} - 7\frac{\partial^2 z}{\partial x \partial y} + 12\frac{\partial^2 z}{\partial y^2} = e^{x-y} .$$

Ans.: $z = f_1(y + 3x) + f_2(y + 4x) + \frac{1}{20}e^{x-y}$.

Q.No.9.: Solve the following partial differential equation:

$$\left(D^2 + 2DD' + D'^2\right)z = e^{2x+3y} .$$

Ans.: $z = f_1(y - x) + xf_2(y - x) + \frac{1}{25}e^{2x+3y}$.

Q.No.10.: Solve the following partial differential equation:

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = \cos(3x + y) .$$

Ans.: $z = f_1(y - 3x) + f_2(y + 2x) - \frac{1}{6}\cos(3x + y)$.

Q.No.11.: Solve the following partial differential equation:

$$\left(D^3 - 4D^2D' + 4DD'^2\right)z = 6\sin(3x + 2y) .$$

Ans.: $z = f_1(y) + f_2(y + 2x) + xf_3(y + 2x) + 2\cos(3x + 2y)$.

Q.No.12.: Solve the following partial differential equation:

$$\frac{\partial^2 z}{\partial x^2} - 3\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = e^{2x+3y} + \sin(x - 2y) .$$

Ans.: $z = f_1(y+x) + f_2(y+2x) + \frac{1}{4}e^{2x+3y} - \frac{1}{15}\sin(x-2y)$.

Q.No.13.: Solve the following partial differential equation:

$$\frac{\partial^2 z}{\partial x^2} + 3\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = 12xy.$$

Ans.: $z = f_1(y-x) + f_2(y-2x) + 2x^3y - \frac{3}{2}x^4$.

Q.No.14.: Solve the following partial differential equation:

$$\left(D^2 - 6DD' + 9D'^2\right)z = 6x + 2y.$$

Ans.: $z = f_1(y+3x) + xf_2(y+3x) + x^2(3x+y)$.

Q.No.15.: Solve the following partial differential equation:

$$\left(D^2 - 6DD' + 9D'^2\right)z = 12x^2 + 36xy.$$

Ans.: $z = f_1(y+3x) + xf_2(y+3x) + 10x^4 + 6x^3y$.

Q.No.16.: Solve the following partial differential equation:

$$\left(D^2 - 2DD' + D'^2\right)z = e^{x+y}.$$

Ans.: $z = f_1(y+x) + xf_2(y+x) + \frac{1}{2}x^2e^{x+y}$.

Q.No.17.: Solve the following partial differential equation:

$$r + 2s + t = 2(y-x) + \sin(x-y).$$

Ans.: $z = f_1(y-x) + xf_2(y-x) + x^2y - x^3 + \frac{1}{2}x^2\sin(x-y)$

Q.No.18.: Solve the following partial differential equation:

$$r + s - 6t = \cos(2x+y).$$

Ans.: $z = f_1(y-3x) + f_2(y+2x) + \frac{1}{5}x\sin(2x+y)$.

Q.No.19.: Solve the following partial differential equation:

$$\frac{\partial^3 z}{\partial x^3} - 4\frac{\partial^3 z}{\partial x^2 \partial y} + 4\frac{\partial^3 z}{\partial x \partial y^2} = 4\sin(2x+y).$$

Ans.: $z = f_1(y) + f_2(y+2x) + xf_3(y+2x) - x^2\cos(2x+y)$.

Q.No.20.: Solve the following partial differential equation:

$$4r - 4s + t = 16\log(x + 2y).$$

Ans.: $z = f_1(2y + x) + xf_2(2y + x) + 2x^2 \log(x + 2y)$.

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7th Topic

Partial Differential Equations

Non-homogeneous linear partial differential equations

Prepared by:
Prof. Sunil
Department of Mathematics & Scientific Computing
NIT Hamirpur (HP)

Non-homogeneous linear partial differential equations:

Definition: Consider a partial differential equation of the form

$$\left(a_0\frac{\partial^n z}{\partial x^n}+a_1\frac{\partial^n z}{\partial x^{n-1}\partial y}+a_2\frac{\partial^n z}{\partial x^{n-2}\partial y^2}+\dots\dots\dots+a_n\frac{\partial^n z}{\partial y^n}\right)+$$
$$\left(b_0\frac{\partial^{n-1} z}{\partial x^{n-1}}+b_1\frac{\partial^{n-1} z}{\partial x^{n-2}\partial y}+b_2\frac{\partial^{n-1} z}{\partial x^{n-3}\partial y^2}+\dots\dots\dots+b_{n-1}\frac{\partial^{n-1} z}{\partial y^{n-1}}\right)+$$
$$\dots\dots\dots\dots\dots+\left(m_0\frac{\partial z}{\partial x}+m_n\frac{\partial z}{\partial y}\right)+n_0 z=F(x,y).$$

i.e. $f(D,D')z=F(x,y)$

Here $a_0,a_1,a_2,\dots\dots\dots,a_n$, $b_0,b_1,b_2,\dots\dots\dots,b_{n-1}$, m_0,m_1,n_0 are all constants. In this equation the dependent variable z and its derivatives are linear, since each term in the LHS contains z or its derivatives. Since RHS is not zero, so this equation is called a ***non-homogeneous linear partial differential equation of the nth order with constant co-efficients***.

Some authors call an equation non-homogeneous if LHS of the equation is not of the same order. Then definition is like this:

In the equation $f(D,D')z=F(x,y)$, (i)

if the polynomial $f(D, D')$ in D, D' is not homogeneous, then (i) is called a ***non-homogeneous linear partial differential equation***.

Complete solution:

As in the case of homogeneous linear partial differential equations,

its complete solution is = C.F. + P.I.

To find the particular integral (P.I.):

The method to find P.I. is exactly the same as those for homogeneous linear partial differential equations.

To find complementary function (C.F.):

We resolve $\phi(D, D')$ into linear factors of the form $D - mD' - a$.

To find the solution of $(D - mD' - a)z = 0$. (ii)

This can also be written as $p - mq = az$.

Lagrange's auxiliary equations are $\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{az}$.

From the first two members $dy + mdx = 0$. $\therefore y + mx = b$.

From the first and last members $\frac{dz}{z} = adx$. $\therefore \log z = ax + \log c \Rightarrow z = ce^{ax}$.

$\therefore$ The complete solution of (ii) is $z = e^{ax} f(y + mx)$.

Hence, the C.F. of (i), i. e. the complete solution of

$(D - m_1 D' - a_1)(D - m_2 D' - a_2)............(D - m_n D' - a_n)z = 0$ is

$z = e^{a_1 x} f_1(y + m_1 x) + e^{a_2 x} f_2(y + m_2 x) + ... + e^{a_n x} f_n(y + m_n x)$.

Also, in the case of repeated factor, for example $(D - mD' - a)^3 z = 0$

We have $z = e^{ax} f_1(y + mx) + xe^{ax} f_2(y + mx) + x^2 e^{ax} f_3(y + mx)$.

Now let us solve some non-homogeneous linear partial differential equation:

Q.No.1.: Solve the following non-homogeneous linear partial differential equation:

$$\left(D^2 + 2DD' + D'^2 - 2D - 2D'\right)z = \sin(x + 2y).$$

Sol.: The given equation is $\left(D^2 + 2DD' + D'^2 - 2D - 2D'\right)z = \sin(x + 2y)$.

This is a non-homogeneous linear partial differential equation with constant co-efficients.

Step 1. To find the complementary function

Here $f(D, D')z = (D + D' - 0)(D + D' - 2)z$.

Note: Since the solution of $(D - m_1D' - a_1)(D - m_2D' - a_2)...........(D - m_nD' - a_n)z = 0$

is $z = e^{a_1x}f_1(y + m_1x) + e^{a_2x}f_2(y + m_2x) + ... + e^{a_nx}f_n(y + m_nx)$.

Thus, C.F. $= f_1(y - x) + e^{2x}f_2(y - x)$.

Step 2. To find the particular integral

$$\text{P.I.} = \frac{1}{D^2 + 2DD' + D'^2 - 2D - 2D'}\sin(x + 2y) = \frac{1}{-1 + 2(-2) + (-4) - 2D - 2D'}\sin(x + 2y)$$

$$= -\frac{1}{2(D + D') + 9}\sin(x + 2y) = -\frac{2(D + D') - 9}{4(D^2 + 2DD' + D'^2) - 81}\sin(x + 2y)$$

$$= -\frac{2(D + D') - 9}{4[-1 + 2(-2) - 4] - 81}\sin(x + 2y)$$

$$= \frac{1}{117}[2\{\cos(x + 2y) + 2\cos(x + 2y)\} - 9\sin(x + 2y)]$$

$$= \frac{1}{39}[2\cos(x + 2y) - 3\sin(x + 2y)].$$

Step 3. To find the complete solution

Now, since the complete solution is z = C.F. + P.I.

$$\therefore \quad z = f_1(y - x) + e^{2x}f_2(y - x) + \frac{1}{39}[2\cos(x + 2y) - 3\sin(x + 2y)].\text{Ans.}$$

Q.No.2.: Solve the following non-homogeneous linear partial differential equation:

$$(D^2 - D'^2 + 3D' - 3D)z = e^{x+2y} + xy.$$

Sol.: The given equation is $(D^2 - D'^2 + 3D' - 3D)z = e^{x+2y} + xy$.

This is a non-homogeneous linear partial differential equation with constant co-efficients.

Step 1. To find the complementary function

Here $f(D, D')z = (D^2 - D'^2 + 3D' - 3D)z = (D - D')(D + D' - 3)z$

$= (D - D' - 0)[D + D' - 3]z$.

Note: Since the solution of $(D - m_1D' - a_1)(D - m_2D' - a_2)...........(D - m_nD' - a_n)z = 0$

is $z = e^{a_1 x} f_1(y + m_1 x) + e^{a_2 x} f_2(y + m_2 x) + \dots\dots\dots\dots + e^{a_n x} f_n(y + m_n x)$.

Thus, C.F. $= f_1(y+x) + e^{3x} f_2(y-x)$.

Step 2. To find the particular integral

(i) P.I. corresponding to e^{x+2y}

$$= \frac{1}{(D-D')(D+D'-3)} e^{x+2y} = \frac{1}{(D+D'-3)(1-2)} e^{x+2y} = -\frac{1}{(D+D'-3)} e^{x+2y}$$

$$= -e^x \frac{1}{(1+D'-3)} e^{2y} = -e^x \frac{1}{D'-2} e^{2y} = -e^x . e^{2y} \frac{1}{D'+2-2} .1$$

$$= -e^{x+2y} \frac{1}{D'} .1 = -e^{x+2y} \int .1 dy = -y e^{x+y} .$$

(ii) P.I. corresponding to xy

$$= \frac{1}{(D-D')(D+D'-3)} .xy = -\frac{1}{3D}\left(1 - \frac{D'}{D}\right)^{-1}\left(1 - \frac{D}{3} - \frac{D'}{3}\right)^{-1} xy$$

$$= -\frac{1}{3D}\left(1 + \frac{D'}{D} + \dots\dots\right)\left(1 + \frac{D}{3} + \frac{D'}{3} + \frac{2DD'}{9} + \dots\dots\dots\right) xy$$

$$= -\frac{1}{3D}\left(1 + \frac{D}{3} + \frac{D'}{3} + \frac{D'}{D} + \frac{D'}{3} + \frac{2DD'}{9} \dots\dots\dots\right) xy$$

$$= -\frac{1}{3D}\left(xy + \frac{y}{3} + \frac{x}{3} + \frac{1}{D}x + \frac{x}{3} + \frac{2}{9}\right) = -\frac{1}{3}\left(\frac{x^2 y}{2} + \frac{xy}{3} + \frac{x^2}{3} + \frac{2x}{9} + \frac{x^3}{6}\right).$$

Step 3. To find the complete solution

Now, since the complete solution is z = C.F. + P.I.

$$\therefore z = f_1(y+x) + e^{3x} f_2(y-x) - y e^{x+2y} - \frac{1}{3}\left(\frac{x^2 y}{2} + \frac{xy}{3} + \frac{x^2}{2} + \frac{2x}{9} + \frac{x^3}{6}\right).$$

Q.No.3.: Solve the following non-homogeneous linear partial differential equation:

$$(D - 3D' - 2)^3 z = 6e^{2x} \sin(3x + y).$$

Sol.: The given equation is $(D - 3D' - 2)^3 z = 6e^{2x} \sin(3x + y)$.

