1^{st} Topic

Fourier Series

Importance, Definitions of Fourier series

Euler's formulae

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Introduction:

Jean Baptiste Joseph Fourier (21 March 1768 – 16 May 1830) was a French mathematician and physicist, best known for initiating the investigation of Fourier series and their application to problems of heat transfer. The Fourier transform and Fourier's Law are also named in his honour. Fourier is also generally credited with the discovery of the greenhouse effect.

Fourier series introduced in 1807 by Fourier (after works by Euler and Danial Bernoulli) was one of the most important developments in applied mathematics. Fourier series is an infinite series representation of periodic function in terms of the trigonometric sine and cosine functions.

In mathematics, a Fourier series decomposes a periodic function or periodic signal into a sum of simple oscillating functions, namely sines and cosines (or complex exponentials). The study of Fourier series is a branch of Fourier analysis. Fourier series were introduced by Joseph Fourier for the purpose of solving the heat equation in a metal plate.

Heat Equation:

The heat equation is an important partial differential equation which describes the distribution of heat (or variation in temperature) in a given region over time. For a function $u(x,y,z,t)$ of three spatial variables (x,y,z) and the time variable t, the heat equation is

$$\frac{\partial u}{\partial t} - \alpha \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) = 0$$

or equivalently

$$\frac{\partial u}{\partial t} = \alpha \nabla^2 u$$

where α is a constant.

Note: The heat equation predicts that if a hot body is placed in a box of cold water, the temperature of the body will decrease, and eventually (after infinite time, and subject to no external heat sources) the temperature in the box will equalize.

Solution of heat equation prior to Fourier's work:

Prior to Fourier's work, there was no known solution to the heat equation in a general situation, although particular solutions were known if the heat source behaved in a simple way, in particular, if the heat source was a sine or cosine wave. These simple solutions are now sometimes called eigensolutions. Fourier's idea was to model a complicated heat source as a superposition (or linear combination) of simple sine and cosine waves, and to write the solution as a superposition of the corresponding eigensolutions. This superposition or linear combination is called the Fourier series.

Fourier series is named in honour of Joseph Fourier (1768-1830), who made important contributions to the study of trigonometric series, after preliminary investigations by Leonhard Euler, Jean le Rond d'Alembert, and Daniel Bernoulli. He applied this technique to find the solution of the heat equation, publishing his initial results in his 1807 *Mémoire sur la propagation de la chaleur dans les corps solides* and 1811, and publishing his *Théorie analytique de la chaleur* in 1822.

Original motivation:

Although the original motivation was to solve the heat equation, it later became obvious that the same techniques could be applied to a wide array of mathematical and physical problems.

Applications:

The Fourier series has many applications in

- communication engineering,
- electrical engineering,
- vibration analysis,
- acoustics,
- optics,
- signal processing,
- image processing,
- quantum mechanics, and
- econometrics.

Fourier series is also very useful in the study of

- heat conduction,
- mechanics,
- concentration of chemicals and pollutants (impurities),
- electrostatics, and
- in areas unheard of in Fourier's days such as computing and
- CAT scan (computer assisted tomography-medical technology that uses X-Rays and computers to produce 3-dimensional images of the human body).
- Fourier series is very powerful method to solve ordinary and partial differential equations particularly with periodic functions appearing as non-homogeneous terms.

Additional validity:

As we know, Taylor's series expansion is valid only for functions, which are continuous and differentiable. But Fourier series is possible not only for continuous functions, but for periodic functions, functions discontinuous in their values and derivatives.

Further, because of periodic nature, Fourier series constructed for one period is valid for all values.

Drawbacks in Fourier's days:

From a modern point of view, Fourier's results are somewhat informal, due to the lack of a precise notion of function and integral in the early nineteenth century.

Later, Dirichlet and Riemann expressed Fourier's results with greater accuracy and formality.

Periodic functions:

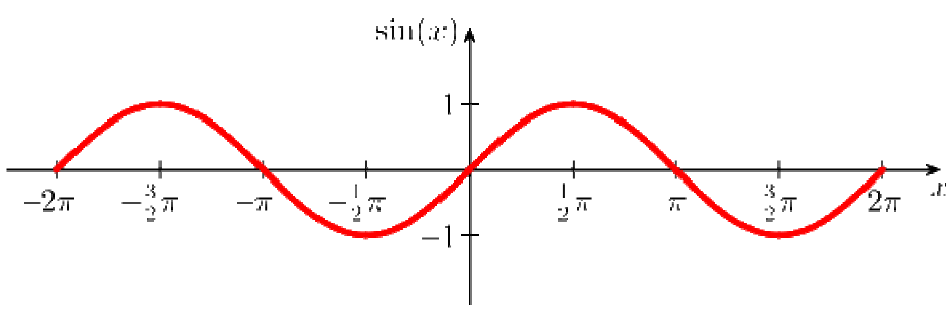
A function f(x) which satisfies the relation $f(x+T)=f(x)$ for all x and for some positive number T, is called a **periodic function**. The smallest positive number T, for which this relation holds, is called the **period** of f(x).

If T is the period, then $f(x)=f(x+T)=f(x+2T)=\text{..........}=f(x+nT)=\text{.....}$

Also $f(x)=f(x-T)=f(x-2T)=\text{..........}=f(x-nT)=\text{.....}$

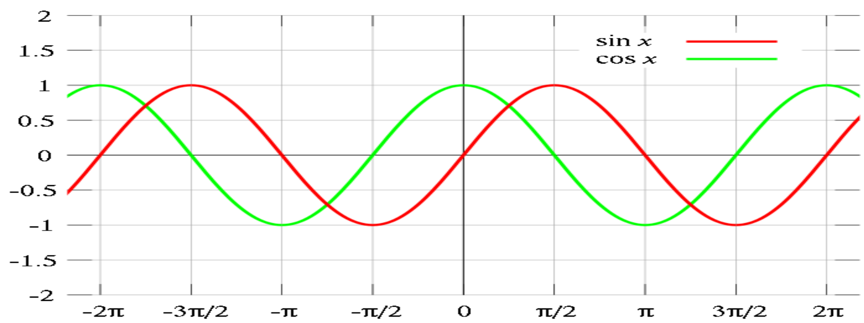
$\therefore f(x)=f(x\pm nT)$, where n is a positive integer.

Thus, f(x) repeats itself after periods of T.



A graph of the sine function, showing two complete periods.

Geometrically, a periodic function can be defined as a function whose graph exhibits translational symmetry. Specifically, a function f is periodic with period P if the graph of f is invariant under translation in the x-direction by a distance of P. This definition of periodic can be extended to other geometric shapes and patterns, such as periodic tessellations of the plane.



A plot of $f(x) = \sin(x)$ and $g(x) = \cos(x)$; both functions are periodic with period 2π.

Aperiodic functions:

A function that is not periodic is called **aperiodic**.

Trigonometric series:

Trigonometric series is a functional series of the form

$$\frac{1}{2}a_0 + a_1 \cos x + a_2 \cos 2x + \ldots\ldots\ldots\ldots + b_1 \sin x + b_2 \sin 2x + \ldots\ldots\ldots$$

or $$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx\ ,$$

where the coefficients $a_0,\ a_n,\ b_n$ $(n = 1,2,3,\ldots\ldots)$ are called the **coefficients**.

Fourier series:

Most of the single valued functions, which occur in many physical and engineering problems, can be expressed in the form

$$\frac{1}{2}a_0 + a_1 \cos x + a_2 \cos 2x + \ldots\ldots\ldots\ldots + b_1 \sin x + b_2 \sin 2x + \ldots\ldots\ldots$$

within a desired range of values of the variable.

Then, such a series is known as the Fourier series.

The individual terms in Fourier series are known as harmonics.

Euler's Formulae: [Fourier-Euler Formulae]

The Fourier series for the function $f(x)$ in the interval $\alpha < x < \alpha + 2\pi$ is given by

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx},$$

where $a_0 = \dfrac{1}{\pi}\displaystyle\int_{\alpha}^{\alpha+2\pi} f(x)\,dx$,

$$a_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\cos nx\,dx \, ,$$

$$b_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\sin nx\,dx \, .$$

Leonhard Paul Euler
(17-04-1707 to 18-09-1783)

These formulae of a_0, a_n, b_n are known as **Euler's Formulae.**

Note: For getting more symmetric formulae for the coefficients, we write $\dfrac{a_0}{2}$ instead of a_0.

To establish these formulae, the following definite integrals will be required:

1. $$\int_{\alpha}^{\alpha+2\pi} \cos nx\,dx = \left|\frac{\sin nx}{n}\right|_{\alpha}^{\alpha+2\pi} = 0 \qquad (n \neq 0)$$

2. $$\int_{\alpha}^{\alpha+2\pi} \sin nx\,dx = -\left|\frac{\cos nx}{n}\right|_{\alpha}^{\alpha+2\pi} = 0 \qquad (n \neq 0)$$

3. $$\int_{\alpha}^{\alpha+2\pi} \cos mx \cos nx\,dx = \frac{1}{2}\int_{\alpha}^{\alpha+2\pi} [\cos(m+n)x + \cos(m-n)x]dx$$
$$= \frac{1}{2}\left|\frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n}\right|_{\alpha}^{\alpha+2\pi} = 0 \qquad (m \neq n)$$

4. $$\int_{\alpha}^{\alpha+2\pi} \cos^2 nx\,dx = \left|\frac{x}{2} + \frac{\sin 2nx}{4n}\right|_{\alpha}^{\alpha+2\pi} = \pi \qquad (n \neq 0)$$

5. $$\int_{\alpha}^{\alpha+2\pi} \sin mx \cos nx\,dx = -\frac{1}{2}\left[\frac{\cos(m-n)x}{m-n} + \frac{\cos(m+n)x}{m+n}\right]_{\alpha}^{\alpha+2\pi} = 0 \qquad (m \neq n)$$

6. $$\int_{\alpha}^{\alpha+2\pi} \sin nx \cos nx\,dx = \left|\frac{\cos 2nx}{2n}\right|_{\alpha}^{\alpha+2\pi} = 0 \qquad (n \neq 0)$$

7. $$\int_{\alpha}^{\alpha+2\pi} \sin mx \sin nx\,dx = \frac{1}{2}\left[\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n}\right]_{\alpha}^{\alpha+2\pi} = 0 \qquad (m \neq n)$$

8. $$\int_{\alpha}^{\alpha+2\pi} \sin^2 nx\,dx = \left|\frac{x}{2} - \frac{\sin 2nx}{4n}\right|_{\alpha}^{\alpha+2\pi} = \pi \qquad (n \neq 0)$$

Proof: Let $f(x)$ be represented in the interval $(\alpha,\ \alpha+2\pi)$ by Fourier series:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx\,. \qquad \text{(i)}$$

In finding the coefficients a_0, a_n and b_n, we assume that the series on the RHS of (i) is uniformly convergent for $\alpha < x < \alpha + 2\pi$ and it can be integrated term by term in the given interval.

To determine the coefficient a_0:

Integrate both sides of (i) w.r.t. x from $x=\alpha$ to $x=\alpha+2\pi$. Then, we get

$$\int_{\alpha}^{\alpha+2\pi} f(x)\,dx = \frac{1}{2}a_0\int_{\alpha}^{\alpha+2\pi} dx + \int_{\alpha}^{\alpha+2\pi}\left(\sum_{n=1}^{\infty} a_n\cos nx\right)dx + \int_{\alpha}^{\alpha+2\pi}\left(\sum_{n=1}^{\infty} b_n\sin nx\right)dx$$

$$= \frac{1}{2}a_0(\alpha+2\pi-\alpha)+0+0 = a_0\pi\,. \qquad \text{[by integrals (1) and (2) above]}$$

Hence $\boxed{a_0 = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\,dx}$.

To determine the coefficient a_n for n = 1, 2, 3,.......:

Multiply each side of (i) by $\cos nx$ and integrate w.r.t. x from $x=\alpha$ to $x=\alpha+2\pi$.

Then, we get

$$\int_{\alpha}^{\alpha+2\pi} f(x)\cos nx\,dx = \frac{1}{2}a_0\int_{\alpha}^{\alpha+2\pi}\cos nx\,dx + \int_{\alpha}^{\alpha+2\pi}\left(\sum_{n=1}^{\infty} a_n\cos nx\right)\cos nx\,dx$$

$$+ \int_{\alpha}^{\alpha+2\pi}\left(\sum_{n=1}^{\infty} b_n\sin nx\right)\cos nx\,dx$$

$$= 0+\pi a_n+0\,. \qquad \text{[by integrals (1),(3), (4), (5) and (6)]}$$

Hence $\boxed{a_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\cos nx\,dx}$.

To determine the coefficient b_n for n = 1, 2, 3,......:

Multiply each side of (i) by $\sin nx$ and integrate w.r.t. x from $x=\alpha$ to $x=\alpha+2\pi$.

Then, we get

$$\int_{\alpha}^{\alpha+2\pi} f(x)\sin nx\,dx = \frac{1}{2}a_0\int_{\alpha}^{\alpha+2\pi}\sin nx\,dx + \int_{\alpha}^{\alpha+2\pi}\left(\sum_{n=1}^{\infty} a_n\cos nx\right)\sin nx\,dx$$

$$+ \int_{\alpha}^{\alpha+2\pi}\left(\sum_{n=1}^{\infty} b_n\sin nx\right)\sin nx\,dx$$

$$= 0+0+\pi b_n\,. \qquad \text{[by integrals (2),(5), (6), (7) and (8)]}$$

Hence $\boxed{b_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\sin nx\,dx}$.

Thus $a_0 = \dfrac{1}{\pi}\displaystyle\int_{\alpha}^{\alpha+2\pi} f(x)dx$, $a_n = \dfrac{1}{\pi}\displaystyle\int_{\alpha}^{\alpha+2\pi} f(x)\cos nx dx$ and $b_n = \dfrac{1}{\pi}\displaystyle\int_{\alpha}^{\alpha+2\pi} f(x)\sin nx dx$.

These formulae of a_0, a_n, b_n are known as **Euler's (or Fourier-Euler) formulae.**

The coefficients a_0, a_n and b_n, are known as **Fourier coefficients** of f(x).

Remarks:

1.: Putting $\alpha = 0$, the interval becomes $0 < x < 2\pi$, and the formula (1) reduce to

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx, \quad a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx, \quad b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx.$$

2.: Putting $\alpha = -\pi$, the interval becomes $-\pi < x < \pi$ and the formula (1) take the form

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx.$$

Q.: What is the significance of the coefficient a_0?

Ans.: a_0 is an additive constant, i.e. changing it results in a shift of the graph in y-direction. Furthermore, $\dfrac{a_0}{2}$ is the mean value of the function represented by the series, taken over the interval $[0, 2\pi]$.

Note: Practically all interesting functions with period 2π may be written in this form. Interpreting x as time, the coefficients a_n, b_n may be interpreted as representing the contributions of frequency n to a given signal.

Now let us expand the following functions as a Fourier series. In all these problems, f(x) is assumed to have the period 2π.

Q.No.1.: Expand in a Fourier series, the function $f(x) = x$ in the interval $0 < x < 2\pi$.

Sol.: Let $f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ (i)

be the required Fourier series.

Here $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx = \frac{1}{\pi}\int_0^{2\pi} xdx = \frac{1}{\pi}\left|\frac{x^2}{2}\right|_0^{2\pi} = \frac{1}{\pi}\frac{(4\pi^2 - 0)}{2} = 2\pi.$.