This is a non-homogeneous linear partial differential equation with constant co-efficients.

Step 1. To find the complementary function

Here $f(D, D')z = (D - 3D' - 2)^3 z$.

Note: Since the solution of $(D-mD'-a)^3 z=0$

is $z=e^{ax}f_1(y+mx)+xe^{ax}f_2(y+mx)+x^2e^{ax}f_3(y+mx)$.

Thus, C. F. $=e^{2x}f_1(y+3x)+xe^{2x}f_2(y+3x)+x^2e^{2x}f_3(y+3x)$.

Step 2. To find the particular integral

$$\text{P.I.} =\frac{1}{(D-3D'-2)^3}6e^{2x}\sin(3x+y)=6e^{2x}\frac{1}{(D+2-3D'-2)^3}\sin(3x+y)$$

$$=6e^{2x}\frac{1}{(D-3D')^3}\sin(3x+y)=6e^{2x}.\frac{x^3}{6}.\sin(3x+y)=x^3e^{2x}\sin(3x+y).$$

Step 3. To find the complete solution

Now, since the complete solution is z = C.F. + P.I.

$\therefore z=e^{2x}f_1(y+3x)+xe^{2x}f_2(y+3x)+x^2e^{2x}f_3(y+3x)+x^3e^{2x}\sin(3x+y)$.

Q.No.4.: Solve the following non-homogeneous linear partial differential equation:

$$(D^3-3DD'+D'+4)z=e^{2x+y}.$$

Sol.: The given equation is $(D^3-3DD'+D'+4)z=e^{2x+y}$.

This is a non-homogeneous linear partial differential equation with constant co-efficients.

Step 1. To find the complementary function

Here $(D^3-3DD'+D'+4)$ cannot be resolved into linear factors in D and D'.

Let $z=Ae^{hx+ky}$.

$\therefore (D^3-3DD'+D'+4)z=A(h^3-3hk+k+4)e^{hx+ky}$.

Here $(D^3-3DD'+D'+4)z=0$, iff $h^3-3hk+k+4=0$.

Thus, C.F. $=\sum Ae^{hx+ky}$, where $h^3-3hk+k+4=0$.

Step 2. To find the particular integral

$$\text{P.I.} =\frac{1}{D^3-3DD'+D'+4}e^{2x+y}=\frac{e^{2x+y}}{2^3-3(2)(1)+1+4}=\frac{1}{7}e^{2x+y}.$$

Step 3. To find the complete solution

Now, since the complete solution is z = C.F. + P.I.

$\therefore z = \sum Ae^{hx+ky} + \frac{1}{7}e^{2x+y}$, where $h^3 - 3hk + k + 4 = 0$.

Equation reducible to Partial Differential Equations with constant co-efficients:

An equation in which the **co-efficient** of derivative of any order say k is a multiple of the **variable** of the degree k, then it can be reduced to linear partial differential equation with constant co-efficients in the following way.

Let $x = e^X$, $y = e^Y$ so that $X = \log x$, $Y = \log y$.

Then $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}.\frac{\partial X}{\partial x} = \frac{1}{x}.\frac{\partial z}{\partial X} \Rightarrow x\frac{dz}{dx} = \frac{dz}{dX}$ $\quad\therefore x\frac{\partial}{\partial x} = D\left(= \frac{\partial}{\partial X}\right)$.

Now $x\frac{\partial}{\partial x}\left(x^{k-1}\frac{\partial^{k-1}z}{\partial x^{k-1}}\right) = x^k\frac{\partial^k z}{\partial x^k} + (k-1)x^{k-1}\frac{\partial^{k-1}z}{\partial x^{k-1}}$

$$\Rightarrow x^k\frac{\partial^k z}{\partial x^k} = \left(x\frac{\partial}{\partial x} - k + 1\right)x^{k-1}\frac{\partial^{k-1}z}{\partial x^{k-1}}$$

Putting k = 2, 3, ……….., we get

$$x^2\frac{\partial^2 z}{\partial x^2} = (D-1)x\frac{\partial z}{\partial x} = (D-1)Dz$$

$$x^3\frac{\partial^3 z}{\partial x^3} = (D-2)x^2\frac{\partial^2 z}{\partial x^2} = (D-2)(D-1)Dz \text{ etc.}$$

Similarly, $y\frac{dz}{dy} = D'z$, $y^2\frac{\partial^2 z}{\partial y^2} = (D'-1)D'z$, $y^3\frac{\partial^3 z}{\partial y^3} = (D'-2)(D'-1)D'z$ etc.

and $xy\frac{\partial^2 z}{\partial x\partial y} = DD'z$........

Substituting in the given equation it reduces to $\psi(D - D')z = V$, which is a linear partial differential equation with constant co-efficients.

Q.No.5.: Solve the following partial differential equations:

$$x^2\frac{\partial^2 z}{\partial x^2} - 4xy\frac{\partial^2 z}{\partial x\partial y} + 4y^2\frac{\partial^2 z}{\partial y^2} + 6y\frac{\partial z}{\partial y} = x^3y^4.$$

Sol.: The given equation is $x^2\frac{\partial^2 z}{\partial x^2} - 4xy\frac{\partial^2 z}{\partial x\partial y} + 4y^2\frac{\partial^2 z}{\partial y^2} + 6y\frac{\partial z}{\partial y} = x^3y^4$.

This equation is not a linear partial differential equation with constant co-efficients, but it can be reduced to linear partial differential equation with constant co-efficients in the following way.

Substituting $x = e^X$, $y = e^Y$ and denoting $D = \frac{\partial}{\partial X}$, $D' = \frac{\partial}{\partial Y}$.

Given equation reduces to $[D(D-1) - 4DD' + 4(D'-1)D' + 6D']z = e^{3X+4Y}$

$$\Rightarrow (D-2D')(D-2D'-1)z = e^{3X+4Y}.$$

Now this is a non-homogeneous linear partial differential equation with constant co-efficients.

Step 1. To find the complementary function

Here $f(D,D')z = (D-2D')(D-2D'-1)z$.

Note: Since the solution of $(D-m_1D'-a_1)(D-m_2D'-a_2)\ldots\ldots\ldots(D-m_nD'-a_n)z = 0$

is $z = e^{a_1x}f_1(y+m_1x) + e^{a_2x}f_2(y+m_2x) + \ldots\ldots\ldots + e^{a_nx}f_n(y+m_nx)$.

Thus, $\text{C.F.} = f_1(Y+2X) + e^X f_2(Y+2X)$

$$= f_1(\log y + 2\log x) + e^{\log x} f_2(\log y + 2\log x)$$

$$= f_1\left[\log\left(yx^2\right)\right] + xf_2\left[\log\left(yx^2\right)\right]$$

$$= g_1\left(yx^2\right) + xg_2\left(yx^2\right).$$

Step 2. To find the particular integral

$$\text{P.I.} = \frac{1}{(D-2D')(D-2D'-1)}e^{3X+4Y} = \frac{e^{3X+4Y}}{[3-2(4)][3-2(4)-1]} = \frac{1}{30}x^3y^4.$$

Step 3. To find the complete solution

Now, since the complete solution is z = C.F. + P.I.

$$\therefore z = g_1\left(yx^2\right) + xg_2\left(yx^2\right) + \frac{1}{30}x^3y^4.$$

Q.No.6.: Solve the following partial differential equations:

$$\left(D^2 - DD' + D' - 1\right)z = \cos(x+2y).$$

Sol.: The given equation is $\left(D^2 - DD' + D' - 1\right)z = \cos(x+2y)$.

This is a non-homogeneous linear partial differential equation with constant co-efficients.

Step 1. To find the complementary function

Here $f(D,D')z = \left(D^2 - DD' + D' - 1\right)z = (D+1)(D-1)z - D'(D-1)z$

$$= (D-1)(D-D'+1)z = (D-0D'-1)\left[D-D'-(-1)\right]z.$$

Note: Since the solution of $(D-m_1D'-a_1)(D-m_2D'-a_2)..........(D-m_nD'-a_n)z=0$

is $z = e^{a_1x}f_1(y+m_1x)+e^{a_2x}f_2(y+m_2x)+..........+e^{a_nx}f_n(y+m_nx)$.

Thus, C.F. $= e^{x}f_1(y+0x)+e^{-x}f_2(y+x) \Rightarrow e^{x}f_1(y)+e^{-x}f_2(y+x)$.

Step 2. To find the particular integral

$$\text{P.I.} = \frac{1}{D^2-DD'+D'-1}\cos(x+2y)$$

$$= \frac{1}{-1^2-(-1.2)+D'-1}\cos(x+2y) \qquad [\text{putting } D^2=-1^2 \text{ and } DD'=-1.2]$$

$$= \frac{1}{D'}\cos(x+2y) = \int_{x\ \text{constant}} \cos(x+2ydy) = \frac{1}{2}\sin(x+2y).$$

Step 3. To find the complete solution

Now, since the complete solution is z = C.F. + P.I.

$\therefore z = e^{x}f_1(y)+e^{-x}f_2(y+x)+\frac{1}{2}\sin(x+2y)$. Ans.

Q.No.7.: Solve the following partial differential equations: $\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial x\partial y}+\frac{\partial z}{\partial y}-z=e^{-x}$.

Sol.: The given equation is $\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial x\partial y}+\frac{\partial z}{\partial y}-z=e^{-x}$.

This equation in symbolic form can be written as $\left(D^2+DD'+D'-1\right)z=e^{-x}$.

This is a non-homogeneous linear partial differential equation with constant co-efficients.

Step 1. To find the complementary function

Here $f(D,D')z = \left(D^2+DD'+D'-1\right)z = (D+1)(D+D'-1)z$.

Note: Since the solution of $(D-m_1D'-a_1)(D-m_2D'-a_2)..........(D-m_nD'-a_n)z=0$

is $z = e^{a_1x}f_1(y+m_1x)+e^{a_2x}f_2(y+m_2x)+..........+e^{a_nx}f_n(y+m_nx)$.

Thus, C.F. $= e^{-x}f_1(y)+e^{x}f_2(y-x)$.

Step 2. To find the particular integral

$$\text{P.I.} = \frac{1}{D^2 + DD' + D' - 1} e^{-x} = \frac{1}{(D+1)(D+D'-1)} e^{-x}$$

$$= \frac{1}{D+1} \frac{1}{(-1+0-1)} e^{-x} = -\frac{1}{2} \frac{1}{D+1} e^{-x} = -\frac{1}{2} x \frac{1}{1} e^{-x} = -\frac{1}{2} x e^{-x}.$$

Step 3. To find the complete solution

Now, since the complete solution is z = C.F. + P.I.

$$\therefore\ z = e^{-x} f_1(y) + e^{x} f_2(y - x) - \frac{1}{2} x e^{-x}.\ \text{Ans.}$$

Q.No.8.: Solve the following partial differential equations:

$$(D - D' - 1)(D - D' - 2)z = e^{2x-y}.$$

Sol.: The given equation is $(D - D' - 1)(D - D' - 2)z = e^{2x-y}$**.**

This is a non-homogeneous linear partial differential equation with constant co-efficients.

Step 1. To find the complementary function

Here $f(D, D')z = (D - D' - 1)(D - D' - 2)z$.

Note: Since the solution of $(D - m_1 D' - a_1)(D - m_2 D' - a_2)\ldots\ldots\ldots\ldots(D - m_n D' - a_n)z = 0$

is $z = e^{a_1 x} f_1(y + m_1 x) + e^{a_2 x} f_2(y + m_2 x) + \ldots\ldots\ldots\ldots + e^{a_n x} f_n(y + m_n x)$.

Thus, C.F. $= e^{x} f_1(y + x) + e^{2x} f_2(y + x)$.

Step 2. To find the particular integral

$$\text{P.I.} = \frac{1}{(D - D' - 1)(D - D' - 2)} e^{2x-y} = \frac{1}{(2+1-1)(2+1-2)} e^{2x-y} = \frac{1}{2} e^{2x-y}.$$

Step 3. To find the complete solution

Now, since the complete solution is z = C.F. + P.I.

$$\therefore\ z = e^{x} f_1(y + x) + e^{2x} f_2(y + x) + \frac{1}{2} e^{2x-y}.\ \text{Ans.}$$

Q.No.9.: Solve the following partial differential equations:

$$(D + D' - 1)(D + 2D' - 3)z = 4 + 3x + 6y.$$

Sol.: The given equation is $(D + D' - 1)(D + 2D' - 3)z = 4 + 3x + 6y$**.**

This is a non-homogeneous linear partial differential equation with constant co-efficients.

Step 1. To find the complementary function

Here $f(D,D')z = (D+D'-1)(D+2D'-3)z$.

Note: Since the solution of $(D-m_1D'-a_1)(D-m_2D'-a_2)..........(D-m_nD'-a_n)z=0$

is $z = e^{a_1x}f_1(y+m_1x)+e^{a_2x}f_2(y+m_2x)+..........+e^{a_nx}f_n(y+m_nx)$.

Thus, C.F. $= e^x f_1(y-x)+e^{3x}f_2(y-2x)$.

Step 2. To find the particular integral

$$\text{P.I.} = \frac{1}{(D+D'-1)(D+2D'-3)}(4+3x+6y)$$

$$= \frac{1}{3}(1-D-D')^{-1}\left(1-\frac{D}{3}-\frac{2}{3}D'\right)^{-1}(4+3x+6y)$$

$$= \frac{1}{3}[1+D+D'+.....]\left[1+\frac{D}{3}+\frac{2}{3}D'+.......\right](4+3x+6y)$$

$$= \frac{1}{3}\left[1+\frac{D}{3}+\frac{2}{3}D'+D+D'+.....\right][4+3x+6y]$$

$$= \frac{1}{3}\left[4+3x+6y+\frac{1}{3}.3+\frac{2}{3}.6+3+6\right] = \frac{1}{3}[3x+6y+18] = x+2y+6.$$

Step 3. To find the complete solution

Now, since the complete solution is z = C.F. + P.I.

$\therefore z = e^x f_1(y-x)+e^{3x}f_2(y-2x)+x+2y+6$. Ans.

Q.No.10.: Solve the following partial differential equations:

$$\frac{\partial^2 z}{\partial x^2}-\frac{\partial^2 z}{\partial x\partial y}+\frac{\partial z}{\partial x} = x^2+y^2.$$

Sol.: The given equation is $\frac{\partial^2 z}{\partial x^2}-\frac{\partial^2 z}{\partial x\partial y}+\frac{\partial z}{\partial x} = x^2+y^2$.

This equation in symbolic form can be written as $(D^2-DD'+D)z = x^2+y^2$.

This is a non-homogeneous linear partial differential equation with constant co-efficients.

Step 1. To find the complementary function

Here $f(D,D')z = (D^2-DD'+D)z = D(D-D'+1)z = (D-0D'-0)(D-D'+1)z$.

Note: Since the solution of $(D-m_1D'-a_1)(D-m_2D'-a_2)..........(D-m_nD'-a_n)z=0$

is $z = e^{a_1 x} f_1(y + m_1 x) + e^{a_2 x} f_2(y + m_2 x) + \ldots\ldots\ldots\ldots + e^{a_n x} f_n(y + m_n x)$.

Thus, C.F. $= f_1(y) + e^{-x} f_2(y + x)$.

Step 2. To find the particular integral

$$\text{P.I.} = \frac{1}{D(D - D' + 1)}\left(x^2 + y^2\right) = \frac{1}{D}(1 + D - D')^{-1}\left(x^2 + y^2\right)$$

$$= \frac{1}{D}\left[1 - (D - D') + (D - D')^2 - \ldots\ldots\ldots\right]\left(x^2 + y^2\right)$$

$$= \frac{1}{D}\left[1 - D + D' + D^2 + D'^2 - 2DD' - \ldots\ldots\ldots\right]\left(x^2 + y^2\right)$$

$$= \frac{1}{D}\left(x^2 + y^2 - 2x + 2y + 2 + 2\right) = \frac{x^3}{3} + xy^2 - 2\frac{x^2}{2} + 2xy + 2x + 2x$$

$$= \frac{1}{3}x^3 - x^2 + xy^2 + 2xy + 4x\,.$$

Step 3. To find the complete solution

Now, since the complete solution is z = C.F. + P.I.

$$\therefore\ z = f_1(y) + e^{-x} f_2(y + x) + \frac{1}{3}x^3 - x^2 + xy^2 + 2xy + 4x\,. \text{ Ans.}$$

Q.No.11.: Solve the following partial differential equations:

$$\left(2DD' + D'^2 - 3D'\right)z = 3\cos(3x - 2y).$$

Sol.: The given equation is $\left(2DD' + D'^2 - 3D'\right)z = 3\cos(3x - 2y)$**.**

This is a non-homogeneous linear partial differential equation with constant co-efficients.

Step 1. To find the complementary function

Here $f(D, D')z = D'[2D + D' - 3]z = (D' - 0)[2D + D' - 3]z$.

Note: Since the solution of $(D - m_1 D' - a_1)(D - m_2 D' - a_2)\ldots\ldots\ldots\ldots(D - m_n D' - a_n)z = 0$

is $z = e^{a_1 x} f_1(y + m_1 x) + e^{a_2 x} f_2(y + m_2 x) + \ldots\ldots\ldots\ldots + e^{a_n x} f_n(y + m_n x)$.

Thus, C.F. $= f_1(x) + e^{3x} f_2(2y - x)$.

Step 2. To find the particular integral

$$\text{P.I.} = \frac{1}{2DD' + D'^2 - 3D'}3\cos(3x - 2y) = \frac{3}{2(6) - 4 - 3D'}\cos(3x - 2y)$$

$$=\frac{3}{8-3D'}\cos(3x-2y)=\frac{3(8+3D')}{64-9D'^2}\cos(3x-2y)$$

$$=\frac{3}{100}(8+3D')\cos(3x-2y)=\frac{3}{50}[4\cos(3x-2y)+3\sin(3x-2y)]$$

Step 3. To find the complete solution

Now, since the complete solution is z = C.F. + P.I.

$\therefore\ z=f_1(x)+e^{3x}f_2(2y-x)+\frac{3}{50}[4\cos(3x-2y)+3\sin(3x-2y)]$. Ans.

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Home Assignments

Q.No.1.: Solve the following partial differential equation:

$$DD'(D+2D'+1)z=0.$$

Ans.: $z=f_1(y)+f_2(-x)+e^{-x}f_3(y-2x)$.

Q.No.2.: Solve the following partial differential equation:

$$r+2s+t+2p+2q=0$$

Ans.: $z=e^{-x}f_1(y-x)+xe^{-x}f_2(y-x)$.

Q.No.3.: Solve the following partial differential equation:

$$D(D-2D'-3)z=e^{x+2y}.$$

Ans.: $z=f_1(y)+e^{3x}f_2(y+2x)-\frac{1}{6}e^{x+2y}$.

Q.No.4.: Solve the following partial differential equation:

$$(D-D'-1)(D-D'-2)z=e^{x+2y}+x.$$

Ans.: $z=e^{x}f_1(y+x)+e^{2x}f_2(y+x)+\frac{1}{2}e^{2x-y}+\frac{1}{4}(2x+3)$.

Q.No.5.: Solve the following partial differential equation:

$$(D+D'-1)(D+2D'-3)z=4+3x+6y.$$

Ans.: $z = e^{x}f_1(y-x) + e^{3x}f_2(y-2x) + x + 2y + 6$.

Q.No.6.: Solve the following partial differential equation:

$$\left(D^2 + DD' + D' - 1\right)z = \cos(x+2y) + e^{y}.$$

Ans.: $z = e^{x}f_1(y) + e^{-x}f_2(y+x) + \frac{1}{2}\sin(x+2y) - xe^{y}$.

Q.No.7.: Solve the following partial differential equation:

$$\left(D^2 - D'\right)z = 2y - x^2.$$

Ans.: $z = \sum Ae^{hx+h^2y} - y^2 - \frac{x^4}{12}$.

Q.No.8.: Solve the following partial differential equation:

$$x^2r + 2xys + y^2t = 0.$$

Ans.: $z = f_1\left(\frac{y}{x}\right) + xf_2\left(\frac{y}{x}\right)$.

Q.No.9.: Solve the following partial differential equation:

$$x^2r - 3xys + 2y^2t + px + 2qy = x + 2y.$$

Ans.: $z = f_1(xy) + f_2\left(x^2y\right) + x + y$.

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8th Topic

Partial Differential Equations

Non-linear partial differential equation of second order

(Monge's Method)

Prepared by:
Prof. Sunil
Department of Mathematics & Scientific Computing
NIT Hamirpur (HP)

PARTIAL DIFFERENTIAL EQUATION OF THE FIRST ORDER:

A differential equation involving first order partial derivatives p and q only is called a **partial differential equation of the first order**.

NON-LINEAR PARTIAL DIFFERENTIAL EQUATION:

A partial differential equation, which is not linear, is known as non-linear P.D.E.

e.g. $\left(\frac{\partial^2 u}{\partial x^2}\right)^2 + u^2\left(\frac{\partial u}{\partial y}\right) = f(x, y)$ is non-linear in u and of second order.

NON-LINEAR PARTIAL DIFFERENTIAL EQUATION OF THE FIRST ORDER:

A partial differential equation, which involves **first order** partial derivatives p and q of **degree (power) higher than one** and/or product of terms p and q, is called a **non-linear partial differential equation.**

NON-LINEAR PARTIAL DIFFERENTIAL EQUATION OF THE SECOND ORDER:

Equation, which at least included one of the partial derivatives r, s, t but none of higher order is called a non-linear partial differential equation of second order.

The most general second order non-linear partial differential equation in two independent variables, and z as the dependent variable has the form

$F(x, y, z, p, q, r, s, t) = 0$,

whose solution can be obtained by Monge's method in special cases.

Here we study a method due to Monge's to solve a non-linear partial differential equation of the form

$Rr + Ss + Tt = V$,

where R, S, T and V are functions of p and q as well as x, y, z.

This equation is non-linear (also referred to as "quasilinear" or "uniform non-linear".

Monge's Method of integrating $Rr + Ss + Tt = V$:

The given equation is $Rr + Ss + Tt = V$. (i)

Since, we know that

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = pdx + qdy$$

$$dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy = rdx + sdy \Rightarrow r = \frac{dp - sdy}{dx}.$$

$$dq = \frac{\partial q}{\partial x}dx + \frac{\partial q}{\partial y}dy = sdx + tdy \Rightarrow t = \frac{dq - sdx}{dy}.$$

Substituting the values of r and t in (i) and rearranging the terms, we get

$$R\left(\frac{dp - sdy}{dx}\right) + Ss + T\left(\frac{dq - sdx}{dy}\right) = V$$

$$(Rdpdy + Tdqdx - Vdxdy) - s\left[R(dy)^2 - Sdydx + T(dx)^2\right] = 0. \quad \text{(ii)}$$

Thus, we obtain

$$R(dy)^2 - Sdydx + T(dx)^2 = 0. \quad \text{(iii)}$$

$$Rdpdy + Tdqdx - Vdxdy = 0, \quad \text{(iv)}$$

These two simultaneous equations (iii) and (iv) are known as **Monge's (subsidiary) equations**. Subsidiary equations mean auxiliary/additional/contributory/secondary equations. By solving the Monge's auxiliary equations (iii) and (iv), one or two intermediate integrals $u_1 = f(v_1)$ and/or $u_2 = f(v_2)$ of (i) are obtained.