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx dx = \frac{1}{\pi}\int_0^{2\pi} \underset{\mathrm{I}}{x}\underset{\mathrm{II}}{\cos nx}\, dx$$

$$= \frac{1}{\pi}\left[(x)\left(\frac{\sin nx}{n}\right) - (1)\left(\frac{-\cos nx}{n^2}\right)\right]_0^{2\pi} = \frac{1}{\pi}\left[\left(2\pi(0) + \frac{\cos 2n\pi}{n^2}\right) - \left(0 + \frac{1}{n^2}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{1}{n^2} - \frac{1}{n^2}\right] = 0.$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx dx = \frac{1}{\pi}\int_0^{2\pi} \underset{\mathrm{I}}{x}\underset{\mathrm{II}}{\sin nx}\, dx \qquad \left[\int \mu v dx = \mu.v_1 - \mu' v_2 + \mu''.v_3 - \mu'''.v_4 + ...\right]$$

$$= \frac{1}{\pi}\left[(x).\left(\frac{-\cos nx}{n}\right) - (1)\left(\frac{-\sin nx}{n^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-2\pi\frac{1}{n} + 0 - (-0 + 0)\right] = -\frac{2}{n}. \qquad \left[\begin{array}{l}\cos 2n\pi = (-1)^{2n} = 1 \\ \sin 2n\pi = 0\end{array}\right]$$

Hence, from (i), we get

$$f(x) = \frac{1}{2}.2\pi + \sum_{n=1}^{\infty}\left(0 + \left(\frac{-2}{n}\right)\sin nx\right)$$

$\boxed{\Rightarrow f(x) = \pi - 2\sum_{n=1}^{\infty}\frac{\sin nx}{n}}$, is the required Fourier series.

Q.No.2.: Prove that for all values of x between $-\pi$ and π,

$$\frac{1}{2}x = \sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + ...$$

Sol.: Here $f(x)=\frac{1}{2}x$, $-\pi<x<\pi$.

As f(x) is an odd function.

Hence, the required Fourier series is $f(x)=\sum_{n=1}^{\infty} b_n \sin nx$. (i)

Now $b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx=\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{2}x\sin nx dx$

$$=\frac{1}{\pi}\int_0^{\pi}\underset{\mathrm{I}}{x}\sin nx\,dx \qquad \left[\because \int_{-\pi}^{\pi} x\sin nx dx=2\int_0^{\pi} x\sin nx dx \atop (x\sin nx \text{ is even function})\right]$$

$$=\frac{1}{\pi}\left[x\left(\frac{-\cos nx}{n}\right)-(1)\left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$=\frac{1}{\pi}\left[\frac{-\pi}{n}(-1)^n\right]=-\frac{(-1)^n}{n}=\frac{(-1)^{n+1}}{n} \qquad \left[\sin n\pi=0,\ n\in Z \atop \cos n\pi=(-1)^n,\ n\in Z\right]$$

Hence, from (i), we get

$$f(x)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx$$

$\boxed{\Rightarrow f(x)=\sin x-\frac{\sin 2x}{2}+\frac{\sin 3x}{3}-.........}$, is the required Fourier series.

Q.No.3.: Prove that $x^2=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}(-1)^n\frac{\cos nx}{n^2}$, $-\pi<x<\pi$.

Hence, show that **(i)** $\sum\frac{1}{n^2}=\frac{\pi^2}{6}$,

(ii) $\sum\frac{(-1)^{n-1}}{n^2}=\frac{\pi^2}{12}$,

(iii) $\sum\frac{1}{(2n-1)^2}=\frac{\pi^2}{8}$.

or

Develop a Fourier series for the function $f(x)=x^2$ in the interval $-\pi<x<\pi$.

Hence, show that **(i)** $\sum \frac{1}{n^2} = \frac{\pi^2}{6}$, **(ii)** $\sum \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}$**, (iii)** $\sum \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$.

Sol.: The Fourier series is given by

$$f(x) = x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx . \qquad \text{(i)}$$

Here $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 dx = \frac{1}{\pi}\left|\frac{x^3}{3}\right|_{-\pi}^{\pi} = \frac{1}{\pi}\frac{2\pi^3}{3} = \frac{2\pi^2}{3}$.

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx dx = \frac{1}{\pi}\left[\frac{x^2 \sin nx}{n}\right]_{-\pi}^{\pi} - \frac{2}{n\pi}\int_{-\pi}^{\pi} x \sin nx dx$$

$$= \frac{1}{\pi}\left[\frac{x^2 \sin nx}{n}\right]_{-\pi}^{\pi} - \frac{2}{\pi n}\left[\left(\frac{-x\cos nx}{n}\right)_{-\pi}^{\pi} + \frac{1}{n}\int_{-\pi}^{\pi} \cos nx dx\right]$$

$$= \frac{1}{\pi}\left[\frac{x^2 \sin nx}{n} + \frac{2x}{n^2}\cos nx - \frac{2}{n^3}\sin nx\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\pi^2.0}{n} + \frac{2\pi}{n^2}(-1)^n - \frac{2}{n^3}(0) - \frac{\pi^2.0}{n} + \frac{2\pi}{n^2}(-1)^n + \frac{2}{n^3}.0\right] = \frac{4(-1)^n\pi}{\pi n^2} = \frac{4(-1)^n}{n^2}.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \sin nx dx = \frac{1}{\pi}\left\{\left[\frac{-x^2\cos nx}{n}\right]_{-\pi}^{\pi} + \frac{2}{n}\int_{-\pi}^{\pi} x\cos nx dx\right\}$$

$$= \frac{1}{\pi}\left\{\left[\frac{-x^2\cos nx}{n}\right]_{-\pi}^{\pi} + \frac{2}{n}\left[\left(\frac{x\sin nx}{n}\right)_{-\pi}^{\pi} - \frac{1}{n}\int_{-\pi}^{\pi}\sin nx dx\right]\right\}$$

$$= \frac{1}{\pi}\left[\frac{-x^2\cos nx}{n} + \frac{2x}{n^2}\sin nx + \frac{2}{n^3}\cos nx\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-\pi^2}{n}(-1)^n + \frac{2\pi.0}{n^2} + \frac{2}{n^3}(-1)^n + \frac{\pi^2(-1)^n}{n} - \frac{2\pi.0}{n^2} - \frac{2(-1)^n}{n^3}\right] = 0.$$

$\Rightarrow b_n = 0$.

Substituting the values of a_i's and b_i's in (i), we get the required Fourier series of

$f(x) = x^2$ for $-\pi < x < \pi$ as

$$x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$\Rightarrow x^2 = \frac{2\pi^2}{3.2} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos nx + 0$$

$$\boxed{\Rightarrow x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx}. \qquad \text{(ii)}$$

To show (i): $\sum \frac{1}{n^2} = \frac{\pi^2}{6}$.

Putting $x = \pi$ in equation (i), we obtain

$$\pi^2 = \frac{\pi^2}{3} + 4\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\ldots\ldots\ldots\right]$$

$$\Rightarrow 4\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\ldots\ldots\ldots\right] = \pi^2 - \frac{\pi^2}{3} = \frac{2\pi^2}{3}$$

$$\Rightarrow \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\ldots\ldots\ldots = \frac{\pi^2}{6}$$

$\boxed{\Rightarrow \sum \frac{1}{n^2} = \frac{\pi^2}{6}}$, which is the required result.

To show (ii): $\sum \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}$.

Putting x = 0 in (i), we get

$$0 = \frac{\pi^2}{3} + 4\left[-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \ldots\ldots\ldots\right] \Rightarrow 4\left[-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \ldots\ldots\ldots\right] = -\frac{\pi^2}{3}$$

$$\Rightarrow 4\sum \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{3}$$

$\boxed{\Rightarrow \sum \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}}$, which is the required result.

To show (iii): $\sum \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$.

Adding results (i) and (ii), we get

$$\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots\ldots\ldots\right)+\left(\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\ldots\ldots\right)=\frac{\pi^2}{6}+\frac{\pi^2}{12}$$

$$\Rightarrow 2\left(\frac{1}{1^2}+\frac{1}{3^2}+\ldots\ldots\ldots\right)=\frac{\pi^2}{4}\Rightarrow\left(\frac{1}{1^2}+\frac{1}{3^2}+\ldots\ldots\ldots\right)=\frac{\pi^2}{8}$$

$\boxed{\Rightarrow \sum \dfrac{1}{(2n-1)^2}=\dfrac{\pi^2}{8}}$, which is the required result.

Q.No.4.: $f(x)=x+x^2$ for $-\pi<x<\pi$ and $f(x)=\pi^2$ for $x=\pm\pi$.

Expand f(x) in Fourier series and show that

$$x+x^2=\frac{\pi^2}{3}+\sum_{n=1}^{\infty}(-1)^n\left\{\frac{4}{n^2}\cos nx-\frac{2}{n}\sin nx\right\}.$$

Hence, show that $\dfrac{1}{1^2}-\dfrac{1}{2^2}+\dfrac{1}{3^2}-\ldots\ldots=\dfrac{\pi^2}{12}$.

Sol.: The Fourier series is given by

$$f(x)=x+x^2=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx\,. \qquad \text{(i)}$$

Here $a_0=\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi}\left(x+x^2\right)dx=\dfrac{1}{\pi}\left[\dfrac{x^2}{2}+\dfrac{x^3}{3}\right]_{-\pi}^{\pi}=\dfrac{1}{\pi}\left[\dfrac{\pi^2}{2}+\dfrac{\pi^3}{3}-\dfrac{\pi^2}{2}+\dfrac{\pi^3}{3}\right]=\dfrac{2\pi^2}{3}$.

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}\left(x+x^2\right)\cos nx\,dx=\frac{1}{\pi}\left\{\left[\left(x+x^2\right)\frac{\sin nx}{n}\right]_{-\pi}^{\pi}-\int_{-\pi}^{\pi}(1+2x)\frac{\sin nx}{n}dx\right\}$$

$$=\frac{1}{\pi}\left\{\left[\left(x+x^2\right)\frac{\sin nx}{n}\right]_{-\pi}^{\pi}-\left[\frac{-(2x+1)}{n}\frac{\cos nx}{n}\right]_{-\pi}^{\pi}-\int_{-\pi}^{\pi}\frac{2\cos nx}{n^2}dx\right\}$$

$$=\frac{1}{\pi}\left\{\left[\left(x+x^2\right)\frac{\sin nx}{n}+\frac{(1+2x)\cos nx}{n^2}\right]_{-\pi}^{\pi}-\left[\frac{2\sin nx}{n^3}\right]_{-\pi}^{\pi}\right\}$$

$$=\frac{1}{\pi}\left[\frac{\left(\pi+\pi^2\right)\sin n\pi}{n}+\frac{(1+2\pi)\cos n\pi}{n^2}-\frac{2\sin n\pi}{n^3}\right]$$

$$-\frac{1}{\pi}\left[\frac{\left(-\pi+\pi^2\right)\sin(-n\pi)}{n}+\frac{(1-2\pi)\cos(-n\pi)}{n^2}-\frac{2\sin(-n\pi)}{n^3}\right]$$

$$= \frac{1}{\pi}\left[\frac{(1+2\pi)(-1)^n}{n^2} - \frac{(1-2\pi)(-1)^n}{n^2}\right]$$

$$= \frac{1}{\pi}\left[\frac{(1+2\pi-1+2\pi)}{n^2}(-1)^n\right] = \frac{1}{\pi}\times\frac{4\pi}{n^2}(-1)^n = \frac{4}{n^2}(-1)^n.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\sin nx\,dx = \frac{1}{\pi}\left\{\left[-(x+x^2)\frac{\cos nx}{n}\right]_{-\pi}^{\pi} + \int_{-\pi}^{\pi}(1+2x)\frac{\cos nx}{n}dx\right\}$$

$$= \frac{1}{\pi}\left\{\left[-(x+x^2)\frac{\cos nx}{n}\right]_{-\pi}^{\pi} + \left[\frac{(1+2x)\sin nx}{n^2}\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\frac{2\sin nx}{n^2}dx\right\}$$

$$= \frac{1}{\pi}\left[\frac{-(x+x^2)\cos nx}{n} + \frac{(1+2x)\sin nx}{n^2} + \frac{2\cos nx}{n^3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-(\pi+\pi^2)\cos n\pi}{n} + \frac{(1+2\pi)\sin n\pi}{n^2} + \frac{2}{n^3}\cos n\pi\right]$$

$$-\frac{1}{\pi}\left[\frac{-(-\pi+\pi^2)\cos(-n\pi)}{n} + \frac{(1-2\pi)\sin(-n\pi)}{n^2} + \frac{2}{n^3}\cos(-n\pi)\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{-(-1)^n(\pi+\pi^2)}{n} + \frac{2}{n^3}(-1)^n\right) - \left(\frac{-(-1)^n(-\pi+\pi^2)}{n} + \frac{2}{n^3}(-1)^n\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{-(-1)^n(\pi+\pi^2)}{n} + \frac{2}{n^3}(-1)^n + \frac{(-\pi+\pi^2)}{n}(-1)^n - \frac{2}{n^3}(-1)^n\right]$$

$$= \frac{1}{n\pi}\left[(-\pi-\pi^2-\pi+\pi^2)(-1)^n\right] = -\frac{1}{\pi}\times\frac{2\pi}{n}(-1)^n.$$

$$\Rightarrow b_n = \frac{-2}{n}(-1)^n.$$

Substituting the values of a_0, a_n and b_n in (i), we get the required Fourier series of

$f(x) = x + x^2$, in the range $-\pi < x < \pi$, as

$$x + x^2 = \frac{1}{2}\times\frac{2\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n\cos nx + \sum_{n=1}^{\infty}-\frac{2}{n}(-1)^n\sin nx$$

$$\boxed{\Rightarrow x + x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} (-1)^n \left[\frac{4}{n^2}\cos nx - \frac{2}{n}\sin nx\right]}.$$

Deduction: Put x = 0, in above, we get

$$0 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \Rightarrow 0 = \frac{\pi^2}{3} + 4\left(-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \ldots\ldots\right)$$

$$\boxed{\Rightarrow \frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \ldots\ldots}$$

Q.No.5.: Find a Fourier series to represent $x - x^2$ from $x = -\pi$ to $x = \pi$.

Hence, show that $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ldots = \frac{\pi^2}{12}$.

Sol.: The Fourier series is given by

$$f(x) = x - x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx . \qquad \text{(i)}$$

Here $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} (x - x^2)dx = = \frac{1}{\pi}\left|\frac{x^2}{2} - \frac{x^3}{3}\right|_{-\pi}^{\pi} = -\frac{2\pi^2}{3}$.