Solving these integrals, we will get the values of p and q.

Substituting p and q in $dz = pdx + qdy$, which on integration yields the required **general solution** of (i) involving two arbitrary functions.

In general, the quadratic (iii) can be resolved into two (or one repeated) equations.

Case I: Suppose (iii) can be resolved as

$$R(dy)^2 - Sdydx + T(dx)^2 = (A_1dy + B_1dx)(A_2dy + B_2dx) = 0$$

where $A_1B_2 \neq A_2B_1$.

Then we have two systems

$$A_1dy + B_1dx = 0 \text{ and } Rdpdy + Tdqdx - Vdxdy = 0 \quad \text{(v)}$$

$$A_2dy + B_2dx = 0 \text{ and } Rdpdy + Tdqdx - Vdxdy = 0. \quad \text{(vi)}$$

Integrating (v), we get two integrals $u_1 = a$ and $v_1 = b$.

Thus, we get an intermediate integral of (i) as $u_1 = f_1(v_1)$. (vii)

Similarly, from (vi), we get another intermediate integral of (i) as $u_2 = f_2(v_2)$. (viii)

Solving (vii) and (viii), determine p and q as functions of x and y and z.

Substituting p and q in $dz = pdx + qdy$,

which on integration yields the required general solution of (i) involving two arbitrary functions.

Note:

1. Some cases, the second intermediate integral can be obtained from the first one by inspection.
2. When inspection fails, rearrange the first intermediate integral in the form $Pp + Qq = R^*$ and solve by Lagrange's method.

Case II: Suppose (iii) is a perfect square, i.e.

$$R(dy)^2 - Sdydx + T(dx)^2 = (Ady + Bdx)^2 = 0,$$

then we have only one system

$$Ady + Bdx = 0 \text{ and } Rdpdy + Tdqdx - Vdxdy = 0 \quad \text{(ix)}$$

Solving (ix), we get only one intermediate integral $u = f(v)$ of the form $Pp + Qq = R^*$, which is integrated using Lagrange's method (or Charpit's method).

Now let us solve some non-linear partial differential equation of second order by Monge's method:

Q.No.1.: Solve the following non-linear partial differential equation of second order by Monge's method: $(x-y)(xr-xs-ys+yt)=(x+y)(p-q)$.

Sol.: Rewriting the given non-linear partial differential equation of second order in the standard form $Rr+Ss+Tt=V$ and find the values of R, S, T and V, then substitute these values in Monge's auxiliary equations

$R(dy)^2-Sdydx+T(dx)^2=0$,

$Rdpdy+Tdqdx-Vdxdy=0$.

Thus $(x-y)(xr-xs-ys+yt)=(x+y)(p-q) \Rightarrow xr-(x+y)s+yt=\dfrac{(x+y)}{(x-y)}(p-q)$.

Here $R=x,\ S=-(x+y),\ T=y$ and $V=\dfrac{x+y}{x-y}(p-q)$.

Thus, Monge's auxiliary equations become

$$xdy^2+(x+y)dydx+ydx^2=0, \tag{i}$$

$$xdpdy+ydqdx-\frac{x+y}{x-y}(p-q)dydx=0. \tag{ii}$$

Here equation (i) can be factorized as

$xdy^2+(x+y)dydx+ydx^2=0 \Rightarrow (xdy+ydx)(dx+dy)=0$

$\Rightarrow (xdy+ydx)=0$ and/or $(dx+dy)=0$.

Integrating, we get $xy=c$ and/or $x+y=c$.

Taking $xy=c$ and dividing each term of (ii) by xdy or its equivalent $-ydx$, we get

$dp-dq-\dfrac{dx-dy}{x-y}(p-q)=0 \Rightarrow \dfrac{d(p-q)}{p-q}-\dfrac{d(x-y)}{x-y}=0$.

Integrating, we obtain

$\log(p-q)-\log(x-y)=\log c \Rightarrow \dfrac{(p-q)}{(x-y)}=c$.

Hence, a first integral of the given equation is $p-q=(x-y)\phi(xy)$,

which is a Langrange's linear equation.

Its auxiliary equations are $\frac{dx}{1} = \frac{dy}{-1} = \frac{dz}{(x-y)\phi(xy)}$.

From the first two equations, we have $x + y = a$.

Using this, we have $dz = -\phi(ax - x^2).(a - 2x)dx$.

Integrating, we get $z = \phi_1(ax - x^2) + b$.

Writing $b = \phi_2(a)$ and $a = x + y$.

Thus, the general solution involving two arbitrary functions is

$z = \phi_1(xy) + \phi_2(x + y)$,

where ϕ_1, ϕ_2 are two arbitrary functions.

Remarks: We can also started with the integral $x + y = c$ and divided each term of (ii) by dx or $-dy$, we would have arrived at the same solution.

Q.No.2.: Solve the following non-linear partial differential equation of second order by Monge's method: $y^2 r - 2ys + t = p + 6y$.

Sol.: Rewriting the given non-linear partial differential equation of second order in the standard form $Rr + Ss + Tt = V$ and find the values of R, S, T and V, then substitute these values in Monge's auxiliary equations

$R(dy)^2 - Sdydx + T(dx)^2 = 0$,

$Rdpdy + Tdqdx - Vdxdy = 0$.

Here $R = y^2$, $S = -2y$, $T = 1$ and $V = p + 6y$.

Thus, Monge's auxiliary equations become

$$ydy^2 + 2ydydx + dx^2 = 0 , \qquad \text{(i)}$$

$$y^2dpdy + dqdx - (p + 6y)dydx = 0 . \qquad \text{(ii)}$$

Here equation (i) is a perfect square and can be written as

$$(ydy + dx)^2 = 0 \Rightarrow y^2 + 2x = c . \qquad \text{(iii)}$$

Putting $ydy = -dx$ in (ii), we get

$ydp - dq + (p + 6y)dy = 0 \Rightarrow (ydp + pdy) - dq + 6ydy = 0$.

Integrating, we get $py - q + 3y^2 = a$.

Combining this with (iii), we get the intermediate integral $py - q + 3y^2 = \phi\left(y^2 + 2x\right)$, which is a Langrange's linear equation.

The subsidiary equations are $\frac{dx}{y} = \frac{dy}{-1} = \frac{dz}{\phi\left(y^2 + 2x\right) - 3y^2}$.

From the first two equations, we get $\boxed{y^2 + 2x = c}$.

Using this, we have $dz + \left[\phi(c) - 3y^2\right]dy = 0$.

Integrating, we get $z + y\phi(c) - y^3 = b$.

Writing $b = \psi(c)$ and $c = y^2 + 2x$.

Thus, the general solution involving two arbitrary functions is

$z = y^3 - y(\phi)\left(y^2 + 2x\right) + \psi\left(y^2 + 2x\right)$,

where ϕ, ψ are two arbitrary functions.

Q.No.3.: Solve the following non-linear partial differential equation of second order by

Monge's method: $y^2 r + 2sxy + x^2 t + px + qy = 0$.

Sol.: Rewriting the given non-linear partial differential equation of second order in the standard form $Rr + Ss + Tt = V$ and find the values of R, S, T and V, then substitute these values in Monge's auxiliary equations

$R(dy)^2 - Sdydx + T(dx)^2 = 0$,

$Rdpdy + Tdqdx - Vdxdy = 0$.

Here $R = y^2$, $S = 2xy$, $T = x^2, V = -(px + qy)$.

Thus, Monge's auxiliary equations become

$y^2 dy^2 - 2xydxdy + x^2 dx^2 = 0$. (i)

$y^2 dpdy + x^2 + dqdx + (px + qy)dxdy = 0$, (ii)

Here equation (i) is a perfect square and can be written as

$(ydy - xdx)^2 = 0 \Rightarrow ydy - xdx = 0$. (iii)

Integrating, we get $x^2 - y^2 = c_1$. (iv)

From (ii) and (iii), we get $(ydp + pdy) + (xdq + qdx) = 0 \Rightarrow yp + xq = c_1$ (v)

One intermediate integral is

$yp + xq = f\left(x^2 - y^2\right) \Rightarrow yp + xq = f(c_1)$,

which is a Langrange's linear equation.

The subsidiary equations are $\frac{dx}{y} = \frac{dy}{x} = \frac{dz}{f(c_1)}$.

From the first two members, we get $xdx - ydy = 0 \Rightarrow x^2 - y^2 = c_1$

From the last two members, we get $\frac{dy}{x} = \frac{dz}{f(c_1)} \Rightarrow dz = \frac{f(c_1)}{\sqrt{c_1 + y^2}} dz$.

Integrating, we get

$$z = f(c_1)\sinh^{-1}\left\{\frac{y}{\sqrt{c}}\right\} + c_2 \Rightarrow z - f(c_1)\sinh^{-1}\left\{\frac{y}{\sqrt{c}}\right\} = c_2$$

Thus, the general solution involving two arbitrary functions is $c_2 = g\left(c_1\right)$

$$\Rightarrow z = f\left(x^2 - y^2\right)\sinh^{-1}\left\{\frac{y}{\sqrt{x^2 - y^2}}\right\} + g\left(x^2 - y^2\right),$$

where f, g are two arbitrary functions.

Q.No.4.: Solve the following non-linear partial differential equation of second order by Monge's method: $r = a^2 t$.

Sol.: Rewriting the given non-linear partial differential equation of second order in the standard form $Rr + Ss + Tt = V$ and find the values of R, S, T and V, then substitute these values in Monge's auxiliary equations

$$R(dy)^2 - Sdydx + T(dx)^2 = 0,$$

$$Rdpdy + Tdqdx - Vdxdy = 0.$$

The given equation is $r - a^2 t = 0$.

Here $R = 1,\ T = -a^2,\ S = 0,\ V = 0.$

Thus, Monge's auxiliary equations become

$$dy^2 - a^2 dx^2 = 0, \qquad \text{(i)}$$

$$dpdy - a^2 dqdx = 0. \qquad \text{(ii)}$$

Here equation (i) can be factorized as

$$(dy - adx)(dy + adx) = 0$$

$dy - adx = 0$ (iii)

$dy + adx = 0$ (iv)

From (ii) and (iii), we have $dp(adx) - a^2 dqdx = 0 \Rightarrow dp - adq = 0$ (v)

From (iii) and (v), we get $y - ax = c_1$ and $p - aq = c_2$.

$$\therefore p - aq = f(y - ax). \qquad \text{(vi)}$$

Similarly from (iv) and (ii), we get $p - aq = g(y + ax)$. (vii)

Solving (vi) and (vii) for p and q, we get

$$p = \frac{1}{2}\{f(y - ax) + g(y + ax)\},$$

and $q = \dfrac{1}{2a}\{g(y + ax) - f(y - ax)\}$.

Substituting the values of p and q in $dz = pdx + qdy$, we get

$$dz = \frac{1}{2}\{f(y - ax) + g(y + ax)\}dx + \frac{1}{2a}\{g(y + ax) - f(y - ax)\}dy$$

$$= \frac{1}{2}\{g(y + ax)(dy + adx) - f(y - ax)(dy - adx)\}.$$

Integrating, we get $z = \dfrac{1}{2a}\{G(y + ax) - F(y - ax)\} \Rightarrow z = G_1(y + ax) + F_1(y - ax)$.