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (x - x^2)\cos nx dx = \frac{1}{\pi}\left[\left|(x - x^2)\frac{\sin nx}{n}\right|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} (1 - 2x)\frac{\sin nx}{n}dx\right]$$

$$= \frac{1}{\pi}\left[\left|(x - x^2)\frac{\sin nx}{n}\right|_{-\pi}^{\pi} - \left|(1 - 2x)\times\left(-\frac{\cos nx}{n^2}\right)\right|_{-\pi}^{\pi} + \int_{-\pi}^{\pi} (-2)\left(-\frac{\cos nx}{n^2}\right)dx\right]$$

$$= \frac{1}{\pi}\left[(x - x^2)\frac{\sin nx}{n} - (1 - 2x)\times\left(-\frac{\cos nx}{n^2}\right) + (-2)\left(-\frac{\sin nx}{n^3}\right)\right]_{-\pi}^{\pi} = \frac{-4(-1)^n}{n^2}.$$

$$\left[\because \cos n\pi = (-1)^n\right]$$

$$\therefore a_1 = \frac{4}{1^2},\ a_2 = \frac{-4}{2^2},\ a_3 = \frac{4}{3^2},\ a_4 = \frac{-4}{4^2}, \ldots\ldots \text{ etc.}$$

Finally, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (x - x^2)\sin nx dx$

$$= \frac{1}{\pi}\left[\left(x - x^2\right)\left(-\frac{\cos nx}{n}\right) - (1-2x)\times\left(-\frac{\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{-2(-1)^n}{n}.$$

$$\therefore b_1 = \frac{2}{1},\ b_2 = \frac{-2}{2},\ b_3 = \frac{2}{3},\ b_4 = \frac{-2}{4}, \ldots\ldots \text{etc.}$$

Substituting the values of a_i's and b_i's in (i), we get the required Fourier series of

$f(x) = x - x^2$ from $x = -\pi$ to $x = \pi$ as

$$x - x^2 = -\frac{\pi^2}{3} + 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + \ldots\ldots\ldots\right]$$
$$+ 2\left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \ldots\ldots\ldots\right].\ \text{Ans.}$$

2nd Part:

Putting x = 0, in the above relation, we get

$$0 = -\frac{\pi^2}{3} + 4\left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\right)$$

$$\boxed{\Rightarrow \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots = \frac{\pi^2}{12}}$$, which is the required result.

Remarks:

In the above example, we have used the result $\sin n\pi = 0$ and $\cos n\pi = (-1)^n$.

Also $\sin\left(n + \frac{1}{2}\right)\pi = (-1)^n$ and $\cos\left(n + \frac{1}{2}\right)\pi = 0$.

Q.No.6.: Obtain the Fourier series of $f(x) = \frac{(\pi - x)}{2}$ in the interval $(0,\ 2\pi)$.

Deduce $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots$

Sol.: Let $f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right)$ (i)

be the required Fourier series. Then

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx = \frac{1}{\pi}\int_0^{2\pi} \frac{1}{2}(\pi - x)dx = \frac{1}{2\pi}\left[\frac{(\pi-x)^2}{2(-1)}\right]_0^{2\pi} = -\frac{1}{4\pi}\left(\pi^2 - \pi^2\right) = 0.$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx dx = \frac{1}{\pi}\int_0^{2\pi} \frac{1}{2}\underset{\mathrm{I}}{(\pi - x)}\underset{\mathrm{II}}{\cos nx}\, dx$$

Integrating by parts, we get

$$a_n = \frac{1}{2\pi}\left[(\pi - x)\left(\frac{\sin nx}{n}\right) - (-1)\left(\frac{-\cos nx}{n^2}\right)\right]_0^{2\pi} = \frac{1}{2\pi}\left[\left(0 - \frac{1}{n^2}\right) - \left(0 - \frac{1}{n^2}\right)\right] = 0.$$

Also $b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx dx = \frac{1}{\pi}\int_0^{2\pi} \frac{1}{2}\underset{\mathrm{I}}{(\pi - x)}\underset{\mathrm{II}}{\sin nx}\, dx$

Integrating by parts, we get

$$b_n = \frac{1}{2\pi}(\pi - x)\sin x dx = \frac{1}{2\pi}\left[(\pi - x)\left(-\frac{\cos nx}{n}\right) - (-1)\frac{\sin nx}{n^2}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\left((-\pi)\left(-\frac{1}{n}\right) - 0\right) - \left(-\frac{\pi}{n} - 0\right)\right] = \frac{1}{2n}\left(\frac{\pi}{n} + \frac{\pi}{n}\right) = \frac{1}{n}.$$

Hence, from (i), we get

$$\boxed{f(x) = \frac{1}{2}.0 + \sum_{n=1}^{\infty}\left(\frac{1}{n}\sin nx\right) = \sum_{n=1}^{\infty}\frac{\sin nx}{n}}$$, is the required Fourier series.

2nd Part: Deduce $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots..$

Put $x = \frac{\pi}{2}$, then Fourier series becomes

$$\frac{\pi - \frac{\pi}{2}}{2} = \frac{\pi}{4} = \sum_{n=1}^{\infty}\frac{1}{n}\sin n\frac{\pi}{2}$$

$$\boxed{\Rightarrow \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots..}$$

Q.No.7.: If $f(x) = \left[\frac{\pi - x}{2}\right]^2$ in the range 0 to 2π, then show that $f(x) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\frac{\cos nx}{n^2}$.

Also, deduce that **(i).** $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

(ii). $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ldots = \frac{\pi^2}{12}$,

(iii). $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots = \frac{\pi^2}{8}$.

Sol.: The Fourier series is given by

$$f(x) = \left[\frac{\pi - x}{2}\right]^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx\,. \qquad \text{(i)}$$

Here $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx = \frac{1}{\pi}\int_0^{2\pi}\left(\frac{\pi - x}{2}\right)^2 dx = \frac{1}{4\pi}\left[\int_0^{2\pi} \pi^2 dx - 2\pi\int_0^{2\pi} xdx + \int_0^{2\pi} x^2 dx\right]$

$$= \frac{1}{4\pi}\left[\pi^2 |x|_0^{2\pi} - 2\pi\left|\frac{x^2}{2}\right|_0^{2\pi} + \left|\frac{x^3}{3}\right|_0^{2\pi}\right]$$

$$= \frac{1}{4\pi}\left[2\pi^3 - 4\pi^3 + \frac{8\pi 3}{3}\right] = \frac{\pi^2}{2} - \pi^2 + \frac{2\pi^2}{3} = \frac{\pi^2}{6}\,.$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \operatorname{cox} nxdx = \frac{1}{\pi}\int_0^{2\pi}\left(\frac{\pi - x}{2}\right)^2 \cos nxdx$$

$$= \frac{1}{4\pi}\left[\pi^2\int_0^{2\pi} \cos nxdx - 2\pi\int_0^{2\pi} x\cos nxdx + \int_0^{2\pi} x^2 \cos nxdx\right]$$

$$= \frac{1}{4\pi}\left[\pi^2\left|\frac{\sin nx}{n}\right|_0^{2\pi} - 2\pi\int_0^{2\pi} x\cos nxdx + \int_0^{2\pi} x^2 \cos nxdx\right].$$

Let $I_1 = \int_0^{2\pi} x\cos nxdx = \frac{x\sin nx}{n} - \frac{1}{n}\int_0^{2\pi} \sin nxdx = \left|\frac{x\sin nx}{n} + \frac{1}{n^2}\cos nx\right|_0^{2\pi}$

$$= \left[\frac{2\pi\sin n2\pi}{n} + \frac{\cos n2\pi}{n^2}\right] - \left[\frac{\cos(0)n}{n^2}\right] = \frac{1}{n^2} - \frac{1}{n^2} = 0\,.$$

$$I_2 = \int_0^{2\pi} x^2 \cos nxdx = \frac{x^2 \sin nx}{n} - \frac{2}{n}\int_0^{2\pi} x\sin nxdx = \left|\frac{x^2 \sin nx}{n}\right|_0^{2\pi} - \frac{2}{n}\int_0^{2\pi} x\sin nxdx\,.$$

Let $I_3 = \int_0^{2\pi} x \sin nx dx$

$$\therefore I_3 = \int_0^{2\pi} x \sin nx dx = \frac{-x\cos nx}{n} - \frac{1}{n}\int_0^{2\pi} \frac{1}{n}\int (-\cos nx)dx = \left|\frac{-x\cos nx}{n} + \frac{1}{n}\frac{\sin nx}{n}\right|_0^{2\pi}$$

$$\therefore I_2 = \left|\frac{x^2 \sin nx}{n} - \frac{2}{n}\left(\frac{-x\cos nx}{n} + \frac{1}{n}\frac{\sin nx}{n}\right)\right|_0^{2\pi} = \frac{4\pi}{n^2}.$$

Thus $a_n = \frac{1}{4\pi}\left[0 + \frac{4\pi}{n^2} + 0\right] = \frac{4\pi}{n^2} \times \frac{1}{4\pi} = \frac{1}{n^2}$.

It is clear that $b_n = 0 \Rightarrow \frac{1}{\pi}\int_0^{2\pi}\left(\frac{\pi - x}{2}\right)^2 \sin nx dx = 0$.

Substituting the values of a_0, a_n and b_n in (i), we get the required Fourier series of

$f(x) = \left[\frac{\pi - x}{2}\right]^2$ in the range 0 to 2π, as

$$\boxed{f(x) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}}. \qquad \text{(ii)}$$

2nd Part:

(i) To show: $\boxed{\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}}$.

Putting x = 0 in (ii), we get

$$\frac{\pi^2}{4} = \frac{\pi^2}{12} + \left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\ldots\ldots\right]$$

$$\Rightarrow \frac{\pi^2}{4} - \frac{\pi^2}{12} = \left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\ldots\ldots\right]$$

$\Rightarrow \frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$, which is the required result.

(ii) To show: $\boxed{\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ldots = \frac{\pi^2}{12}}$.

Put $x = \pi$ in (ii), we get

$$0=\frac{\pi^2}{12}+\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}=\frac{\pi^2}{12}+\left(1+\frac{1}{2^2}-\frac{1}{3^2}+\frac{1}{4^2}-\dots\dots\dots\right)$$

$\Rightarrow \frac{\pi^2}{12}=1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\dots\dots$, which is the required result.

(iii) To show: $\boxed{\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\dots\dots=\frac{\pi^2}{8}}$.

Adding (i) and (ii), we get

$$\frac{\pi^2}{6}+\frac{\pi^2}{12}=\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}-\dots\dots\dots\right)+\left(1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\dots\dots\dots\right)$$

$$\Rightarrow \frac{\pi^2}{4}=2\left(1+\frac{1}{3^2}+\frac{1}{5^2}+\dots\dots\right)$$

$\Rightarrow \frac{\pi^2}{8}=1+\frac{1}{3^2}+\frac{1}{5^2}+\dots\dots$, which is the required result.

Q.No.8.: Prove that for $-\pi<x<\pi$, $\frac{\left(\pi^2-x^2\right)x}{12}=\frac{\sin x}{1^3}-\frac{\sin 2x}{2^3}+\dots\dots$.

Sol.: Given $f(x)=\frac{x\left(\pi^2-x^2\right)}{12}$, $-\pi<x<\pi$.

Now $f(-x)=-x\frac{\left(\pi^2-x^2\right)}{12}=-f(x)\Rightarrow$ f(x) is odd function.

$\therefore a_0=0,\quad a_n=0.$

Thus $f(x)=\sum_{n=1}^{\infty} b_n \sin nx$ (i)

be the required Fourier series.

Now $b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx=\frac{1}{\pi}\int_{-\pi}^{\pi} x\frac{\left(\pi^2-x^2\right)}{12}\sin nx dx=\frac{1}{12\pi}\int_{-\pi}^{\pi}\left(\pi^2x-x^3\right)\sin nx dx$

$$=\frac{1}{6\pi}\int_0^{\pi}\underset{\mathrm{I}}{\left(\pi^2x-x^3\right)}\underset{\mathrm{II}}{\sin nx}\, dx \quad \left[\left(\pi^2x-x^3\right)\sin nx \text{ is even function}\right]$$

Integrating by parts, we get

$$=\frac{1}{6\pi}\left[\left\{\left(\pi^2x-x^3\right)\left(\frac{-\cos nx}{n}\right)\right\}_0^\pi-\int_0^\pi\left(\pi^2-3x^2\right)\left(\frac{-\cos nx}{n}\right)dx\right]$$

$$=\frac{1}{6\pi}\left[0-\int_0^\pi\left(\pi^2-3x^2\right)\left(\frac{-\cos nx}{n}\right)dx\right]=\frac{1}{6\pi}\left[\int_0^\pi\left(\pi^2-3x^2\right)\left(\frac{\cos nx}{n}\right)dx\right]$$

$$=\frac{1}{6\pi}\left[\left\{\left(\pi^2-3x^2\right)\left(\frac{\sin nx}{n^2}\right)\right\}_0^\pi-\int_0^\pi(-6x)\left(\frac{\sin nx}{n^2}\right)dx\right]=\frac{1}{6\pi}\left[0-\int_0^\pi(-6x)\left(\frac{\sin nx}{n^2}\right)dx\right]$$

$$=\frac{1}{6\pi}\left[\int_0^\pi(6x)\left(\frac{\sin nx}{n^2}\right)dx\right]=\frac{1}{6\pi}\left[\left\{(6x)\left(\frac{-\cos nx}{n^3}\right)\right\}_0^\pi-\int_0^\pi(-6)\left(\frac{-\cos nx}{n^3}\right)dx\right]$$

$$=\frac{1}{6\pi}\left[\left\{(6\pi)\left(\frac{-(-1)^n}{n^3}\right)-0\right\}+\int_0^\pi 6\left(\frac{\cos nx}{n^3}\right)dx\right]=\frac{1}{6\pi}\left[\left\{(6\pi)\left(\frac{(-1)^{n+1}}{n^3}\right)\right\}+6\left(\frac{\sin nx}{n^4}\right)_0^\pi\right]$$

$$=\frac{1}{6\pi}\left[\left\{(6\pi)\left(\frac{(-1)^{n+1}}{n^3}\right)\right\}+0\right]=\frac{(-1)^{n+1}}{n^3}\qquad\left[\begin{array}{l}\sin n\pi=0,\ n\in Z\\ \sin 0=0\end{array}\right]$$

Hence, from (i), we get

$$\boxed{x\left(\frac{\pi^2-x^2}{12}\right)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^3}\sin nx=\frac{\sin x}{1^3}-\frac{\sin 2x}{2^3}+\ldots\ldots}.$$

Q.No.9.: Obtain the Fourier series to represent e^x in the interval $0<x<2\pi$.

Sol.: Let $f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}\left(a_n\cos nx+b_n\sin nx\right)$ (i)

be the required Fourier series.

Now $a_0=\frac{1}{\pi}\int_0^{2\pi}e^x dx=\frac{1}{\pi}\left|e^x\right|_0^{2\pi}=\frac{\left(e^{2\pi}-1\right)}{\pi}$.

$$a_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\cos nx\,dx=\frac{1}{\pi}\int_0^{2\pi}e^x\cos nx\,dx$$

$$=\frac{1}{\pi}\left[\frac{e^x}{1+n^2}\left(\cos nx+n\sin nx\right)\right]_0^{2\pi}\qquad\left[\because\int e^{ax}\cos bx\,dx=\frac{e^{ax}}{a^2+b^2}\left(a\cos bx+b\sin bx\right)\right]$$

$$=\frac{1}{\pi}\left[\frac{e^{2\pi}}{1+n^2}(1+0)-\frac{1}{1+n^2}(1+0)\right]=\frac{e^{2\pi}-1}{\pi\left(1+n^2\right)}.$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx dx = \frac{1}{\pi}\int_0^{2\pi} e^x \sin nx dx$$

$$= \frac{1}{\pi}\left[\frac{e^x}{1+n^2}(\sin nx - n\cos nx)\right]_0^{2\pi} \quad \left[\because \int e^{ax}\sin bx dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)\right]$$

$$= \frac{1}{\pi(1+n^2)}e^{2\pi}(0-n) - \frac{1}{\pi}\left[\frac{1}{1+n^2}(-n)\right] = \frac{n(1-e^{2\pi})}{\pi(1+n^2)}.$$

Hence, from (i), we get

$$\boxed{f(x) = \frac{1}{2\pi}(e^{2\pi}-1) + \frac{e^{2\pi}-1}{\pi}\sum_{n=1}^{\infty}\left(\frac{\cos nx}{1+n^2} - \frac{n\sin nx}{1+n^2}\right)}.$$

Q.No.10.: Find the Fourier series to represent e^x in the interval $(-\pi, \pi)$.

Sol.: Let $f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ (i)

be the required Fourier series.

Now $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x dx = \frac{1}{\pi}\left|e^x\right|_{-\pi}^{\pi} = \frac{1}{\pi}\left(e^{\pi} - e^{-\pi}\right) = \frac{2}{\pi}\sinh\pi$

$$\left[\sinh\theta = \frac{e^{\theta}-e^{-\theta}}{2}\right]$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \cos nx dx$$

$$= \frac{1}{\pi}\left[\frac{e^x}{1+n^2}(\cos nx + n\sin nx)\right]_{-\pi}^{\pi} \quad \left[\begin{array}{l}\because \int e^{ax}\cos bx dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) \\ \text{Here } a=1,\ b=n\end{array}\right]$$

$$= \frac{1}{\pi}\left[\frac{e^x}{1+n^2}\left((-1)^n + 0\right) - \frac{e^{-x}}{1+n^2}\left((-1)^n + 0\right)\right]$$

$$= \frac{1}{\pi}\frac{(-1)^n}{(1+n^2)}\left(e^{\pi} - e^{-\pi}\right) \quad \left[\begin{array}{l}\cos(-n\pi) = \cos n\pi = (-1)^n \\ \text{Also } \sin n\pi = 0,\ n \in Z\end{array}\right]$$

$$= \frac{2\sinh\pi(-1)^n}{\pi(1+n^2)}. \quad \left[\sinh\pi = \frac{e^{\pi}-e^{-\pi}}{2}\right]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \sin nx dx$$

$$= \frac{1}{\pi}\left[\frac{e^x}{1+n^2}(\sin nx - n\cos nx)\right]_{-\pi}^{\pi} \qquad \left[\begin{array}{l}\because \int e^{ax}\sin bx dx = \dfrac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) \\ \text{Here } a = 1,\ b = n\end{array}\right]$$

$$= \frac{1}{\pi}\left[\frac{e^{\pi}}{1+n^2}\left(0-(-1)^n\right) - \frac{e^{-\pi}}{1+n^2}\left(-n(-1)^n\right)\right] = \frac{-(-1)^n n}{\pi\left(1+n^2\right)}\left(e^{\pi}-e^{-\pi}\right) = \frac{-2n(-1)^n \sinh\pi}{\pi\left(1+n^2\right)}.$$

Hence, from (i), we get

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$$

$$\Rightarrow f(x) = \frac{1}{2}.\left(\frac{2}{\pi}\sinh\pi\right) + \sum_{n=1}^{\infty}\left(\left\{\frac{2\sinh\pi}{\pi\left(1+n^2\right)}(-1)^n\right\}\cos nx + \left\{\frac{-2n(-1)^n}{\pi\left(1+n^2\right)}(\sinh x)\right\}\sin nx\right)$$

$$= \frac{2\sinh\pi}{\pi}\left[\frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{(-1)^n\cos nx}{\left(1+n^2\right)} - \frac{n(-1)^n}{1+n^2}\sin nx\right)\right]$$

$$= \frac{2\sinh\pi}{\pi}\left[\frac{1}{2} - \left(\frac{1}{2}\cos x - \frac{\cos 2x}{5} + \frac{\cos 3x}{10} - .....\right)\right.$$

$$\left. - \left(\frac{1}{2}\sin x - \frac{2}{5}\sin 2x + \frac{3}{10}\sin 3x - .....\right)\right].$$

Q.No.11.: Obtain the Fourier series for $f(x) = e^{-x}$ in the interval $0 < x < 2\pi$.

Sol.: The Fourier series is given by

$$e^{-x} = \frac{a_0}{2} + \sum_{n=2}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n\sin nx\,. \qquad \text{(i)}$$

Here $a_0 = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} f(x)dx = \dfrac{1}{\pi}\int_0^{2\pi} e^{-x}dx = \dfrac{1}{\pi}\left|-e^{-x}\right|_0^{2\pi} = \dfrac{1-e^{-2\pi}}{\pi}$.

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\cos nx dx$$

$$= \frac{1}{\pi\left(n^2+1\right)}\left|e^{-x}(-\cos nx + n\sin nx)\right|_0^{2\pi} = \left(\frac{1-e^{-2\pi}}{\pi}\right).\frac{1}{n^2+1}.$$

$$\therefore a_1 = \left(\frac{1-e^{-2\pi}}{\pi}\right)\frac{1}{2},\ a_2 = \left(\frac{1-e^{-2\pi}}{\pi}\right)\frac{1}{5},\ a_3 = \left(\frac{1-e^{-2\pi}}{\pi}\right)\frac{1}{10}\ \ ....\text{etc.}$$

Finally, $b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\sin nx dx$

$$= \frac{1}{\pi(n^2+1)}\left|e^{-x}(-\sin nx - n\cos nx)\right|_0^{2\pi} = \left(\frac{1-e^{-2\pi}}{\pi}\right).\frac{n}{n^2+1}.$$

$$\therefore b_1 = \left(\frac{1-e^{-2\pi}}{\pi}\right).\frac{1}{2},\ b_2 = \left(\frac{1-e^{-2\pi}}{\pi}\right).\frac{2}{5},\ b_3 = \left(\frac{1-e^{-2\pi}}{\pi}\right).\frac{3}{10},\ldots\ \text{etc.}$$

Substituting the values of a_i's and b_i's in (i), we get the required Fourier series of $f(x) = e^{-x}$ in the interval $0 < x < 2\pi$ as

$$\boxed{e^{-x} = \left(\frac{1-e^{-2\pi}}{\pi}\right)\left\{\begin{array}{l}\frac{1}{2}+\left(\frac{1}{2}\cos x+\frac{1}{5}\cos 2x+\frac{1}{10}\cos 3x+.............\right)\\ \quad +\left(\frac{1}{2}\sin x+\frac{2}{5}\sin 2x+\frac{3}{10}\sin 3x+.............\right)\end{array}\right\}}. \text{ Ans.}$$

Q.No.12.: Obtain a Fourier series to represent e^{-ax} from $x = -\pi$ to $x = \pi$.

Hence, derive series for $\frac{\pi}{\sinh \pi}$.

Sol.: The Fourier series is given by

$$f(x) = e^{-ax} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx. \tag{i}$$

Here $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}dx = \frac{1}{\pi}\left[-\frac{e^{-ax}}{a}\right]_{-\pi}^{\pi} = \frac{-1}{\pi a}\left[e^{-ax}\right]_{-\pi}^{\pi} = \frac{-1}{\pi a}\left[e^{-a\pi} - e^{a\pi}\right]$

$$= \left[\frac{e^{a\pi} - e^{-a\pi}}{a\pi}\right] = \frac{2\sinh a\pi}{a\pi}.$$

Since we know that $\sinh x = \frac{e^x - e^{-x}}{2} \Rightarrow 2\sinh x = e^x - e^{-x} \Rightarrow e^{a\pi} - e^{-a\pi} = 2\sinh a\pi$.

Also $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}\cos nx dx = I$ (say).

$$\text{Then } I=\frac{1}{\pi}\left[\cos nx\left(\frac{e^{-ax}}{-a}\right)\Bigg|_{-\pi}^{\pi}-\int_{-\pi}^{\pi}\left(\frac{e^{-ax}}{-a}\right)(-n\sin nx)\,dx\right]$$

$$=\left(\frac{-e^{ax}\cos nx}{\pi a}\right)_{-\pi}^{\pi}-\frac{n}{\pi a}\left[\sin nx\left(\frac{e^{-ax}}{-a}\right)_{-\pi}^{\pi}-\int_{-\pi}^{\pi}\left(\frac{e^{-ax}}{-a}\right).n\cos nxdx\right]$$

$$=\left[\frac{-e^{-ax}\cos nx}{a\pi}+\frac{n\sin nx.e^{-ax}}{a^2\pi}\right]_{-\pi}^{\pi}-\frac{n^2}{a^2\pi}\int_{-\pi}^{\pi}e^{-ax}\cos nxdx$$

$$=\left[\frac{-e^{-ax}\cos nx}{a\pi}+\frac{n\sin nxe^{-ax}}{a^2\pi}\right]_{-\pi}^{\pi}-\frac{n^2}{a^2}[I] \qquad \left[\because I=\frac{1}{\pi}\int_{-\pi}^{\pi}e^{-ax}\cos nxdx\right]$$

$$\Rightarrow\left[I+\frac{n^2}{a^2}.I\right]=\left[\frac{n\sin nxe^{-ax}}{a^2\pi}-\frac{e^{-ax}\cos nx}{a\pi}\right]_{-\pi}^{\pi}$$

$$\Rightarrow I=\frac{a^2}{\left(a^2+n^2\right)}\left[\frac{n\sin nxe^{-ax}}{a^2\pi}-\frac{e^{-ax}\cos nx}{a\pi}\right]_{-\pi}^{\pi}$$

$$=\frac{1}{\pi\left(a^2+n^2\right)}\left[ne^{-a\pi}\sin n\pi-ae^{-a\pi}\cos n\pi+n\sin n\pi e^{+a\pi}+ae^{+a\pi}\cos n\pi\right]$$

$$=\frac{1}{\pi\left(a^2+n^2\right)}\left[0-ae^{-a\pi}(-1)^n-0+ae^{a\pi}(-1)^n\right]=\frac{1}{\pi\left(a^2+n^2\right)}\left[ae^{-a\pi}-ae^{+a\pi}\right](-1)^n$$

$$=\frac{(-1)^n a}{\pi\left(n^2+a^2\right)}[2\sinh a\pi].$$

Now put $n=1, 2, 3, \ldots\ldots\ldots\ldots$, we get

$$a_1=\frac{-2a\sinh a\pi}{\pi\left(a^2+1^2\right)},\quad a_2=\frac{2a\sinh a\pi}{\pi\left(a^2+2^2\right)},\quad a_3=\frac{-2a\sinh a\pi}{\pi\left(a^2+3^2\right)},\quad \ldots\ldots\ldots\text{etc.}$$

Similarly, $b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}e^{-ax}\sin nxdx=I$ (say).

$$\therefore I=\frac{1}{\pi}\left[\sin nx\left(\frac{e^{-ax}}{-a}\right)_{-\pi}^{\pi}-\int_{-\pi}^{\pi}\left(\frac{e^{-ax}}{-a}\right).n\cos nxdx\right]$$

$$=\left[\frac{-e^{-ax}\sin nx}{a\pi}\right]_{-\pi}^{\pi}+\frac{n}{a\pi}\left[\cos nx\left(\frac{e^{-ax}}{-a}\right)_{-\pi}^{\pi}-\int_{-\pi}^{\pi}\left(\frac{e^{-ax}}{-a}\right)(-n)\sin nx\,dx\right]$$

$$=\left[\frac{-e^{-ax}\sin nx}{a\pi}\right]_{-\pi}^{\pi}-\left[\frac{ne^{-ax}\cos nx}{\pi a^2}\right]_{-\pi}^{\pi}-\frac{n^2}{a^2\pi}\int_{-\pi}^{\pi}e^{-ax}\sin nx\,dx\,.$$

Thus $\left[I+\frac{n^2}{a^2}.I\right]=\left[\frac{-e^{-ax}\sin nx}{a\pi}-\frac{ne^{-ax}\cos nx}{\pi a^2}\right]_{-\pi}^{\pi}$

$$\Rightarrow I=\frac{a^2}{a^2+n^2}\left[\frac{-e^{-ax}\sin nx}{a\pi}-\frac{ne^{-ax}\cos nx}{\pi a^2}\right]_{-\pi}^{\pi}$$

$$\therefore I=\frac{a^2}{\left(a^2+n^2\right)}\times\frac{1}{a^2\pi}\left[-ae^{-ax}\sin nx-ne^{-ax}\cos nx\right]_{-\pi}^{\pi}$$

$$=\frac{1}{\pi\left(a^2+n^2\right)}\left[-ae^{-a\pi}\sin n\pi-ne^{-a\pi}\cos n\pi-ae^{a\pi}\sin n\pi+ne^{a\pi}\cos n\pi\right]$$

$$=\frac{1}{\pi\left(a^2+n^2\right)}\left[0-ne^{-a\pi}(-1)^n+0+ne^{a\pi}(-1)^n\right]=\frac{(-1)^n}{\pi\left(a^2+n^2\right)}\left[ne^{a\pi}-ne^{-a\pi}\right]$$

$$=\frac{n(-1)^n}{\pi\left(a^2+n^2\right)}\left[e^{a\pi}-e^{-a\pi}\right]\Rightarrow b_n=I=\frac{n(-1)^n\,2\sinh a\pi}{\pi\left(a^2+n^2\right)}.\qquad\left[\because 2\sinh a\pi=e^{a\pi}-e^{-a\pi}\right]$$

Now putting $n=1,\ 2,\ 3,..........$, we get

$$b_1=\frac{-2\sinh a\pi}{\pi\left(a^2+1^2\right)},\quad b_2=\frac{2(2\sinh a\pi)}{\pi\left(a^2+2^2\right)},\quad b_3=\frac{3(-2\sinh a\pi)}{\pi\left(a^2+3^2\right)}\ \text{..... etc.}$$

Substituting the values of a_i's and b_i's in (i), we get the required Fourier series of

$f(x)=e^{-ax}$ from $x=-\pi$ to $x=\pi$ as

$$e^{-ax}=\frac{2\sinh a\pi}{2\pi a}-\frac{2\sinh\pi a}{\pi}\left[\frac{a\cos x}{\left(a^2+1^2\right)}-\frac{a\cos 2x}{\left(a^2+2^2\right)}+\frac{a\cos 3x}{\left(a^2+3^2\right)}+..............\right]$$

$$+\frac{2\sinh\pi a}{2\pi a}\left[\frac{-\sin x}{\left(a^2+1^2\right)}+\frac{2\sin 2x}{\left(a^2+2^2\right)}-\frac{3\sin 3x}{\left(a^2+3^2\right)}+..............\right]$$

$$\boxed{\Rightarrow e^{-ax} = \frac{2\sinh a\pi}{\pi}\left[\begin{array}{l}\left(\frac{1}{2a} - \frac{a\cos x}{a^2+1^2} + \frac{a\cos 2x}{a^2+2^2} - \frac{a\cos 3x}{a^2+3^2} + \ldots\ldots\ldots\ldots\ldots\ldots\right) \\ -\left(\frac{\sin x}{a^2+1^2} - \frac{2\sin 2x}{a^2+2^2} + \frac{3\sin 3x}{a^2+3^2} - \ldots\ldots\ldots\ldots\ldots\right)\end{array}\right]}$$

2^{nd} Part:

By putting $x = 0$, $a = 1$ in the above relation, we get

$$e^{-1(0)} = \frac{2\sinh\pi}{\pi}\left[\begin{array}{l}\left(\frac{1}{2} - \frac{\cos 0}{1^2+1^2} + \frac{\cos 0}{2^2+1^2} - \frac{\cos 0}{3^2+1^2} + \ldots\ldots\ldots\ldots\ldots\ldots\right) \\ -\left(\frac{\sin 0}{1^2+1^2} - \frac{2\sin 0}{2^2+1^2} + \frac{3\sin 0}{3^2+1^2} - \ldots\ldots\ldots\ldots\ldots\right)\end{array}\right]$$

$$\Rightarrow 1 = \frac{2\sinh\pi}{\pi}\left[\left(\frac{1}{2} - \frac{1}{2} + \frac{1}{2^2+1^2} - \frac{1}{3^2+1^2} + \ldots\ldots\right) + (0)\right]$$

$$\boxed{\Rightarrow \frac{\pi}{2\sinh\pi} = \frac{1}{2^2+1} - \frac{1}{3^2+1} + \frac{1}{4^2+1} - \ldots\ldots},$$

which is the required result.