Thus, the general solution involving two arbitrary functions is

$z = G_1(y + ax) + F_1(y - ax)$,

where G_1 and F_1 are two arbitrary functions.

Q.No.5.: Solve the following non-linear partial differential equation of second order by

Monge's method: $q(q^2 + s) = pt$ or $qs - pt = q^3$.

Sol.: Rewriting the given non-linear partial differential equation of second order in the standard form $Rr + Ss + Tt = V$ and find the values of R, S, T and V, then substitute these values in Monge's auxiliary equations

$R(dy)^2 - Sdydx + T(dx)^2 = 0$,

$Rdpdy + Tdqdx - Vdxdy = 0$.

Here $R = 0$, $S = -q$, $T = p$, $V = q^3$.

Thus, Monge's auxiliary equations become

$pdqdx - q^3dxdy = 0$,

$qdxdy + pdx^2 = 0$.

$\Rightarrow pdq - q^3dy = 0$, (i)

and $qdy + pdx = 0$. (ii)

But $dz = pdx + qdy = 0 \Rightarrow z = c_1$.

Combining (i) and (ii), we get

$$pdq - q^2(-pdx) = 0 \Rightarrow dq + q^2dx = 0 \Rightarrow \frac{dq}{q^2} + dx = 0 \Rightarrow -\frac{1}{q} + x = c_2$$

$$\Rightarrow \frac{dy}{dz} - x = -f(z), \quad \text{where } c_2 = f(z).$$

Integrating, we get $y = xz - \int f(z)dz + c_3 \Rightarrow y = xz + g(z) + h(z)$.

Thus, the general solution involving two arbitrary functions is

$y = xz + g(z) + h(z)$,

where g, h are two arbitrary functions.

Q.No.6.: Solve the following non-linear partial differential equation of second order by Monge's method: $r - t\cos^2 x + p\tan x = 0$.

Sol.: Rewriting the given non-linear partial differential equation of second order in the standard form $Rr + Ss + Tt = V$ and find the values of R, S, T and V, then substitute these values in Monge's auxiliary equations

$R(dy)^2 - Sdydx + T(dx)^2 = 0$,

$Rdpdy + Tdqdx - Vdxdy = 0$.

Here R = 1, S = 0, $t = -\cos^2 x$, $V = -p\tan x$

Thus, Monge's auxiliary equations become

$dy^2 - \cos^2 xdx^2 = 0$. (i)

$dpdy - \cos^2 xdqdx + p\tan xdxdy = 0$, (ii)

Here equation (i) can be factorized as

$(dy - \cos xdx)(dy + \cos xdx) = 0$

which yields $dy - \cos xdx = 0$ (iii)

or $dy + \cos x dx = 0$. (iv)

From (iii), we have $y - \sin x = c_1$.

Again from (iii) and (ii), we have

$dp(\cos x dx) - \cos^2 x dq dx + p \tan x dx \cos x dx = 0 \quad \Rightarrow dp - \cos x dq + p \tan x dx = 0$

$\Rightarrow \sec x dp - dq + p \tan x \sec x dx = 0 \Rightarrow d(p \sec x) - dq = 0 \Rightarrow p \sec x - q = c_2$

$\therefore p \sec x - q = f(y - \sin x)$. (v)

Similarly from (iv) and (ii), we have

$p \sec x + q = g(y = \sin x)$. (vi)

Solving (v) and (vi), we get

$$p = \frac{1}{2}\cos x[f(y - \sin x) + g(y + \sin x)],$$

$$q = \frac{1}{2}[g(y + \sin x) - f(y - \sin x)].$$

Substituting the values of p and q in $dz = pdx + qdy$, we get

$$dz = \frac{1}{2}\cos x[f(y - \sin x) + g(y + \sin x)]dx + \frac{1}{2}[g(y + \sin x) - f(y - \sin x)]dy$$

$$= \frac{1}{2}[g(y + \sin x)(dy + \cos x dx) - f(y - \sin x)(dy - \cos x dx)].$$

Integrating, we get $z = \frac{1}{2}[G(y + \sin x) + F(y - \sin x)]$.

Thus, the general solution involving two arbitrary functions is

$$z = \frac{1}{2}[G(y + \sin x) + F(y - \sin x)],$$

where G, F are two arbitrary functions.

Q.No.8.: Solve the following non-linear partial differential equation of second order by Monge's method: $r + (a + b)s + abt = xy$.

Sol.: Rewriting the given non-linear partial differential equation of second order in the standard form $Rr + Ss + Tt = V$ and find the values of R, S, T and V, then substitute these values in Monge's auxiliary equations

$R(dy)^2 - Sdydx + T(dx)^2 = 0$,

$Rdpdy + Tdqdx - Vdxdy = 0$.

Here R = 1, S = a + b, T = ab, V = xy.

Thus, Monge's auxiliary equations become

$$(dy)^2 - (a+b)dxdy + ab(dx)^2 = 0, \tag{i}$$

$$dpdy + abdqdx - xydxdy = 0. \tag{ii}$$

Here equation (i) can be factorized as

$$(dy - adx)(dy - bdx) = 0$$

which yields $dy - adx = 0$, (iii)

and $dy - bdx = 0$. (iv)

Integrating (iii) and (iv), we get

$$y - ax = c_1, \tag{v}$$

$$y - bx = c_2. \tag{vi}$$

Substitute $dy = adx$ from (iii) in (ii), then

$$dp(adx) + abdqdx - xydx(adx) = 0$$

$$\Rightarrow dp + bdq - xydx = 0.$$

Integrating, we get

$$p + bq - y\frac{x^2}{2} = \text{constant} = \psi_1(y - ax). \tag{vii}$$

Similarly using $dy = bdx$ from (iv) in (ii), we get

$$dp(bdx) + abdqdx - xydx(bdx) = 0 \Rightarrow dp = adq - xydx = 0$$

Integrating, we get

$$p + aq - y\frac{x^2}{2} = \text{constant} = \psi_2(y - bx) \tag{viii}$$

Now solve (vii) and (viii) for p and q.

Multiply (vii) by 'a' and (viii) by 'b' and subtract them

$$(b-a)p - (b-a)\frac{x^2y}{2} = b\psi_2 - a\psi_1$$

$$\Rightarrow p = \frac{b\psi_2 - a\psi_1}{b-a} + \frac{x^2y}{2}. \tag{ix}$$

Similarly, subtracting (viii) from (vii), we get

$$q=\frac{\psi_2-\psi_1}{a-b}. \tag{x}$$

Since we know that $dz = pdx + qdy$.

Substituting p and q from (ix) and (x), we have

$$dz=\left(\frac{b\psi_2-a\psi_1}{b-a}+\frac{x^2y}{2}\right)dx+\left(\frac{\psi_2-\psi_1}{a-b}\right)dy$$

$$\Rightarrow (a-b)dz=-\psi_1(dy-adx)+\psi_2(dy-bdx)+(a-b)\frac{x^2y}{2}dx$$

Integrating, we get $z=\phi_1(y-ax)+\phi_2(y-bx)+\dfrac{x^3y}{6}$.

Thus, the general solution involving two arbitrary functions is

$$z=\phi_1(y-ax)+\phi_2(y-bx)+\frac{x^3y}{6},$$

where ϕ_1, ϕ_2 are two arbitrary functions.

Q.No.9.: Solve the following non-linear partial differential equation of second order by Monge's method: $q^2r-2pqs+p^2t=pt-qs$.

Sol.: Rewriting the given non-linear partial differential equation of second order in the standard form $Rr+Ss+Tt=V$ and find the values of R, S, T and V, then substitute these values in Monge's auxiliary equations

$$R(dy)^2-Sdydx+T(dx)^2=0,$$

$$Rdpdy+Tdqdx-Vdxdy=0.$$

Rewriting the equation in the standard form

$$q^2r-q(2p-1)s+p(p-1)t=0$$

Here $R=q^2$, $S=q-2pq$, $T=p^2-p$, $V=0$.

Thus, Monge's auxiliary equations become

$$q^2(dy)^2+(2pq-q)dxdy+\left(p^2-p\right)(dx)^2=0$$

$$q^2dpdy+p(p-1)dqdx=0$$

Here equation (i) can be factorized as

$$q^2(dy)^2+(2pq-q)dxdy+(p^2-p)(dx)^2=0$$

$$\Rightarrow q^2(dy)^2+q(p-1)dxdy+pqdxdy+p(p-1)(dx)^2=0$$

$$\Rightarrow [pdx+qdy][qdy+(p-1)dx]=0$$

which yields $pdx+qdy=dz=0\Rightarrow z=c_1$

and $qdy+(p-1)dx=qdy+pdx-dx=dz-dx=0\Rightarrow z-x=c_2$.

The second Monge's equation is

$$q^2dpdy+p(p-1)dqdx=0.$$

Since $pdx+qdy=0$, substitute $qdy=-pdx$.

Then, $qdp(-pdx)+p(p-1)dqdx=0\Rightarrow -qdp+pdq-dq=0$

Rewriting , $\dfrac{qdp-pdq}{q^2}+\dfrac{dq}{q^2}=0\Rightarrow d\left(\dfrac{p}{q}\right)+d\left(-\dfrac{1}{q}\right)=0$

Integrating, we get $\dfrac{p}{q}-\dfrac{1}{q}=\text{constant}=\psi_1(z)\Rightarrow p-\psi_1(z)q=1$,

which is a Langrange's equation, with auxiliary equations $\dfrac{dx}{1}=\dfrac{dy}{-\psi_1}=\dfrac{dz}{1}$.

From first and third, $x=z=\text{constant}=c_2$

From second and third, $\psi_1(z)dz+dy=0$.

Integrating, we get

$$\psi_2(z)+y=\text{constant}=\psi_3(x-z).$$

Thus, the general solution involving two arbitrary functions is

$$y=\psi(z)+\phi(x-z),$$

where ψ, ϕ are two arbitrary functions.

Q.No.10.: Solve the following non-linear partial differential equation of second order by Monge's method: $x^2r-2xs+t+q=0$.

Sol.: Rewriting the given non-linear partial differential equation of second order in the standard form $Rr+Ss+Tt=V$ and find the values of R, S, T and V, then substitute these values in Monge's auxiliary equations

$R(dy)^2 - Sdydx + T(dx)^2 = 0$,

$Rdpdy + Tdqdx - Vdxdy = 0$.

Here $R = x^2$, $S = -2x$, $T = 1$, $V = -q$.

Thus, Monge's auxiliary equations become

$$x^2(dy)^2 + 2xdxdy + (dx)^2 = 0$$

$$x^2dpdy + dqdx + qdxdy = 0$$

Here equation (i) is a perfect square and can be written as

$$x^2(dy)^2 + 2xdxdy + (dx)^2 = 0 \Rightarrow (xdy + dx)^2 = 0.$$

Then $xdy + dx = 0 \Rightarrow dy + \frac{dx}{x} = 0$

Integrating, we get $y + \log x = c_1$.

Now the second Monge's equation is $x^2dpdy + dqdx + qdxdy = 0$

Substituting $xdy = -dx$

$$xdp(-dq) + dqdx + qdx\left(\frac{dx}{-x}\right) = 0 \Rightarrow -xdp + dq - q\frac{dx}{x} = 0$$

i.e., $-dp + \frac{dq}{x} - q\frac{dx}{x^2} = 0 \Rightarrow -dp + d\left(\frac{q}{x}\right) = 0$

Integrating, we get $p = \frac{q}{x} + c_1(y + \text{In } x) \Rightarrow p - \frac{q}{x} = c_1$,

which is Lagrange's equation with auxiliary equations $\frac{dx}{1} = \frac{dy}{-\frac{1}{x}} = \frac{dz}{c_1}$

From one and two, $\frac{1}{x}dx + dy = 0 \Rightarrow y + \log x = c$.

From one and three, $dz = c_1dx \Rightarrow z = c_1x + c_2$.

Thus, the general solution involving two arbitrary functions is

$$z = x\psi_1(y + \log x) + \psi_2 y + \log x,$$

where ψ_1, ψ_2 are two arbitrary functions.

Q.No.11.: Solve the following non-linear partial differential equation of second order by

Monge's method: $\left(e^{x}-1\right)(qr-ps)=pqe^{x}$.

Sol.: Rewriting the given non-linear partial differential equation of second order in the standard form $Rr+Ss+Tt=V$ and find the values of R, S, T and V, then substitute these values in Monge's auxiliary equations

$R(dy)^2-Sdydx+T(dx)^2=0$,

$Rdpdy+Tdqdx-Vdxdy=0$.

Rewriting in the standard form

$\left(e^{x}-1\right)q.r+p\left(1-e^{x}\right)s=pqe^{x}$

Here $R=q\left(e^{x}-1\right)$, $S=p\left(1-e^{x}\right)$, $T=0$, $V=pqe^{x}$.

Thus, Monge's auxiliary equations become

$q\left(e^{x}-1\right)(dy)^2-p\left(1-e^{x}\right)dxdy=0$, (i)

$q\left(e^{x}-1\right)dydp-pqe^{x}dxdy=0$. (ii)

Here equation (i) can be factorized as

$q\left(e^{x}-1\right)(dy)^2-p\left(1-e^{x}\right)dxdy=0$

$\left(e^{x}-1\right)dy[qdy+pdx]=0$

so $\left(e^{x}-1\right)dy=0$, $pdx+qdy=dz=0$

Integrating, we get $y=c_1$, $z=c_2$.

Now the Monge's second equation is

$q\left(e^{x}-1\right)dydp-pqe^{x}dxdy=0$

Substituting $qdy=-pdx$, we get

$\left(e^{x}-1\right)(-pdx).dp-pqe^{x}dxdy=0 \Rightarrow \left(e^{x}-1\right)dp+qe^{x}dy=0$

i.e. $\left(e^{x}-1\right)dp-pe^{x}dx=0$

$$\frac{dp}{p}=\frac{e^{x}dx}{e^{x}-1}=\frac{d\left(e^{x}-1\right)}{\left(e^{x}-1\right)}$$

Integrating $p=c_1\left(e^{x}-1\right) \Rightarrow \frac{\partial z}{\partial x}=p=\psi_1(z)\left(e^{x}-1\right)$

Integrating $\frac{dz}{\psi_1(z)} = \left(e^x - 1\right)dx$

$\psi_2(z) = e^x - x + \psi_3(y)$

$\therefore x = \psi_3(y) - \psi_2(z) + e^x$. Ans.

Thus, the general solution involving two arbitrary functions is

$x = \psi_3(y) - \psi_2(z) + e^x$,

where ψ_3, ψ_2 are two arbitrary functions.

Q.No.12.: Solve the following non-linear partial differential equation of second order by Monge's method: $(1-q)^2 r - 2(2-p-2q+pq)s + (2-p)^2 t = 0$.

Sol.: Rewriting the given non-linear partial differential equation of second order in the standard form $Rr + Ss + Tt = V$ and find the values of R, S, T and V, then substitute these values in Monge's auxiliary equations

$R(dy)^2 - Sdydx + T(dx)^2 = 0$,

$Rdpdy + Tdqdx - Vdxdy = 0$.

Here $R = (1-q)^2$. $S = -2(2-p-2q+pq)$, $T = (2-p)^2$, $V = 0$.

Thus, Monge's auxiliary equations become

$$(1-q)^2(dy)^2 + 2(2-p-2q+pq)dxdy + (2-p)^2(dx)^2 = 0, \quad \text{(i)}$$

$$(1-q)^2 dydp + (2-p)^2 dxdq = 0. \quad \text{(ii)}$$

Here equation (i) is a perfect square and can be written as

$(1-q)^2(dy)^2 + 2(2-p-2q+pq)dxdy + (2-p)^2(dx)^2 = 0$

$[(1-q)dy + (2-p)dx]^2 = 0 \Rightarrow (1-q)dy + (2-p)dx = 0$

$dy - qdy + 2dx - pdx = dy + 2dx - (pdx + qdy) = 0$

i.e. $dy + 2dx - dz = 0$

Since $dz = pdx + qdy$.

Integrating, we get $y + 2x - z = c_1$.

Substituting $(1-q)dy = -(2-p)dx$ in the Monge's second equation.

$(1-q)^2 dydp + (2-p)^2 dxdq = 0$

reduces to $(1-q)dp[-(2-p)dx] + (2-p)^2 dxdq = 0$

$\Rightarrow (1-q)dp + (2-p)dq = 0$

i.e. $\dfrac{dp}{p-2} = \dfrac{dq}{q-1}$

Integrating, we get $(p-2) = c_2(q-1)$

$q-1 = c_3(p-2) = \psi(y+2x-z)(p-2)$

Then $c_3 p - q = 2c_3 - 1$

Which is Lagrange's equation with subsidiary equations

$$\frac{dx}{c_3} = \frac{dy}{-1} = \frac{dz}{2c_3 - 1}$$

From first and second

$dx + c_3 dy = 0$

which on integration gives

$x + y.\psi(y+2x-z) = \phi(y+x+z)$.

Thus, the general solution involving two arbitrary functions is

$x + y.\psi(y+2x-z) = \phi(y+x+z)$,

where ψ, ϕ are two arbitrary functions.

Q.No.13.: Solve the following non-linear partial differential equation of second order by Monge's method: $r - 3s - 10t = -3$.

Sol.: Rewriting the given non-linear partial differential equation of second order in the standard form $Rr + Ss + Tt = V$ and find the values of R, S, T and V, then substitute these values in Monge's auxiliary equations

$R(dy)^2 - Sdydx + T(dx)^2 = 0$,

$Rdpdy + Tdqdx - Vdxdy = 0$.

Here R = 1, $S = -3$, $T = -10$, $V = -3$.

Thus, Monge's auxiliary equations become

$$(dy)^2 + 3dxdy - 10(dx)^2 = 0, \qquad \text{(i)}$$

$dydp - 10dxdq + dxdy = 0$. (ii)

Equation (i) can be factorized as

$(dy + 5dx)(dy - 2dx) = 0$.

Thus $dy + dx = 0$ (iii)

and $dy - 2dx = 0$ (iv)

which on integration gives

$y + 5x = c_1$ (v)

$y - 2x = c_2$ (vi)

Substituting $dy = -5dx$ from (iii) in (ii), we get

$(-5dx)dp - 10dxdq + 3dx(-5dx) = 0$

$\Rightarrow dp + 2dq + 3dx = 0$

Integrating, we get

$p + 2q + 3x = c_3$ (vii)

Substituting $dy = 2dx$ from (iv) in (ii), we get

$(2dx)dp - 10dxdq + 3dx(2dx) = 0$

$\Rightarrow dp - 5dq + 3dx = 0$

Integrating, we get

$p - 5q + 3x = c_4$ (viii)

Thus $p + 2q + 3x = f(y - 5x)$ (ix)

and $p - 5q + 3x = g(y - 2x)$ (x)

Solving (ix) and (x), we get

$7p = 5f + 2g - 21x$

$7q = f - g$

Substituting p and q in

$dz = pdx + qdy$

$7dz = (5f + 2g - 21x)dx + (f - g)dy$

$7dz = (5dx + dy)f(5x + y) - dy - 2dxg(y - 2x) - 21xdx$

Integrating, we get $z = \phi_1(5x + y) + \phi_2(y - 2x) - \frac{3x^2}{2}$.

Thus, the general solution involving two arbitrary functions is

$$z = \phi_1(5x + y) + \phi_2(y - 2x) - \frac{3x^2}{2},$$

where ϕ_1, ϕ_2 are two arbitrary functions.

Q.No.14.: Solve the following non-linear partial differential equation of second order by Monge's method: $2x^2r - 5xys + 2y^2t + 2(px + qy) = 0$.

Sol.: Rewriting the given non-linear partial differential equation of second order in the standard form $Rr + Ss + Tt = V$ and find the values of R, S, T and V, then substitute these values in Monge's auxiliary equations

$$R(dy)^2 - Sdydx + T(dx)^2 = 0,$$

$$Rdpdy + Tdqdx - Vdxdy = 0.$$

Here $R = 2x^2$, $S = -5xy$, $T = 2y^2$ and $V = -(px + qy)$.

Thus, Monge's auxiliary equations become

$$2x^2dy^2 + 5xy\,dx\,dy + 2y^2dx^2 = 0. \quad \text{(i)}$$

$$2x^2dp\,dy + 2y^2dq\,dx + 2(px + qy)dx\,dy = 0, \quad \text{(ii)}$$

Here equation (i) can be factorized as

$$(2x\,dy + ydx)(xdy + 2y\,dx) = 0$$

$$\therefore 2xdy + ydx = 0 \quad \text{(iii)}$$

and $xdy + 2ydx = 0$ (iv)

from (iv), we have

$$\frac{dy}{y} + 2\frac{dx}{x} = 0 \Rightarrow \log y + 2\log x = \log A, \qquad yx^2 = A.$$

Also from (iv) and (ii), we have

$$2xdp - ydq + 2pdx - qdy = 0 \Rightarrow (2xdp + 2pdx) - (ydq + qdy) = 0$$

$$\Rightarrow 2px - qy = B.$$

$$\therefore 2px - qy = f_1(yx^2) \quad \text{(v)}$$

is one intermediate integral.

Similarly from (iii) and (ii) another intermediate integral is

$$px - 2qy = f_2\left(xy^2\right) \qquad \text{(vi)}$$

Solving (v) and (vi), we get

$$p = \frac{1}{3x}\left[2f_1\left(x^2y\right) - f_2\left(xy^2\right)\right]$$

and $q = \frac{1}{3y}\left[f_1\left(x^2y\right) - 2f_2\left(xy^2\right)\right]$.

Substituting these values in $dz = pdx + qdy$, we have

$$dz = \frac{1}{3x}\left[2f_1\left(x^2y\right) - f_2\left(xy^2\right)\right]dx + \frac{1}{3y}\left[f_1\left(x^2y\right) - 2f_2\left(xy^2\right)\right]dy$$

$$= \frac{1}{3}f_1\left(x^2y\right)\left(\frac{2dx}{x} + \frac{dy}{y}\right) - \frac{1}{3}f_2\left(xy^2\right)\left(\frac{dx}{x} + \frac{2dy}{y}\right).$$

$$= \frac{1}{3}f_1\left(x^2y\right)d\left[\log\left(x^2y\right)\right] - \frac{1}{3}f_2\left(xy^2\right)d\left[\log\left(xy^2\right)\right],$$

$$\therefore z = \frac{1}{3}\int f_1\left(x^2y\right)d\left(\log x^2y\right) - \frac{1}{3}\int f_2\left(xy^2\right)d\left(\log xy^2\right).$$

$$\Rightarrow z = \phi_1\left(x^2y\right) + \phi_2\left(xy^2\right).$$

Thus, the general solution involving two arbitrary functions is

$$z = \phi_1\left(x^2y\right) + \phi_2\left(xy^2\right),$$

where ϕ_1, ϕ_2 are two arbitrary functions.

Q.No.15.: Solve the following non-linear partial differential equation of second order by Monge's method: $xy(t - r) + \left(x^2 - y^2\right)(s - 2) = py - qx$.