Q.No.13.: Obtain the Fourier series expansion of $f(x) = e^{ax}$ in $(0, 2\pi)$.

Sol.: The Fourier series is given by

$$f(x) = e^{ax} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx. \qquad \text{(i)}$$

Here $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx = \frac{1}{\pi}\int_0^{2\pi} e^{ax}dx = \left.\frac{e^{ax}}{a\pi}\right|_0^{2\pi} = \frac{e^{2a\pi}-1}{a\pi}$.

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx dx = \frac{1}{\pi}\int_0^{2\pi} e^{ax}\cos nx dx$$

Using $\int e^{ax}\cos bx dx = e^{ax}\frac{[a\cos bx + b\sin bx]}{(a^2+b^2)}$, we get

$$a_n = \frac{1}{\pi}\left[\left. e^{ax}\frac{(a\cos nx + n.\sin nx)}{(a^2+n^2)}\right|_0^{2\pi}\right] = \frac{1}{\pi(a^2+n^2)}\left[ae^{2ax}\cos 2n\pi - ae^0.\cos 0\right]$$

$$= \frac{a}{\pi(a^2+n^2)}\left[e^{2a\pi} - 1\right].$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} e^{ax} \sin nxdx$$

Using $\int e^{ax} \sin bxdx = e^{ax} \frac{[a \sin bx - b \cos bx]}{(a^2 + b^2)}$, we get

$$b_n = \frac{1}{\pi}\left[e^{ax} \frac{(a \sin nx - n \cos nx)}{(a^2 + n^2)}\Bigg|_0^{2\pi} \right] = \frac{n}{\pi(a^2+n^2)}\left[-e^{2a\pi} \cos 2n\pi + 1\right]$$

$$= \frac{n}{\pi(a^2+n^2)}\left[1 - e^{2a\pi}\right].$$

Substituting the values of a_0, a_n and b_n in (i), we get the required Fourier series of

$f(x) = e^{ax}$, in the range $0 < x < 2\pi$, as

$$f(x) = e^{ax} = \frac{e^{2a\pi}-1}{2a\pi} + \sum_{n=1}^{\infty} \frac{a}{\pi(a^2+n^2)}\left[e^{2a\pi}-1\right]\cos nx + \sum_{n=1}^{\infty} \frac{n}{\pi(a^2+n^2)}\left[1-e^{2a\pi}\right]\sin nx$$

$$f(x) = e^{ax} = \left(\frac{e^{2a\pi}-1}{\pi}\right)\left[\frac{1}{2a} + \sum_{n=1}^{\infty} \frac{a}{(a^2+n^2)}\cos nx - \sum_{n=1}^{\infty} \frac{n}{(a^2+n^2)}\sin nx\right] \text{ Ans.}$$

Q.No.14.: Find the Fourier series to represent e^{ax} in the interval $-\pi < x < \pi$.

Sol.: The Fourier series is given by

$$f(x) = e^{ax} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx . \qquad \text{(i)}$$

Here $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{ax}dx = \frac{1}{\pi}\left[\frac{e^{ax}}{a}\right]_{-\pi}^{\pi} = \frac{1}{\pi a}\left(e^{ax} - e^{-ax}\right) = \frac{2 \sinh a\pi}{\pi a}$.

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nxdx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{ax} \cos nxdx = \frac{1}{\pi}\left[\frac{e^{ax}}{a^2+\pi^2}(a \cos nx + n \sin nx)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi(a^2+n^2)}\left[ae^{ax} \cos n\pi - ae^{-ax} \cos n\pi\right] = \frac{a \cos n\pi\left(e^{ax} - e^{-ax}\right)}{\pi(a^2+\pi^2)} = \frac{2a(-1)^n \sinh a\pi}{\pi(a^2+n^2)}.$$

Similarly, $b_n = \frac{2n(-1)^n \sinh a\pi}{\pi(a^2+n^2)}$.

Substituting the values of a_0, a_n and b_n in (i), we get the required Fourier series of

$$e^{ax}=\frac{\sinh a\pi}{a\pi}+\sum_{n=1}^{\infty}\frac{2a(-1)^n\sinh a\pi}{\pi\left(a^2+n^2\right)}\cos nx+\sum_{n=1}^{\infty}\frac{2n(-1)^n\sinh a\pi}{\pi\left(a^2+n^2\right)}\sin nx$$

$$\boxed{e^{ax}=\frac{2\sinh a\pi}{\pi}\left[\begin{array}{l}\left(\frac{1}{2a}-\frac{a\cos x}{a^2+1^2}+\frac{a\cos 2x}{a^2+2^2}-\frac{a\cos 3x}{a^2+3^2}+\ldots\ldots\ldots\ldots\right)\\ -\left(\frac{\sin x}{a^2+1^2}-\frac{2\sin 2x}{a^2+2^2}+\frac{3\sin 3x}{a^2+3^2}-\ldots\ldots\ldots\right)\end{array}\right]}. \text{ Ans.}$$

Q.No.15.: Expand $f(x)=x\sin x$, $0<x<2\pi$, in a Fourier series.

Sol.: The Fourier series is given by $f(x)=x\sin x=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx$.(i)

Here $a_0=\frac{1}{\pi}\int_0^{2\pi}x\sin x dx=\frac{1}{\pi}\left[x(-\cos x)-\int(-\cos x)dx\right]_0^{2\pi}$

$$=\left[\frac{-x\cos x}{\pi}+\frac{\sin x}{\pi}\right]_0^{2\pi}=\frac{1}{\pi}\left[\sin x-x\cos x\right]_0^{2\pi}$$

$$=\frac{1}{\pi}\left[0-2\pi(+1)-0+0\right]=\frac{-2x\pi}{\pi}=-2.$$

$$a_n=\frac{1}{\pi}\int_0^{2\pi}x\sin x.\cos nx dx=\frac{1}{\pi}\int_0^{2\pi}\frac{2}{2}.x\sin x\cos nx dx=\frac{1}{2\pi}\int_0^{2\pi}x.2\sin x\cos nx dx$$

$$=\frac{1}{2\pi}\int_0^{2\pi}x\left[\sin(x+nx)+\sin(x-nx)\right]dx=\frac{1}{2\pi}\int_0^{2\pi}x\sin x(1+n)dx+\frac{1}{2\pi}\int_0^{2\pi}x\sin x(1-n)dx$$

$$=\frac{1}{2\pi}\left[x\frac{(-\cos x(1+n))}{1+n}-\int_0^{2\pi}\frac{(-\cos x(1+n))}{1+n}dx\right]$$

$$+\frac{1}{2\pi}\left[x\frac{(-\cos x(1-n))}{1-n}-\int_0^{2\pi}\frac{(-\cos x(1-n))}{1-n}dx\right]$$

$$=\left[\frac{-x\cos x(1+n)}{2\pi(1+n)}+\frac{\sin x(1+n)}{2\pi(n+1)^2}\right]_0^{2\pi}+\left[\frac{-x\cos x(1-n)}{2\pi(1-n)}+\frac{\sin x(1-n)}{2\pi(n-1)^2}\right]_0^{2\pi}$$

$$=\left[\frac{-2\pi\cos(n+1)2\pi}{2\pi(1+n)}+0+0-0\right]+\left[\frac{-2\pi\cos(1-n)2\pi}{2\pi(1-n)}+0-0+0\right]$$

$$=\left[\frac{(-1)(-1)^{2(n+1)}}{1+n}+\frac{(-1)(-1)^{2(1-n)}}{(1-n)}\right]=\frac{(-1)(-1)^{2n+2}(1-n)+(-1)(-1)^{2-2n}(1+n)}{\left(1-n^2\right)}$$

$$=\frac{-1+n-1-n}{1-n^2}=\frac{-2}{1-n^2}=\frac{2}{n^2-1}.\ (n\neq 1)$$

Thus $a_n=\frac{2}{n^2-1}.(n\neq 1)$.

When n = 1, then

$$a_1=\frac{1}{\pi}\int_0^{2\pi}x\sin x\cos x dx=\frac{1}{\pi}\int_0^{2\pi}\frac{x}{2}\sin 2x dx=\frac{1}{2\pi}\left[\frac{x(-\cos 2x)}{2}-\frac{\sin 2x}{2}\right]_0^{2\pi}$$

$$=\frac{1}{2\pi}\left[\frac{2\pi(-1)-0}{2}\right]=\frac{1}{2\pi}\left(\frac{-2\pi}{2}\right)=-\frac{1}{2}.$$

$$b_n=\frac{1}{\pi}\int_0^{2\pi}x\sin x\sin nx dx=0. \quad \text{(After evaluation)}$$

When n = 1, then

$$b_1=\frac{1}{\pi}\int_0^{2\pi}x\sin x.\sin x dx=\frac{1}{\pi}\int_0^{2\pi}x\sin^2 x dx$$

$$=\frac{1}{\pi}\int_0^{2\pi}x\frac{(1-\cos 2x)}{2}dx=\frac{1}{2\pi}\left[\frac{-x^2}{2}\right]_0^{2\pi}-\left[\frac{\sin 2x}{2}\right]_0^{2\pi}$$

$$=\frac{1}{2\pi}\times\left[\frac{4\pi^2}{2}-0\right]-0=\frac{1}{2\pi}\times\frac{4\pi^2}{2}=\pi.$$

Substituting the values of a_0, a_n and b_n in (i), we get the required Fourier series of $f(x)=x\sin x$, in the range $0<x<2\pi$, as

$$\boxed{x\sin x=-1+\pi\sin x-\frac{1}{2}\cos x+2\sum_{n=2}^{\infty}\frac{\cos nx}{n^2-1}}.$$

Q.No.16.: Prove that, in the range $-\pi<x<\pi$,

$$\cosh ax=\frac{2a}{\pi}\sinh ax\left[\frac{1}{2a^2}+\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+a^2}\cos n\pi\right].$$

Sol.: The Fourier series is given by

$$f(x) = \cosh ax = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx . \qquad \text{(i)}$$

Here $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} \cosh axdx = \frac{2\times 1}{\pi}\int_{0}^{\pi} \cosh axdx$

$$= \frac{1}{\pi}\left[2\int_0^{\pi} \frac{e^{ax}+e^{-ax}}{2}dx\right] = \frac{2}{2\pi}\left[\int_0^{\pi} e^{ax}dx + \int_0^{\pi} e^{-ax}dx\right]$$

$$= \frac{1}{\pi}\left|\frac{e^{ax}}{a} - \frac{e^{-ax}}{a}\right|_0^{\pi} = \frac{1}{\pi a}\left[e^{a\pi} - e^{-a\pi}\right] = \frac{2}{\pi a}\left[\frac{e^{a\pi} - e^{a\pi}}{2}\right] = \frac{2}{\pi a}\sinh a\pi .$$

Thus $a_0 = \frac{2}{\pi a}\sinh a\pi = \frac{2a^2}{\pi}\sinh a\pi\left[\frac{1}{a^2}\right]$.

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (\cosh ax)\cos nxdx = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{e^{ax}+e^{-ax}}{2}\right)\cos nxdx$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{\pi} e^{ax}\cos nxdx + \int_{-\pi}^{\pi} e^{-ax}\cos nxdx\right]. \qquad \text{(i)}$$

Let $I_1 = \int_{-\pi}^{\pi} e^{ax}\cos nxdx = \left[\frac{e^{ax}}{a}\cos nx\right]_{-\pi}^{\pi} - \frac{n}{a}\int_{-\pi}^{\pi} e^{ax}(-\sin nx)dx$

$$= \left[\frac{e^{ax}}{a}\cos nx + \frac{n}{a}\left(\sin nx\frac{e^{ax}}{a} - \frac{n}{a}I_1\right)\right]_{-\pi}^{\pi}$$

$$\Rightarrow \frac{I\left(n^2+a^2\right)}{a^2} = \left[\frac{e^{ax}}{a}\cos nx + \frac{ne^{ax}}{a^2}\sin nx\right]_{-\pi}^{\pi}$$

$$\Rightarrow I_1 = \left[\frac{e^{ax}}{n^2+a^2}(a\cos nx + n\sin nx)\right]_{-\pi}^{\pi} = \frac{1}{n^2+a^2}\left[e^{a\pi}.a(-1)^n - e^{-a\pi}.a(-1)^n\right]$$

$$= \frac{2a(-1)}{n^2+a^2}\left[\frac{e^{a\pi}-e^{-a\pi}}{2}\right] = \frac{2a(-1)^n\sinh a\pi}{\left(a^2+n^2\right)}.$$

Similarly, $I_2 = \int_{-\pi}^{\pi} e^{-ax}\cos nx dx = \left[\frac{e^{-ax}}{n^2+a^2}(-a\cos nx - n\sin nx)\right]_{-\pi}^{\pi}$

$$= \frac{1}{n^2+a^2}\left[e^{-a\pi}(-a)(-1)^n - e^{a\pi}(-a)(-1)^n\right]$$

$$= \frac{e^{-a\pi}}{n^2+a^2}\left[-a(-1)^n\right] - \left[\frac{e^{-a(-\pi)}}{n^2+a^2}(-a)(-1)^n\right]$$

$$= \frac{2(-1)^n(-a)}{n^2+a^2}\left[\frac{e^{-a\pi}-e^{a\pi}}{2}\right] = \frac{2a(-1)^n}{n^2+a^2}\left[\frac{e^{a\pi}-e^{-a\pi}}{2}\right] = \frac{+2a(-1)^n}{(n^2+a^2)}\sinh a\pi.$$

Thus $I_1 + I_2 = \frac{4a(-1)^n}{(n^2+a^2)}\sinh a\pi$.

On keeping the values of $I_1 + I_2$ in equation (i), we get

$$a_n = \frac{2a(-1)^n \sinh a\pi}{\pi(n^2+a^2)}.$$

Also $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{(e^{ax}+e^{-ax})}{2}\sin nx dx = 0$. [as odd function]

Substituting the values of a_0, a_n and b_n in (i), we get the required Fourier series of $f(x) = \cosh ax$, in the range $-\pi < x < \pi$, as

$$\boxed{\cosh ax = \frac{2a\sinh a\pi}{\pi}\left[\frac{1}{2a^2} + \sum_{n=1}^{\infty}\frac{(-1)^n}{(a^2+n^2)}\cos nx\right]}.$$

Q.No.17.: Obtain the Fourier series for $\sqrt{1-\cos 2x}$ in the interval $(0, 2\pi)$.

Sol.: Let $f(x) = \sqrt{1-\cos 2x} = \sqrt{2\sin^2 x} = \sqrt{2}|\sin x|$.

Also $f(-x) = \sqrt{2}|\sin(-x)| = \sqrt{2}|-\sin x| = \sqrt{2}|\sin x| = f(x)$. $\left[\begin{array}{l}|\alpha x| = |\alpha||x| \\ |-1| = 1\end{array}\right]$

$\Rightarrow f(x)$ is even function

$\therefore b_n = 0 \forall n$

Let $f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx$, (i) be the required Fourier series for f(x).