Sol.: Rewriting the given non-linear partial differential equation of second order in the standard form $Rr + Ss + Tt = V$ and find the values of R, S, T and V, then substitute these values in Monge's auxiliary equations

$$R(dy)^2 - Sdydx + T(dx)^2 = 0,$$

$$Rdpdy + Tdqdx - Vdxdy = 0.$$

The given equation can be written as $xyr - \left(x^2 - y^2\right)s - xyt = 2\left(x^2 - y^2\right) - py + qx$

Here $R = xy,\ S = -\left(x^2 - y^2\right),\ T = -xy$ and $V = 2\left(x^2 - y^2\right) - py + qx$.

Thus, Monge's auxiliary equations become

$$xydy^2 + \left(x^2 - y^2\right)dxdy - xydx^2 = 0, \tag{i}$$

$$xydpdy - xydqdx - \left\{2\left(x^2 - y^2\right) + py - qx\right\}dxdy = 0. \tag{ii}$$

Here equation (i) is a perfect square and can be written as

$$xdx + ydy = 0 \tag{iii}$$

and $xdy - ydx = 0$ (iv)

From (i), we have $x^2 + y^2 = A$.

From (ii) and (iii), we have

$$xdp + ydq - 2xdy - 2ydx + pdx + qdy = 0$$

$$\Rightarrow (xdp + pdx) + (ydq + qdy) - 2(xdy + ydx) = 0$$

$$\therefore px + qy - 2xy = B.$$

$\therefore$ One intermediate integral is

$$px + qy - 2xy = f_1\left(x^2 + y^2\right). \tag{v}$$

Similarly from (ii) and (iv) the second intermediate integral is

$$-py + qx = x^2 - y^2 + f_2\left(\frac{y}{x}\right) \tag{vi}$$

Solving (v) and (vi), we have

$$p = y + \frac{1}{\left(x^2 + y^2\right)}\left\{xf_1\left(x^2 + y^2\right) - yf_2\left(\frac{y}{x}\right)\right\}$$

and $q = x + \frac{1}{\left(x^2 + y^2\right)}\left\{yf_1\left(x^2 + y^2\right) + xf_2\left(\frac{y}{x}\right)\right\}$.

Substituting these values in $dz = pdx + qdy$, we have

$$dz = \left[y + \frac{1}{\left(x^2 + y^2\right)}\left\{xf_1\left(x^2 + y^2\right) - yf_2\left(\frac{y}{x}\right)\right\}\right]dx$$

$$+ \left[x + \frac{1}{\left(x^2 + y^2\right)}\left\{yf_1\left(x^2 + y^2\right) + xf_2\left(\frac{y}{x}\right)\right\}\right]dy$$

$$= (ydx + xdy) + f_1(x^2 + y^2).\frac{xdx + ydy}{x^2 + y^2} + f_2\left(\frac{y}{x}\right).\frac{xdy - ydx}{x^2 + y^2}$$

$$= (ydx + xdy) + \frac{1}{2}f_1(x^2 + y^2).d\{\log(x^2 + y^2)\} + f_2\left(\frac{y}{x}\right)d\left\{\tan^{-1}\left(\frac{y}{x}\right)\right\}.$$

$$\therefore z = xy + \frac{1}{2}\int f_1(x^2 + y^2)d\{\log(x^2 + y^2)\} + \int f_2\left(\frac{y}{x}\right).d\left\{\tan^{-1}\frac{y}{x}\right\}$$

$$\Rightarrow z = xy + \phi_1(x^2 + y^2) + \phi_2\left(\frac{y}{x}\right).$$

Thus, the general solution involving two arbitrary functions is

$$\Rightarrow z = xy + \phi_1(x^2 + y^2) + \phi_2\left(\frac{y}{x}\right),$$

where ϕ_1, ϕ_2 are two arbitrary functions.

Q.No.16.: Solve the following non-linear partial differential equation of second order by Monge's method: $t - r\sec^4 y - 2q\tan y = 0$.

Sol.: Rewriting the given non-linear partial differential equation of second order in the standard form $Rr + Ss + Tt = V$ and find the values of R, S, T and V, then substitute these values in Monge's auxiliary equations

$$R(dy)^2 - Sdydx + T(dx)^2 = 0,$$

$$Rdpdy + Tdqdx - Vdxdy = 0.$$

Here $R = -\sec^4 y,\ S = 0,\ T = 1$ and $V = 2q\tan y$.

Thus, Monge's auxiliary equations become

$$dx^2 - \sec^4 ydy^2 = 0. \qquad \text{(i)}$$

$$dqdx - \sec^4 ydpdy - 2q\tan ydxdy = 0, \qquad \text{(ii)}$$

Here equation (i) can be factorized as

$$dx - dy\sec^2 y = 0, \qquad \text{(iii)}$$

$$dx + dy\sec^2 y = 0. \qquad \text{(iv)}$$

From (ii) and (iii), we have

$$dq - dp\sec^2 y - 2q\tan ydy = 0$$

$\Rightarrow dp-\left(\cos^2 y.dq-2q\sin y\cos ydy\right)=0$

Integrating $p-q\cos^2 y=A$.

Also integrating (iii), we have $x-\tan y=B$.

$\therefore p-q\cos^2 y=f_1(x-\tan y)$ (v)

is an intermediate integral.

Similarly from (ii) and (iv) another intermediate integral is

$p+q\cos^2 y=f_2(x+\tan y)$ (vi)

Solving (v) and (vi), we have

$$p=\frac{1}{2}[f_1(x-\tan y)+f_1(x+\tan y)]$$

and $q=\frac{1}{2}\sec^2 y[f_2(x+\tan y)-f_1(x-\tan y)]$.

Substituting these values in $dz=pdx+qdy$, we have

$$dz=\frac{1}{2}[f_1(x-\tan y)+f_2(x+\tan y)]dx+\frac{1}{2}\sec^2 y[f_2(x+\tan y)-f_1(x-\tan y)]dy$$

$$=\frac{1}{2}f_1(x-\tan y)(dx-\sec^2 ydy)+\frac{1}{2}f_2(x+\tan y)(dx+\sec^2 ydy)$$

$\therefore z=\phi_1(x-\tan y)+\phi_2(x+\tan y)$.

Thus, the general solution involving two arbitrary functions is

$z=\phi_1(x-\tan y)+\phi_2(x+\tan y)$,

where ϕ_1, ϕ_2 are two arbitrary functions.

Q.No.17.: Solve the following non-linear partial differential equation of second order by Monge's method: $(1+q)^2r-2(1+p+q+pq)s+(1+p)^2t=0$.

Sol.: Rewriting the given non-linear partial differential equation of second order in the standard form $Rr+Ss+Tt=V$ and find the values of R, S, T and V, then substitute these values in Monge's auxiliary equations

$R(dy)^2-Sdydx+T(dx)^2=0$,

$Rdpdy+Tdqdx-Vdxdy=0$.

Here $R=(1+q)^2$, $S=-2(1+p+q+pq)$, $T=(1+p)^2$ and $V=0$.

Thus, Monge's auxiliary equations become

$(1+q)^2 dy^2 + 2(1+p)(1+q)dxdy + (1+p)^2 dx^2 = 0$. (i)

$(1+q)^2 dpdy + (1+p)^2 dqdx = 0$, (ii)

Here equation (i) is a perfect square and can be written as

$[(1+q)dy + (1+p)dx]^2 = 0$

$\Rightarrow (1+q)dy + (1+p)dx = 0$ (iii)

$\Rightarrow dx + dy + pdx + qdy = 0 \Rightarrow dx + dy + dz = 0 \quad [\because dz = pdx + qdy]$

Integrating, we get $x + y + z = A$ (iv)

Again from (ii) and (iii), we have

$(1+q)dp - (1+p)dq = 0 \Rightarrow \frac{dp}{1+p} - \frac{dq}{1+q} = 0$.

Integrating, we get

$\log(1+p) - \log(1+q) = \log B \Rightarrow (1+p) = B(1+q)$ (v)

From (iv) and (v) the intermediate integral is

$(1+p) = (1+q)f(x+y+z) \Rightarrow p - qf(x+y+z) = f(x+y+z) - 1$.

$\therefore$ The Langrange's subsidiary equations are

$$\frac{dx}{1} = \frac{dy}{-f(x+y+z)} = \frac{dz}{f(x+y+z)-1} = \frac{dx+dy+dz}{0}$$

$\therefore dx + dy + dz = 0 \Rightarrow x + y + z = A$.

Taking the first two members, we have

$dy = -f(A)dx \qquad \therefore y = -xf(A) + B$.

$\therefore y + xf(x+y+z) = F(x+y+z)$.

Thus, the general solution involving two arbitrary functions is

$y + xf(x+y+z) = F(x+y+z)$,

where f, F are two arbitrary functions.

Q.No.18.: Solve the following non-linear partial differential equation of second order by Monge's method: $q^2 r - 2pqs + p^2 t = 0$.

or

Obtain the integral of $q^2r-2pqs+p^2t=0$ in the form $y+xf(z)=F(z)$ and show that this represents a surface generated by straight lines that are parallel to fixed plane.

Sol.: Rewriting the given non-linear partial differential equation of second order in the standard form $Rr+Ss+Tt=V$ and find the values of R, S, T and V, then substitute these values in Monge's auxiliary equations

$$R(dy)^2-Sdydx+T(dx)^2=0,$$

$$Rdpdy+Tdqdx-Vdxdy=0.$$

Here $R=q^2$, $S=-2pq$, $T=p^2$ and $V=0$.

Thus, Monge's auxiliary equations become

$$q^2dy^2+2pq\,dx\,dy+p^2dx^2=0, \tag{i}$$

$$q^2dp\,dy+p^2dq\,dx=0. \tag{ii}$$

Here equation (i) is a perfect square and can be written as

$$(qdy+pdx)^2=0 \Rightarrow pdx+qdy=0 \tag{iii}$$

$$\therefore dz=pdx+qdy=0$$

$$\Rightarrow z=A. \tag{iv}$$

From (ii) and (iii), we have

$$qdp-pdq=0 \Rightarrow \frac{dp}{p}=\frac{dq}{q} \Rightarrow \log p=\log q+\log B$$

$$\therefore p=qB \tag{v}$$

Hence the intermediate integral is

$$p-qf(z)=0.$$

$\therefore$ Langrange's subsidiary equations are

$$\frac{dx}{1}=\frac{dy}{-f(z)}=\frac{dz}{0}.$$

The last member gives, $dz=0$ $\therefore z=c$.

Again from first two members, we have

$$dy+f(z)dx=0 \Rightarrow dy+f(c)dx=0$$

$$\therefore y+xf(c)=c' \quad \Rightarrow y+xf(z)=F(z), \tag{vi}$$

which is the required integral.

The integrals of the given differential equation is the surface (vi) which is the locus of straight lines given by the intersection of planes

$y + xf(c) = F(c)$ and $z = c$.

These lines are all parallel to the planes $z = 0$ as they lie on the plane $z = c$ for verifying values of c.

Q.No.19.: Solve the following non-linear partial differential equation of second order by Monge's method: $(b+cq)^2 r - 2(b+cq)(a+cp)s + (a+cp)^2 t = 0$.

Sol.: Rewriting the given non-linear partial differential equation of second order in the standard form $Rr + Ss + Tt = V$ and find the values of R, S, T and V, then substitute these values in Monge's auxiliary equations

$$R(dy)^2 - Sdydx + T(dx)^2 = 0,$$

$$Rdpdy + Tdqdx - Vdxdy = 0.$$

Here $R = (b+cq)^2$, $S = -2(b+cq)(a+cp)$, $T = (a+cp)^2$ and $V = 0$.