Now $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx = \frac{1}{\pi}\int_0^{2\pi} \sqrt{2}(\sin x)dx = \frac{\sqrt{2}}{\pi}\left[\int_0^{\pi} |\sin x| + \int_{\pi}^{2\pi} |\sin x| dx\right]$

$$= \frac{\sqrt{2}}{\pi}\left[\int_0^{\pi} \sin x dx + \int_{\pi}^{2\pi} (-\sin x)dx\right] \qquad \left[\begin{array}{l} |x| = \begin{cases} x, & x > 0 \\ -x, & x < 0 \end{cases} \\ \text{For } 0 < x < \pi,\ \sin x = +\text{ve and} \\ \text{For } \pi < x < 2\pi,\ \sin x = -\text{ve} \end{array}\right]$$

$$= \frac{\sqrt{2}}{\pi}\left[\left|-\cos x\right|_0^{\pi} + \left|\cos x\right|_{\pi}^{2\pi}\right] = \frac{\sqrt{2}}{\pi}\left(-\cos\pi + \cos 0 + \cos 2\pi - \cos\pi\right)$$

$$= \frac{\sqrt{2}}{\pi}(1+1+1+1) = \frac{4\sqrt{2}}{\pi}. \qquad \text{(ii)}$$

Also $a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx dx = \frac{\sqrt{2}}{\pi}\int_0^{2\pi} |\sin x| \cos nx dx$

$$= \frac{\sqrt{2}}{\pi}\left[\int_0^{\pi} \sin x \cos nx dx + \int_{\pi}^{2\pi} -\sin x \cos nx dx\right]$$

$$= \frac{\sqrt{2}}{\pi}\frac{1}{2}\int_0^{\pi} 2\cos nx \sin x dx - \frac{\sqrt{2}}{\pi}.\frac{1}{2}\int_{\pi}^{2\pi} 2\cos nx \sin x dx$$

$$= \frac{1}{\sqrt{2}\pi}\int_0^{\pi} \left[\sin(n+1)x - \sin(n-1)x\right]dx - \frac{1}{\sqrt{2}\pi}\int_{\pi}^{2\pi} \left[\sin(n+1)x - \sin(n-1)x\right]dx$$

$$= \frac{1}{\sqrt{2}\pi}\left[\frac{-\cos(n+1)x}{(n+1)} + \frac{\cos(n-1)x}{(n-1)}\right]_0^{\pi} - \frac{1}{\sqrt{2}\pi}\left[\frac{-\cos(n+1)x}{(n+1)} + \frac{\cos(n-1)x}{(n-1)}\right]_{\pi}^{2\pi}$$

$$= \frac{1}{\sqrt{2}\pi}\left[\frac{-\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1} - \left(\frac{-1}{n+1} + \frac{1}{n+2}\right)\right]$$

$$-\left[\left(\frac{-\cos 2(n+1)\pi}{n+1} + \frac{\cos 2(n-1)\pi}{n-1}\right) - \left(\frac{-\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1}\right)\right]$$

$$= \frac{1}{\sqrt{2}\pi}\left[\frac{-(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} + \frac{1}{n+1} - \frac{1}{n-1} + \frac{1}{n+1} - \frac{1}{n-1} - \frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1}\right]$$

$$= \frac{2}{\sqrt{2}\pi}\left[\frac{-(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$= \frac{\sqrt{2}}{\pi}\begin{cases} \frac{1}{n+1} - \frac{1}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}; & n \text{ is even} \\ \frac{-1}{n+1} + \frac{1}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}; & n \text{ is odd} \end{cases}$$

$$= \frac{\sqrt{2}}{\pi}\begin{cases} \frac{2}{n+1} - \frac{2}{n-1}, & n \text{ is even} \\ 0, & n \text{ is odd} \end{cases}$$

$$= \frac{-4\sqrt{2}}{\pi(n^2-1)}, \quad n \text{ is even.}$$

Take $n = 2m$, we get $a_n = \dfrac{-4\sqrt{2}}{\pi(4m^2-1)}$, $m = 1, 2, \ldots\ldots$ (iii)

Putting the values of a_0 from (ii) and a_n from (iii) in (i), we get

$$f(x) = \frac{1}{2}.\frac{4\sqrt{2}}{\pi} + \sum_{m=1}^{\infty} \frac{-4\sqrt{2}}{\pi(4m^2-1)}\cos 2mx = \frac{2\sqrt{2}}{\pi} - \frac{4\sqrt{2}}{\pi} + \sum_{m=1}^{\infty} \frac{\cos 2mx}{\pi(4m^2-1)}$$

Thus $\boxed{\sqrt{1-\cos x} = \dfrac{2\sqrt{2}}{\pi} - \dfrac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty} \dfrac{\cos nx}{4n^2-1}}$.

Q.No.18.: Obtain a Fourier expansion for $\sqrt{1-\cos x}$ in the interval $-\pi < x < \pi$.

Sol.: Here $f(x) = \sqrt{1-\cos x} = \sqrt{2\sin^2 \frac{x}{2}} = \sqrt{2}\left|\sin\frac{x}{2}\right|$.

Now $f(-x) = \sqrt{2}\left|\sin\left(-\frac{x}{2}\right)\right| = \sqrt{2}\left|\sin -\frac{x}{2}\right|$ $[\because \sin(-\theta) = -\sin\theta]$

$$= \sqrt{2}\left|\sin\frac{x}{2}\right| = f(x). \qquad [|\alpha x| = |\alpha||x|]$$

$\Rightarrow f(x)$ is even function. $\therefore b_n = 0$.

Let $f(x) = \dfrac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$ (i)

be the required Fourier series.

Now $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} \sqrt{2}\left|\sin\frac{x}{2}\right|dx = \frac{2}{\pi}\int_{0}^{\pi} \sqrt{2}\left|\sin\frac{x}{2}\right|dx$

$$= \frac{2^{3/2}}{\pi}\int_0^{\pi} \sin\frac{x}{2}dx \qquad \left[\begin{array}{l} |x| = \begin{cases} x, & x>0 \\ -x, & x<0 \end{cases} \\ \text{For } 0<x<\pi,\ \sin\frac{x}{2} \text{ is positive } \therefore \left|\sin\frac{x}{2}\right| = \sin\frac{x}{2} \end{array}\right]$$

$$= \frac{2^{3/2}}{\pi}\frac{\left|-\cos\frac{x}{2}\right|_0^{\pi}}{1/2} = \frac{-2}{\pi}.2^{3/2}(0-1) = \frac{4.\sqrt{2}}{\pi}.$$

Also $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \sqrt{2}\left|\sin\frac{x}{2}\right|\cos nx dx \qquad \left[\left|\sin\frac{x}{2}\right|\cos nx \text{ is even function}\right]$

$$= \frac{2}{\pi}.\sqrt{2}\int_0^{\pi} \left|\sin\frac{x}{2}\right|\cos nx dx - 2.\frac{\sqrt{2}}{\pi}\int_0^{\pi} \sin\frac{x}{2}\cos nx dx \qquad \{\text{ For } 0<x<\pi,\ \sin x \text{ is positive}\}$$

$$= \frac{\sqrt{2}}{\pi}\int_0^{\pi}\left[\sin\left(\frac{1}{2}+n\right)x + \sin\left(\frac{1}{2}-n\right)x\right]dx \qquad [2\sin A\cos B = \sin(A+B)+\sin(A-B)]$$

$$= \frac{\sqrt{2}}{\pi}\left[-\frac{\cos\left(n+\frac{1}{2}\right)x}{n+\frac{1}{2}} - \frac{\cos\left(\frac{1}{2}-n\right)x}{\frac{1}{2}-n}\right]_0^{\pi}$$

$$= \frac{\sqrt{2}}{\pi}\left[\left(\frac{-\cos\left(n+\frac{1}{2}\right)\pi}{n+\frac{1}{2}} + \frac{\cos\left(n-\frac{1}{2}\right)\pi}{n-\frac{1}{2}}\right) - \left(\frac{-1}{n+\frac{1}{2}} + \frac{1}{n-\frac{1}{2}}\right)\right]$$

$$= \frac{\sqrt{2}}{\pi}\left(\frac{1}{n+\frac{1}{2}} - \frac{1}{n-\frac{1}{2}}\right) \qquad \left[\begin{array}{l} \cos(n\pi+\theta) = (-1)^n\cos\theta \\ \therefore \cos\left(n\pi+\frac{\pi}{2}\right) = (-1)^n\cos\frac{\pi}{2} \end{array}\right]$$

$$= \frac{-4\sqrt{2}}{\pi(4n^2-1)}.$$

Hence, from (i), we get

$$\sqrt{2}\sin\frac{x}{2}=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx=\frac{1}{2}.\frac{4\sqrt{2}}{\pi}+\sum_{n=1}^{\infty}\frac{-4\sqrt{2}}{\pi\left(4n^2-1\right)}.\cos nx$$

$$\boxed{\Rightarrow\sqrt{2}\sin\frac{x}{2}=\frac{2\sqrt{2}}{\pi}-\frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{\cos nx}{4n^2-1}},$$

is the required Fourier series.

Q.No.19.: Express $f(x)=\cos wx$, $-\pi<x<\pi$, where w is a fraction, as a Fourier series. Hence, prove that $\cot\theta=\frac{1}{\theta}+\frac{2\theta}{\theta^2-\pi^2}+\frac{2\theta}{\theta^2-4\pi^2}+\ldots\ldots\ldots.$

Sol.: Here $f(x)=\cos wx$ is an even function. $\therefore b_n=0$. $\qquad [\cos(-wx)=\cos wx]$

Let $f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}a_n\cos nx$ (i)

be the required Fourier series.

Now $a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx=\frac{1}{\pi}\int_{-\pi}^{\pi}\cos wxdx=\frac{2}{\pi}\int_{0}^{\pi}\cos wxdx$ $\qquad \left[\int_{-a}^{a}f(x)dx=2\int_{0}^{a}f(x)dx \text{ if } f(x) \text{ is even}\right]$

$$=\frac{2}{\pi}\left|\frac{\sin wx}{w}\right|_0^{\pi}=\frac{2}{\pi w}(\sin w\pi).$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nxdx=\frac{1}{\pi}\int_{-\pi}^{\pi}\cos wx\cos nxdx$$

$$=\frac{2}{\pi}\int_{0}^{\pi}\cos wx\cos xdx=\frac{1}{\pi}\int_{0}^{\pi}2\cos wx\cos nxdx \qquad [\cos wx\cos x \text{ is even function}]$$

$$=\frac{1}{\pi}\int_{0}^{\pi}[\cos(w+n)x\cos(w-n)x]dx$$

$$[2\cos A\cos B=\cos(A+B)+\cos(A-B)]$$

$$=\frac{1}{\pi}\left[\frac{\sin(w+n)x}{w+n}+\frac{\sin(w-n)x}{w-n}\right]_0^{\pi}=\frac{1}{\pi}\left[\frac{\sin(w+n)\pi}{w+n}+\frac{\sin(w-n)\pi}{w-n}\right]$$

$$=\frac{1}{\pi}\left[\frac{\sin w\pi\cos n\pi+\cos w\pi\sin n\pi}{w+n}+\frac{\sin w\pi\cos n\pi-\cos w\pi\sin n\pi}{w-n}\right]$$

$$=\frac{1}{\pi}\left[\frac{\sin w\pi\cos n\pi}{w+n}+\frac{\sin w\pi\cos n\pi}{w-n}\right] \qquad [\sin n\pi=0]$$

$$=\frac{\sin w\pi\cos n\pi}{\pi}\left[\frac{1}{(w+n)}+\frac{1}{(w-n)}\right]$$

$$=\frac{\sin w\pi\cos n\pi}{\pi}\left[\frac{w-n+w+n}{(w+n)(w-n)}\right]=\frac{2w\sin w\pi\cos n\pi}{\pi\left(w^2-n^2\right)}$$

$$=\frac{2w(-1)^n\sin w\pi}{\pi\left(w^2-n^2\right)}=\frac{2w(-1)^{n+1}\sin w\pi}{\pi\left(w^2-n^2\right)}.$$

Hence, from (i), we get

$$f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}a_n\cos nx=\frac{\sin w\pi}{w\pi}+\sum_{n=1}^{\infty}\frac{2w(-1)^{n+1}\sin w\pi}{\pi\left(n^2-w^2\right)}\cos nx$$

$$=\frac{\sin w\pi}{w\pi}+\frac{2w}{\pi}\sin w\pi\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\left(n^2-w^2\right)}\cos nx$$

$$=\frac{\sin w\pi}{w\pi}+\frac{2w\sin w\pi}{\pi}\left(\frac{\cos x}{1^2-w^2}-\frac{\cos 2x}{2^2-w^2}+\frac{\cos 3x}{3^2-w^2}+\dots\dots\dots\right)$$

$$=\frac{2w\sin w\pi}{\pi}\left(\frac{1}{2w^2}+\frac{\cos x}{1^2-w^2}-\frac{\cos 2x}{2^2-w^2}+\frac{\cos 3x}{3^2-w^2}+\dots\dots\dots\right)$$

$$\boxed{\therefore \cos wx=\frac{2w\sin w\pi}{\pi}\left(\frac{1}{2w^2}+\frac{\cos x}{1^2-w^2}-\frac{\cos 2x}{2^2-w^2}+\frac{\cos 3x}{3^2-w^2}+\dots\dots\dots\right).}$$

Deduction : Take $x=\pi$, we get

$$\cos wx=2\sin w\pi\times\frac{w}{\pi}\left(\frac{1}{2w^2}+\frac{\cos x}{1^2-w^2}-\frac{\cos 2x}{2^2-w^2}+\dots\dots\dots\right) \qquad \left[\begin{array}{l}\cos\pi=-1\\ \cos 2\pi=1\end{array}\right]$$

$$\Rightarrow \cot w\pi=\frac{1}{w\pi}-\frac{2w}{\left(1^2-w^2\right)\pi}-\frac{2w}{\left(2^2-w^2\right)\pi}-\dots\dots\dots$$

$$\boxed{\Rightarrow \cot\theta=\frac{1}{\theta}+\frac{2\theta}{\theta^2-\pi^2}+\frac{2\theta}{\theta^2-4\pi^2}+\dots..} \qquad \left[\begin{array}{l}w\pi=\theta\\ w=\dfrac{\theta}{\pi}\end{array}\right].$$

Q.No.20.: Find the Fourier series for f(x) in the interval $(-\pi,\ \pi)$, when

$$f(x)=\begin{cases} n+x, & -\pi<x<0 \\ n-x, & 0<x<\pi \end{cases}.$$

Sol.: $f(-x)=\begin{cases} n-x, & -\pi<x<0 \\ n+x, & 0<x<\pi \end{cases}$

$$=\begin{cases} \pi-x, & 0<x<\pi \\ \pi+x, & -\pi<x<0 \end{cases} \qquad [1<2<4 \Rightarrow -1>-2>-4]$$

$= f(x) \Rightarrow$ f(x) is even function. $\therefore b_n=0$.

Let $\quad f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty} a_n\cos nx \qquad$ (i)

be the required Fourier series.

Now $\quad a_0=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx=\frac{1}{\pi}\int_{-\pi}^{0}(\pi+x)dx+\frac{1}{\pi}\int_{0}^{\pi}(\pi-x)dx$

$$=\frac{1}{\pi}\left[\pi x+\frac{x^2}{2}\right]_{-\pi}^{0}+\frac{1}{\pi}\left[nx-\frac{x^2}{2}\right]_0^{\pi}=\frac{1}{\pi}\left[0-\left(-\pi^2+\frac{\pi^2}{2}\right)\right]+\frac{1}{\pi}\left[\pi^2-\frac{\pi^2}{2}-0\right]$$

$$=\frac{1}{\pi}.\frac{\pi^2}{2}+\frac{1}{\pi}.\frac{\pi^2}{2}=\frac{\pi}{2}+\frac{\pi}{2}=\pi.$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx=\frac{1}{\pi}\int_{-\pi}^{0}(\pi+x)\cos nxdx+\frac{1}{\pi}\int_{0}^{\pi}(\pi-x)\cos nxdx$$

$$=\frac{1}{\pi}\left[(\pi+x)\left(\frac{\sin nx}{n}\right)-(1)\left(\frac{-\cos nx}{n^2}\right)\right]_{-\pi}^{0}+\frac{1}{\pi}\left[(\pi-x)\left(\frac{\sin nx}{n}\right)-(-1)\left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$=\frac{1}{\pi}\left[0+\frac{1}{n^2}-\left(0+\frac{\cos nx}{n^2}\right)\right]+\frac{1}{\pi}\left[\left(0-\frac{\cos nx}{n^2}\right)-\left(0-\frac{1}{n^2}\right)\right]$$

$$=\frac{1}{\pi}\left(\frac{2}{n^2}-\frac{2\cos n\pi}{n^2}\right)=\frac{2}{n^2\pi}[1-\cos n\pi]$$

$$=\begin{cases} 0, & n \text{ is even} \\ \dfrac{4}{n^2\pi}, & n \text{ is odd} \end{cases}$$

Hence, from (i), we get

$$\boxed{f(x)=\frac{\pi}{2}+\sum_{n=odd}^{\infty}\frac{4}{n^2\pi}\cos nx=\frac{\pi}{2}+\frac{4}{\pi}\left(\frac{\cos x}{1^2}+\frac{\cos 3x}{3^2}+\frac{\cos 5x}{5^2}+........\right)}.$$

Q.No.21.: Define Fourier series over the interval $-\pi$ to π.

Is it possible to write the Fourier sine series for the function $f(x)$, over the interval $(-\ell,\ell)$?

Sol.: Let f(x) be a function defined in the interval $(-\pi, \pi)$. Then

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos nx+b_n\sin nx\right)$$

where $a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)$, $a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nxdx$, $b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nxdx$

is called Fourier series for f(x).

Second Part: For half-range sine series, f(x) must be defined only in the interval $(0,\ell)$. So that, for developing a Fourier sine series, function can be expended in the interval $(-\ell,0)$, and the extended function becomes odd. But in this case function has already defined in the full interval $(-\ell,\ell)$. So it can be expanded into Fourier series containing both sine and cosine terms.

Hence, we cannot find the Fourier half range sine series for $f(x)$, over the interval $(-\ell,\ell)$.

Home Assignments

No assignment

(Students are advised to solve each problem before moving next topic)

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2^{nd} Topic

Fourier Series

Conditions for a Fourier Expansion

(Dirichlet's conditions)

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Dirichlet's conditions:

Dirichlet conditions are sufficient conditions for a real-valued, periodic function f(x) to be equal the sum of its Fourier series at each point where f is continuous. Moreover, the behavior of the Fourier series at points of discontinuity is determined as well. These conditions are named after Johann Peter Gustav Lejeune Dirichlet.

Johann Peter Gustav Lejeune Dirichlet

13-02-1805 to 05-05-1859

The conditions are:

- f(x) must have a finite number of extrema in any given interval
- f(x) must have a finite number of discontinuities in any given interval
- f(x) must be absolutely integrable over a period.
- f(x) must be bounded

The sufficient conditions for the uniform convergence of a Fourier series are called Dirichlet's conditions. All the functions that normally arise in engineering problems satisfy these conditions and hence they can be expressed as a Fourier series.

Any function f(x) can be developed as a Fourier series

$$\boxed{\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos nx+\sum_{n=1}^{\infty} b_n \sin nx},$$

where $a_0,\ a_n,\ b_n$ **are constants, provided the given function have satisfied the following conditions, which are known as Dirichlet's conditions.**

(i) **f (x) is a periodic, single-valued and finite;**

(ii) **f (x) has a finite number of discontinuities in any one period;**

(iii) **f (x) has at the most a finite number of maxima and minima.**

When these conditions are satisfied, the Fourier series converges to $f(x)$ at every point of continuity. At a point of discontinuity, the sum of the series is equal to the mean of the limits on the right and left

i.e., $\frac{1}{2}[f(x+0)+f(x-0)]$,

where $f(x+0)$ and $f(x-0)$ denote the limit on the right and the limit on the left respectively.

In fact the problem of expressing any function f(x) as a Fourier series depends upon evaluation of the integrals

$$\frac{1}{\pi}\int f(x)\cos nx dx\ ;\ \frac{1}{\pi}\int f(x)\sin nx dx\ ,$$

within the limits $(0,\ 2\pi)$, $(-\pi,\ \pi)$ or $(\alpha,\ \alpha+2\pi)$ as according as f (x) is defined for every value of x in $(0,\ 2\pi)$, $(-\pi,\ \pi)$ or $(\alpha,\ \alpha+2\pi)$.

Now let us examine whether the following functions can be expended in Fourier series in the given interval.

State giving reasons whether the following functions can be expanded in Fourier series in the interval $-\pi \le x \le \pi$.

Q.No.1: cosec x.

Sol.: Now this function has **infinite values** at $x = -\pi,\ 0,\ +\pi$, so we **cannot** develop a Fourier series of this function within this interval.

In this case Dirichlet's conditions no. 1, i.e., f (x) is finite is not satisfied.

Q.No.2: $\sin\frac{1}{x}$.

Sol.: Since this function is **not single valued** at $x = 0$ so we **cannot** develop a Fourier series of this function within this interval.

In this case Dirichlet's conditions no. 1, i.e., f (x) is single valued is not satisfied.

Q.No.3: $f(x) = \frac{(m+1)}{m}$, $\frac{\pi}{m+1} < |x| \le \frac{\pi}{m}$, $m = 1, 2, 3, \ldots\ldots\ldots\ldots \infty$.

Sol.: **Yes**, we can develop a Fourier series of this function because it **satisfies all** the Dirichlet's conditions, i.e.,

(i) f (x) is a periodic, single-valued and finite;

(ii) f (x) has a finite number of discontinuities in any one period;

(iii) f (x) has at the most a finite number of maxima and minima.

Q.No.4.: Is it possible to write the Fourier sine series for the function $f(x) = \cos x$, over the interval $(-\ell, \ell)$?

Sol.: For half range sine series f(x) must be defined in the interval $(0, \ell)$.

Hence, we cannot develop the Fourier half range sine series for $f(x) = \cos x$, over the interval $(-\ell, \ell)$.

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3rd Topic

Fourier Series

Functions having points of discontinuity

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Functions having points of discontinuity:

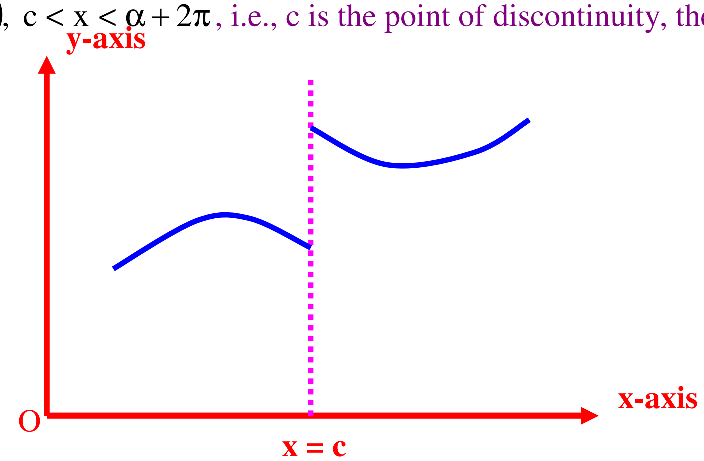
In deriving the Euler's formulae for $a_0,\ a_n,\ b_n$, it was assumed that f(x) was continuous. But in its place, if a function have a finite number of points of finite discontinuity, i.e., its graph consist of a finite number of different curves given by different equations, even then such a function is expressible as a Fourier series.

Example:

If in an interval $(\alpha,\ \alpha + 2\pi)$, f(x) is defined by

$$f(x) = \phi(x), \quad \alpha < x < c$$

$$= \psi(x), \quad c < x < \alpha + 2\pi$$, i.e., c is the point of discontinuity, then



$$\boxed{a_0 = \frac{1}{\pi}\left[\int_{\alpha}^{c} \phi(x)dx + \int_{c}^{\alpha+2\pi} \psi(x)dx\right]},$$

$$\boxed{a_n = \frac{1}{\pi}\left[\int_{\alpha}^{c} \phi(x)\cos nx dx + \int_{c}^{\alpha+2\pi} \psi(x)\cos nx dx\right]},$$

$$\boxed{b_n = \frac{1}{\pi}\left[\int_{\alpha}^{c} \phi(x)\sin nx dx + \int_{c}^{\alpha+2\pi} \psi(x)\sin nx dx\right]}.$$

Value of f(x) at a point of finite discontinuity:

At a point of finite discontinuity x = c, there is finite jump in the graph of function (see fig.). Both the limits on the left [i.e., $f(c-0)$] and the limit on the right [i.e., $f(c+0)$] exit and are different. At such a point, the value of the function f(x) is the **arithmetic mean** of these two limits, i.e., at $x = c$,

$$\boxed{f(x) = \frac{1}{2}[f(c-0) + f(c+0)]}.$$

Now let us develop some Fourier series of functions having some points of discontinuity:

Q.No.1.: Find the Fourier series expansion for f(x), if

$$f(x) = -\pi, \quad -\pi < x < 0,$$
$$= x, \quad 0 < x < \pi,$$

and hence, deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ... = \frac{\pi^2}{8}$.

Sol.: The Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx . \tag{i}$$

Here $a_0 = \frac{1}{\pi}\left[\int_{-\pi}^{0} (-\pi)dx + \int_{0}^{\pi} x dx\right] = \frac{1}{\pi}\left[-\pi\left|x\right|_{-\pi}^{0} + \left|\frac{x^2}{2}\right|_{0}^{\pi}\right] = \frac{1}{\pi}\left(-\pi^2 + \frac{\pi^2}{2}\right) = -\frac{\pi}{2};$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} (-\pi)\cos nx dx + \int_{0}^{\pi} x\cos nx dx\right] = \frac{1}{\pi}\left[-\pi\left|\frac{\sin nx}{n}\right|_{-\pi}^{0} + \left|\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right|_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[0 + \frac{1}{n^2}\cos n\pi - \frac{1}{n^2}\right] = \frac{1}{\pi n^2}(\cos n\pi - 1) = \frac{1}{\pi n^2}\left[(-1)^n - 1\right]$$

$\left[\because \cos n\pi = (-1)^n\right]$

$$\therefore a_1 = \frac{-2}{\pi.1^2},\ \ a_2 = 0,\ a_3 = \frac{-2}{\pi.3^2},\ \ a_4 = 0,\ a_5 = \frac{-2}{\pi.5^2},\ \ ....\ \text{etc.}$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\sin nx\,dx + \int_{0}^{\pi} x\sin nx\,dx\right] = \frac{1}{\pi}\left[\left|\frac{\pi\cos nx}{n}\right|_{-\pi}^{0} + \left|-x\frac{\cos nx}{n} + \frac{\sin nx}{n^2}\right|_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n}(1-\cos n\pi) - \frac{\pi}{n}\cos n\pi\right] = \frac{1}{n}(1 - 2\cos n\pi) = \frac{1}{n}\left[1 - 2(-1)^n\right].$$

$$\therefore b_1 = 3,\ \ b_2 = -\frac{1}{2},\ b_3 = 1,\ \ b_4 = -\frac{1}{4},\ \ ....\text{etc.}$$

Hence substituting the values of a_i's and b_i's in (i), we get the required Fourier series

$$f(x) = -\frac{\pi}{4} - \frac{2}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + ...\right) + 3\sin x - \frac{\sin 2x}{2} + \frac{3\sin 3x}{3} - \frac{\sin 4x}{4} + ...\ \text{(ii)}$$

2^{nd} Part.:

Putting x = 0 in (ii), we get $f(0) = -\frac{\pi}{4} - \frac{2}{\pi}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + ...\infty\right)$ (iii)

Now f(x) is discontinuous at x = 0. Also since $f(0-0) = -\pi$ and $f(0+0) = 0$.

$$\therefore f(0) = \frac{1}{2}\left[f(0-0) + f(0+0)\right] = -\frac{\pi}{2}$$

Hence, (iii) take the form

$$-\frac{\pi}{2} = -\frac{\pi}{4} - \frac{2}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ...\right]$$

$$\Rightarrow \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ... = \frac{\pi^2}{8}.$$

Q.No.2.: Find the Fourier series to represent the function f(x) given by

$$f(x) = x, \qquad \text{for } 0 \le x \le \pi,$$
$$= 2\pi - x \qquad \text{for } \pi \le x \le 2\pi.$$

and hence, deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ...\infty = \frac{\pi^2}{8}$.

Sol.: The Fourier series is given by

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos nx+\sum_{n=1}^{\infty} b_n \sin nx \,. \qquad \text{(i)}$$

Here $a_0=\frac{1}{\pi}\left[\int_0^{\pi} x.dx+\int_{\pi}^{2\pi}(2\pi-x)dx\right]=\frac{1}{\pi}\left[\left|\frac{x^2}{2}\right|_0^{\pi}+\left|2\pi x-\frac{x^2}{2}\right|_{\pi}^{2\pi}\right]$

$$=\frac{1}{\pi}\left[\frac{\pi^2}{2}+(2\pi)^2-\frac{(2\pi)^2}{2}-2\pi^2+\frac{\pi^2}{2}\right]=\frac{1}{\pi}\left[5\pi^2-4\pi^2\right]=\frac{\pi^2}{\pi}=\pi\,.$$

$$a_n=\frac{1}{\pi}\left[\int_0^{\pi} x\cos nx dx+\int_{\pi}^{2\pi}(2\pi-x)\cos nx dx\right]$$

$$=\frac{1}{\pi}\left[\left|\frac{x\sin nx}{n}+\frac{\cos nx}{n^2}\right|_0^{\pi}+\left|\frac{2\pi\sin nx}{n}-\frac{x\sin nx}{n}-\frac{\cos nx}{n^2}\right|_{\pi}^{2\pi}\right]$$

$$=\frac{1}{\pi}\left[\frac{\cos n\pi}{n^2}-\frac{1}{n^2}-\frac{\cos 2n\pi}{n^2}+\frac{\cos n\pi}{n^2}\right]=\frac{1}{\pi}\left[\frac{2\cos n\pi}{n^2}-\frac{\cos 2n\pi}{n^2}-\frac{1}{n^2}\right]$$

$$=\frac{1}{\pi n^2}\left[2\cos n\pi-\cos 2n\pi-1\right]=\frac{1}{\pi n^2}\left[2(-1)^n-(-1)^{2n}-1\right]=\frac{2}{\pi n^2}\left[(-1)^n-1\right].$$

$$\therefore\ a_1=\frac{1}{\pi(1)^2}[-2-1-1]=\frac{-4}{\pi(1)^2},\quad a_2=0,\quad a_3=\frac{1}{\pi(3)^2}(-4),\quad a_4=0,\ \text{..........etc.}$$

$$b_n=\frac{1}{\pi}\left[\int_0^{\pi} x\sin nx dx+\int_{\pi}^{2\pi}(2\pi-x)\sin nx dx\right]$$

$$=\frac{1}{\pi}\left[\left|\frac{-x\cos nx}{n}+\frac{\sin nx}{n^2}\right|_0^{\pi}+\left|\frac{-2\pi\cos nx}{n}+\frac{x\cos nx}{n}-\frac{\sin nx}{n^2}\right|_{\pi}^{2\pi}\right]$$

$$=\frac{1}{\pi}\left[\frac{-\pi\cos n\pi}{n}-\frac{2\pi\cos 2n\pi}{n}+\frac{2\pi\cos 2n\pi}{n}+\frac{2\pi\cos n\pi}{n}-\frac{\pi\cos n\pi}{n}\right]=0$$

Substituting the values of a_i's and b_i's in (i), we get the required Fourier series

$$f(x)=\frac{\pi}{2}-\frac{4}{\pi}\left(\frac{\cos x}{1^2}+\frac{\cos 3x}{3^2}+\frac{\cos 5x}{5^2}+\text{...........}\right). \qquad \text{(ii)}$$

2nd Part:

Putting $x=\pi$, we get

$$f(\pi)=\frac{\pi}{2}+\frac{4}{\pi}\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\ldots\ldots\ldots\right). \tag{iii}$$

Here f(x) is continuous at $x=\pi$

$\therefore f(\pi-0)=\pi$ and $f(\pi+0)=2\pi-\pi=\pi$

$\therefore f(\pi)=\frac{1}{2}[f(\pi-0)+f(\pi+0)]=\frac{2\pi}{2}=\pi$.