Thus, Monge's auxiliary equations become

$$(b+cq)^2 dpdy + (a+cp)^2 dqdx = 0, \qquad \text{(i)}$$

$$(b+cq)^2 dy^2 + 2(b+cq)(a+cp)dxdy + (a+cp)^2 dx^2 = 0. \qquad \text{(ii)}$$

Here equation (i) is a perfect square and can be written as

$$[(b+cq)dy + (a+cp)dx]^2 = 0 \qquad \text{(iii)}$$

$$\Rightarrow bdy + adx + c(pdx + qdy) = 0$$

$$\Rightarrow adx + bdy + cdz = 0$$

$$\therefore ax + by + cz = A.$$

From (i) and (iii), we have

$$(b+cq)dp - (a+cp)dq = 0$$

$$\Rightarrow \frac{dp}{a+cp} - \frac{dq}{b+cq} = 0.$$

Integrating, we get $\frac{a+cp}{b+cq} = B \Rightarrow (a+cp) = (b+cq)B$.

$\therefore$ The intermediate integral is

$a + cp = (b + cq)f(ax + by + cz)$

$\Rightarrow cp - cf(ax + by + cz)q = -a + bf(ax + by + cz)$.

Lagrange's subsidiary equations are

$$\frac{dx}{c} = \frac{dy}{-cf(ax + dy + cz)} = \frac{dz}{-a + bf(ax + by + cz)}$$

Using a, b, c as multipliers,

Each fraction $= \dfrac{adx + bdy + cdz}{0} \Rightarrow ax + by + cz = A$.

Again taking the first two members, we have

$$dx = \frac{dy}{-f(A)} \Rightarrow dy + f(A)dx = 0$$

Integrating, we get $y + f(A)x = A'$.

$\Rightarrow y + xf(ax + by + cz) = F(ax + by + cz)$, which is the required solution.

Q.No.20.: Solve the following non-linear partial differential equation of second order by Monge's method: $q(1+q)r - (p + q + 2pq)s + p(1+p)t = 0$.

Sol.: Rewriting the given non-linear partial differential equation of second order in the standard form $Rr + Ss + Tt = V$ and find the values of R, S, T and V, then substitute these values in Monge's auxiliary equations

$R(dy)^2 - Sdydx + T(dx)^2 = 0$,

$Rdpdy + Tdqdx - Vdxdy = 0$.

Here $R = q(1+q),\ S = -(p + q + 2pq),\ T = p(1+p)t,\ V = 0$.

Thus, Monge's auxiliary equations become

$$q(1+q)dy^2 + (p + q + 2qp)dxdy + p(1+p)dx^2 = 0. \qquad \text{(i)}$$

$$q(1+q)dpdy + p(1+p)dqdx = 0, \qquad \text{(ii)}$$

Here equation (i) can be factorized as

$$(pdx + qdy)\{(1-p)dx + (1+q)dy\} = 0$$

which yields

$$pdx + qdy = 0 \qquad \text{(iii)}$$

$$(1-p)dx + (1+q)dy = 0 \qquad \text{(iv)}$$

From (iii), we have $dz = pdx + qdy = 0 \Rightarrow z = A$.

Also from (ii) and (iii), we have

$$p(1+q)dp - p(1+p)dq = 0 \Rightarrow \frac{dp}{1+p} - \frac{dq}{1+q} = 0.$$

Integrating, we get $\log(1+p) - \log(1+q) = \log B$

$$\Rightarrow (1+p) = (1+q)f_1(z), \qquad \text{(v)}$$

which is one intermediate integral.

Now from (iv), we have

$$qdp - pdq = 0 \Rightarrow \frac{dp}{p} = \frac{dq}{q}$$

Integrating, we get $\log p = \log q + \log B'$

$$\therefore p = qf_2(x+y+z). \qquad \text{(vi)}$$

Solving (v) and (vi), we have

$$p = \frac{(f_1 - 1)f_2}{f_2 - f_1} \quad \text{and} \quad q = \frac{f_1 - 1}{f_2 - f_1}.$$

Substituting these values in $dz = pdx + qdy$, we have

$$dz = \frac{(f_1 - 1)f_2}{f_2 - f_1}dx + \frac{f_1 - 1}{f_2 - f_1}dy$$

$$\Rightarrow (f_2 - f_1)dz = (f_1 - 1)f_2 dx + (f_1 - 1)dy$$

$$\Rightarrow (f_2 - f_1)dz = (f_1 - 1)f_2 dx + (f_1 - 1)(dx + dy + dz) - (f_1 - 1)(dx + dz)$$

$$\Rightarrow (f_1 - 1)dz = (f_1 - 1)(f_2 - 1)dx + (f_1 - 1)(dx + dy + dz)$$

$$\Rightarrow \frac{dz}{f_1 - 1} = dz + \frac{dx + dy + dz}{f_2 - 1}$$

$$\Rightarrow \frac{dz}{f_1(z) - 1} = dx + \frac{dx + dy + dz}{f_2(x+y+z) - 1}$$

Integrating, we get $\phi_1(z) = x + \phi_2(x+y+z)$.

Thus, the general solution involving two arbitrary functions is

$\phi_1(z) = x + \phi_2(x+y+z)$,

where ϕ_1, ϕ_2 are two arbitrary functions.

Q.No.21.: Solve the following non-linear partial differential equation of second order by

Monge's method: $r = t$ (wave equation).

Sol.: Rewriting the given non-linear partial differential equation of second order in the standard form $Rr + Ss + Tt = V$ and find the values of R, S, T and V, then substitute these values in Monge's auxiliary equations

$R(dy)^2 - Sdydx + T(dx)^2 = 0$,

$Rdpdy + Tdqdx - Vdxdy = 0$.

Here $R = 1,\ S = -1,\ T = 0,\ V = 0$.

Put $r = \dfrac{dp - sdy}{dx}$, $t = \dfrac{dq - sdx}{dy}$,

Thus $r = t$ reduces to $\dfrac{dp - sdy}{dx} = \dfrac{dq - sdx}{dy} \Rightarrow dpdy - sdy^2 = dqdx - sdx^2$

$\Rightarrow dpdy - dqdx - s\left(dy^2 - dx^2\right) = 0$.

Thus, Monge's auxiliary equations become

$dpdy - dqdx = 0,$ (ii)

$dy^2 - dx^2 + 0$. (iii)

Here equation (iii) can be factorized as

$dy + dx = 0$ and $dy - dx = 0$ (iv)

$\Rightarrow y + x = a_1$ and $y - x = b_1$ (A)

Using (ii) and (iii), we can find one intermediate integral.

Diving (ii) by $dy = -dx$, we have

$dp + dq = 0 \Rightarrow p + q = f(a_1)$. (v)

Again, using (ii) and (iv), we can find another intermediate integral.

Dividing (ii) by $dy = dx$, we have

$dp - dq = 0 \Rightarrow p - q = f(b_1)$ (vi)

From (v) and (vi), $p = \dfrac{1}{2}[f(a_1) + f(b_1)]$

$q = \dfrac{1}{2}[f(a_1) - f(b_1)]$

Now $dz = pdx + qdy = \dfrac{1}{2}[f(a_1) + f(b_1)]dx + \dfrac{1}{2}[f(a_1) - f(b_1)]dy$

$$=\frac{1}{2}f(a_1)(dx+dy)+\frac{1}{2}f(a_1)(dx-dy).$$

Integrating, we have

$$z=\frac{1}{2}f(a_1)(x+y)+\frac{1}{2}f(a_1)(x-y)=\frac{1}{2}f(x+y)(x+y)-\frac{1}{2}f(x-y)(x-y) \quad \text{[using (A)]}$$

$z=F(x+y)+G(x-y)$, is the required complete solution.

or $z=\phi_1(y-x)+\phi_2(y+x)$**,** is the required complete solution.

Q.No.22.: Solve the following non-linear partial differential equation of second order by Monge's method: $x^2r+2xys+y^2t=0$.

Sol.: Rewriting the given non-linear partial differential equation of second order in the standard form $Rr+Ss+Tt=V$ and find the values of R, S, T and V, then substitute these values in Monge's auxiliary equations

$$R(dy)^2-Sdydx+T(dx)^2=0,$$

$$Rdpdy+Tdqdx-Vdxdy=0.$$

Given equation is $x^2r+2xys+y^2t=0$. (i)

Here $R=x^2,\ S=2xy,\ T=y^2,\ V=0$.

Putting $r=\dfrac{dp-sdy}{dx}$, $t=\dfrac{dq-sdx}{dy}$.

In the given equation (i) , we get $x^2\left(\dfrac{dp-sdy}{dx}\right)+2yxs+y^2\left(\dfrac{dq-sdx}{dy}\right)=0$

$$\Rightarrow x^2dpdy-y^2dqdx+s(-x^2dy^2+2xydydx-y^2dx^2)=0$$

Thus, Monge's auxiliary equations become

$x^2dpdy+y^2dqdx=0$ (ii)

And $x^2dy^2-2xydydx+y^2dx^2=0$ (iii)

Now (iii) can be written as $(xdy-ydx)^2=0\Rightarrow xdy-ydx=0$ (iv)

$$\Rightarrow\frac{xdy-ydx}{x^2}=0\Rightarrow d\left(\frac{y}{x}\right)=0\Rightarrow\frac{y}{x}=c_1\Rightarrow y=c_1x.$$

Using (ii) and (iv), [On dividing(ii) by xdy = ydx], we get

$xdp + ydq = 0 \Rightarrow d(xp + yq) - (pdx + qdy) = 0 \Rightarrow d(xp + yq) - dz = 0$.

Integrating, we get

$xp + yq - z = c_2 = \phi(c_1)$, say, which is Lagrange's equation.

Here $P = x, \quad Q = y, \quad R = z + \phi(c_1)$.

Consider Lagrange's auxiliary equation $\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z + \phi(c_1)}$.

From 2nd and 3rd member, $\frac{dy}{y} = \frac{dz}{z + \phi(c_1)}$.

Integrating, we get

$\log y = \log(z + \phi(c_1)) - \log a \Rightarrow \log ya = \log(z + \phi(c_1)) \Rightarrow z + \phi(c_1) = ya$

$\Rightarrow z + \phi(c_1) = yf(c_1)$, say, $y = f(c_1)$

$\Rightarrow z + \phi\left(\frac{y}{x}\right) = yf\left(\frac{y}{x}\right)$, is the required complete solution. $[y = c_1 x]$.

or $z = \phi_1\left(\frac{y}{x}\right) + y\phi_2\left(\frac{y}{x}\right)$.

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Home Assignments

Q.No.1.: Solve the following non-linear partial differential equation of second order by Monge's method: $(q+1)s = (p+1)t$.

Ans.: $z = \phi_1(x) + \phi_2(x + y + z)$.

Q.No.2.: Solve the following non-linear partial differential equation of second order by Monge's method: $q^2 r - 2pqs + p^2 t = pq^2$.

Ans.: $y = \phi_1(z) + e^x \phi_2(z)$.

Q.No.3.: Solve the following non-linear partial differential equation of second order by

Monge's method: $x^2 - 2xs + t + q = 0$.

Ans.: $z = \phi_1(y + \log x) + x\phi_2(y + \log x)$.

Q.No.4.: Solve the following non-linear partial differential equation of second order by Monge's method: $x\left(r + 2xs + x^2 t\right) = p + 2x^3$.

Ans.: $z = \phi_1\left(x^2 - 2y\right) + \dfrac{x^2}{2}\phi_2\left(x^2 - 2y\right) + \dfrac{x^4}{4}$.

Q.No.5.: Solve the following non-linear partial differential equation of second order by Monge's method: $q(1+q)r - (1+2q)(1+p)s + (1+p)^2 t = 0$

Ans.: $x = \phi_1(x + y + z) + \phi_2(x + z)$

Q.No.6.: Solve the following non-linear partial differential equation of second order by Monge's method: $qr - ps = p^3$.

Ans.: $x = yz - \phi_1(z) + \phi_2(y)$.

Q.No.7.: Solve the following non-linear partial differential equation of second order by Monge's method: $rx^2 - 3sxy + 2ty^2 + px + 2qy = x + 2y$.

Ans.: $z = x + y + \phi_1(xy) + \phi_2\left(x^2 y\right)$.

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