Hence, (iii) takes the form

$$\pi=\frac{\pi}{2}+\frac{4}{\pi}\left[\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\ldots\ldots\ldots\infty\right]\Rightarrow\frac{\pi}{2}\times\frac{\pi}{4}=\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\ldots\ldots\ldots\infty$$

$$\frac{\pi^2}{8}=\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\ldots\ldots\ldots\infty.$$

Q.No.3.: An alternating current after passing through a rectifier has the form

$i=I_0\sin x$ for $0\le x\le\pi$,

$=0$ for $\pi\le x\le 2\pi$,

where I_0 is the maximum and the period is 2π.

Express i as a Fourier series.

Sol.: The Fourier series is given by

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx. \tag{i}$$

Here $a_0=\frac{1}{\pi}\left[\int_0^{\pi}I_0\sin x dx+\int_{\pi}^{2\pi}0.dx\right]=\frac{I_0}{\pi}[-\cos x]_0^{\pi}=\frac{I_0}{\pi}\times 2=\frac{2I_0}{\pi}$.

$$a_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\cos nx dx=\frac{1}{\pi}\left[\int_0^{\pi}I_0\sin x\cos nx dx+0\right]=\frac{I_0}{2\pi}\int_0^{\pi}2\sin x\cos nx dx$$

$$=\frac{I_0}{\pi}\left[\int_0^{\pi}\sin x\cos nx dx\right]=\frac{I_0}{2\pi}\left[\int_0^{\pi}2\sin x\cos nx dx\right]$$

$$=\frac{I_0}{2\pi}\int_0^{\pi}[\sin(1+n)x+\sin(1-n)x]dx$$

$$= \frac{I_0}{2\pi}\left[\frac{-\cos(1+n)x}{(1+n)}\right]_0^{\pi} + \frac{I_0}{2\pi}\left[\frac{-\cos(1-n)x}{(1-n)}\right]_0^{\pi}, \qquad (n \neq 1)$$

$$= \begin{cases} \frac{I_0}{2\pi}\left[-\frac{1}{1+n} - \frac{1}{1-n} + \frac{1}{1+n} + \frac{1}{1-n}\right], & \text{when n is odd} \\ \frac{I_0}{2\pi}\left[\frac{1}{1+n} + \frac{1}{1-n} + \frac{1}{1+n} + \frac{1}{1-n}\right], & \text{when n is even} \end{cases}$$

$$= \begin{cases} 0, & \text{when n is odd} \\ -\frac{2I_0}{\pi(n^2-1)}, & \text{when n is even} \end{cases}$$

When $n = 1$, we have

$$a_1 = \frac{1}{\pi}\int_0^{\pi} I_0 \sin x \cos x dx = \frac{I_0}{2\pi}\int_0^{\pi} \sin 2x dx = \frac{I_0}{2\pi}\left[-\frac{\cos 2x}{2}\right]_0^{\pi} = 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} I_0 \sin x \sin nx dx = \frac{1}{2\pi}\int_0^{2\pi} I_0[\cos(1-n)x - \cos(1+n)x]dx$$

$$= \frac{I_0}{2\pi}\left[\frac{\sin(1-n)x}{(1-n)}\right]_0^{\pi} - \left[\frac{\sin(1+n)x}{(1+n)}\right]_0^{\pi} = 0 \text{ for } n > 1.$$

When n = 1, we get

$$b_1 = \frac{1}{\pi}\int_0^{\pi} I_0 \sin x . \sin x . dx = \frac{1}{\pi}\int_0^{\pi} I_0 \sin^2 x . dx = \frac{1}{\pi}\int_0^{\pi} I_0\left(\frac{1-\cos 2x}{2}\right)dx$$

$$= \frac{I_0}{2\pi}\left[x - \frac{\sin nx}{2}\right]_0^{\pi} = \frac{I_0}{2}(\pi - 0 - 0 + 0) = \frac{\pi I_0}{2\pi} = \frac{I_0}{2}.$$

Substituting the values of a_i's and b_i's in (i), we get the required Fourier series

$$f(x) = \frac{I_0}{\pi} + \frac{I_0}{2}\sin x - \frac{2I_0}{\pi}\sum_{m=1}^{\infty}\frac{\cos 2mx}{4m^2-1}, \text{ by supposing } n = 2m.$$

Q.No.4.: If $f(x) = 0$, for $-\pi < x < 0$,

$= \sin x$, for $0 < x < \pi$,

prove that $f(x) = \frac{1}{\pi} + \frac{\sin x}{2} - \frac{2}{\pi}\sum_{m=1}^{\infty}\frac{\cos 2mx}{4m^2-1}$.

Hence, show that **(i)** $\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{3.7}+..........=\frac{1}{2}$,

(ii) $\frac{1}{1.3}-\frac{1}{3.5}+\frac{1}{3.7}-..........\infty=\frac{1}{4}(\pi-2)$.

Sol.: The Fourier series is given by

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx\,. \qquad \text{(i)}$$

Here $a_0=\frac{1}{\pi}\left[\int_{-\pi}^{0}0.dx+\int_{0}^{\pi}\sin xdx\right]=\frac{1}{\pi}[-\cos x]_0^{\pi}=\frac{1}{\pi}[-(-1)-(-1)]=\frac{2}{\pi}$,

$$a_n=\frac{1}{\pi}\left[\int_0^{\pi}\sin x\cos nxdx\right]=\frac{1}{2\pi}\left[\int_0^{\pi}2\sin x\cos nxdx\right]$$

$$=\frac{1}{2\pi}\int_0^{\pi}[\sin(1+n)x+\sin(1-n)x]dx$$

$$=\frac{1}{2\pi}\left[\frac{-\cos(1+n)x}{(1+n)}\right]_0^{\pi}+\frac{1}{2\pi}\left[\frac{-\cos(1-n)x}{(1-n)}\right]_0^{\pi}, \qquad (n\neq 1)$$

$$=\begin{cases}\frac{1}{2\pi}\left[-\frac{1}{1+n}-\frac{1}{1-n}+\frac{1}{1+n}+\frac{1}{1-n}\right], & \text{when n is odd}\\ \frac{1}{2\pi}\left[\frac{1}{1+n}+\frac{1}{1-n}+\frac{1}{1+n}+\frac{1}{1-n}\right], & \text{when n is even}\end{cases}$$

$$=\begin{cases}0\,, & \text{when n is odd}\\ -\frac{2}{\pi(n^2-1)}\,, & \text{when n is even}\end{cases}$$

When $n=1$, we have

$$a_1=\frac{1}{\pi}\int_0^{\pi}\sin x\cos xdx=\frac{1}{2\pi}\int_0^{\pi}\sin 2xdx=\frac{1}{2\pi}\left[-\frac{\cos 2x}{2}\right]_0^{\pi}=0\,.$$

$$b_n=\frac{1}{\pi}\int_0^{\pi}\sin x\sin nxdx=\frac{1}{2\pi}\int_0^{\pi}2\sin x\sin nxdx$$

$$=\frac{1}{2\pi}\int_0^{\pi}2\cos(1-n)xdx-\frac{1}{2\pi}\int_0^{\pi}2\cos(1+n)xdx$$

$$= \frac{1}{2\pi}\left[\frac{\sin(1-n)x}{(1-n)}\right]_0^\pi - \frac{1}{2\pi}\left[\frac{\sin(1+n)x}{(1+n)}\right]_0^\pi = 0 - 0 = 0.$$

When $n = 1$, we have

$$b_1 = \frac{1}{\pi}\int_0^\pi \sin^2 x dx = \frac{1}{\pi}\int_0^\pi \left(\frac{1-\cos 2x}{2}\right)dx = \frac{1}{2\pi}\int_0^\pi (1-\cos 2x)dx$$

$$= \frac{1}{2\pi}\left[x - \frac{\sin 2x}{2}\right]_0^\pi = \frac{1}{2\pi}[\pi - 0 - 0 + 0] = \frac{\pi}{2\pi} = \frac{1}{2}.$$

Substituting the values of a_i's and b_i's in (i), we get the required Fourier series

$$f(x) = \frac{1}{\pi} - \frac{2}{\pi}\left[\frac{\cos 2x}{2^2-1} + \frac{\cos 4x}{4^2-1} + \frac{\cos 6x}{6^2-1} + \dots\dots\right] + \frac{\sin x}{2}$$

$$\Rightarrow f(x) = \frac{1}{\pi} + \frac{\sin x}{2} - \frac{2}{\pi}\sum_{m=1}^{\infty}\frac{\cos 2mx}{4m^2-1}. \qquad \text{(ii)}$$

This is the required Fourier series expansion.

IInd part:

If x = 0, then we get

$$0 = \frac{1}{\pi} - \frac{2}{\pi}\sum_{m=1}^{\infty}\frac{1}{4m^2-1}$$

$$\Rightarrow \frac{1}{2} = \sum_{m=1}^{\infty}\frac{1}{4m^2-1} = \sum_{m=1}^{\infty}\frac{1}{(2m-1)(2m+1)} = \frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \dots\dots\dots\dots\dots$$

$$\Rightarrow \frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{3.7} + \dots\dots\dots\dots = \frac{1}{2}.$$

IIIrd part:

Now putting $x = \frac{\pi}{2}$ in (ii), we get

$$1 = \frac{1}{\pi} + \frac{1}{2} - \frac{2}{\pi}\sum_{m=1}^{\infty}\frac{\cos m\pi}{4m^2-1} \qquad \left[\because \sin\frac{\pi}{2} = 1\right]$$

$$\Rightarrow \frac{\pi-2}{4} = -\sum_{m=1}^{\infty}\frac{(-1)^m}{(2m-1)(2m+1)} = -\left(-\frac{1}{1.3} + \frac{1}{3.5} - \frac{1}{5.7} + \dots\dots\dots\dots\right)$$

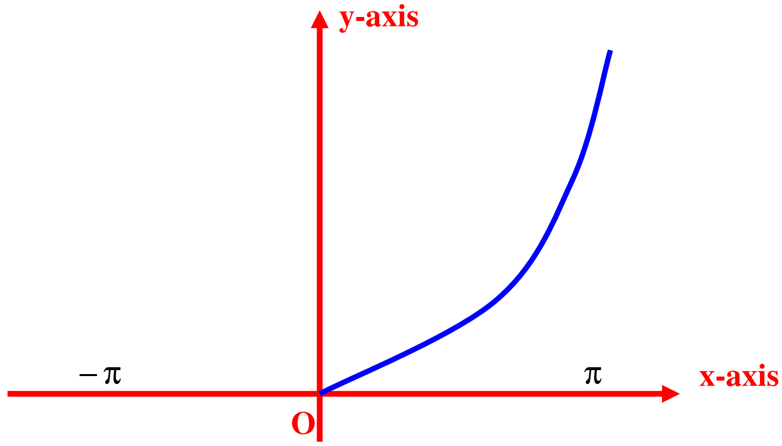
$$\Rightarrow \frac{\pi-2}{4} = \frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \dots\dots\dots\dots\dots\dots$$

Q.No.5.: Draw the graph of the function $f(x) = 0, \quad -\pi < x < 0,$

$$= x^2, \quad 0 < x < \pi.$$

If $f(2\pi + x) = f(x)$, obtain Fourier series of f(x).

Sol.:



The Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx . \tag{i}$$

Here $a_0 = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0.dx + \int_{0}^{\pi} x^2 dx\right] = \frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{1}{\pi}\left[\frac{\pi^3}{3}\right] = \frac{\pi^2}{3}$.

$$a_n = \frac{1}{\pi}\left[0 + \int_0^{\pi} x^2 \cos nx dx\right] = \frac{1}{\pi}\left[\left| x^2 \frac{\sin nx}{n}\right|_0^{\pi} - \int_0^{\pi} 2x.\frac{\sin nx}{n}.dx\right]$$

$$= \frac{1}{\pi}\left[0 - 2\left[\left|\frac{-x\cos nx}{n^2}\right|_0^{\pi} + \int_0^{\pi}\frac{\cos nx}{n^2}dx\right]\right] = \frac{-2}{\pi}\left[\left(-x\frac{\cos nx}{n^2}\right)_0^{\pi} + \left(\frac{\sin nx}{n^3}\right)_0^{\pi}\right]$$

$$= \frac{-2}{\pi}\left[\frac{-\pi(-1)^n}{n^2} + 0\right] = \frac{2}{\pi}\left[\frac{\pi(-1)^n}{n^2}\right] = \frac{2(-1)^n}{n^2}.$$

$\therefore a_1 = \frac{-2}{1^2}, \quad a_2 = \frac{2}{2^2}, \quad a_3 = \frac{-2}{3^2},$ ....etc.

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0.\sin nx.dx + \int_0^{\pi} x^2 \sin nx.dx\right] = \frac{1}{\pi}\left[-x^2\frac{\cos nx}{n}\Big|_0^{\pi} + \int_0^{\pi} 2x\frac{\cos nx}{n}dx\right]$$

$$=\frac{1}{\pi}\left[-\pi^2\frac{(-1)^n}{n}+\frac{2}{n}\cdot\left[\left|x.\frac{\sin nx}{n}\right|_0^{\pi}-\int_0^{\pi}\frac{\sin nx}{n}dx\right]\right]$$

$$=\frac{1}{\pi}\left[-\pi^2\frac{(-1)^n}{n}+\frac{2}{n}\left(0+\frac{\cos nx}{n^2}\right)_0^{\pi}\right]=(-1)^n\left[\frac{-\pi}{n}+\frac{2}{n^3\pi}\right]-\frac{2}{n^3\pi}.$$

$$\therefore b_1=-\frac{1}{\pi}\left(\frac{4}{1^2}-\frac{\pi^2}{1}\right),\ b_2=\frac{1}{\pi}\left(-\frac{\pi^2}{2}\right),\ b_3=-\frac{1}{\pi}\left(\frac{4}{3^2}-\frac{\pi^2}{3}\right),\text{........ etc.}$$

Substituting the values of a_i's and b_i's in (i), we get the required Fourier series

$$f(x)=\frac{\pi^2}{6}-2\left(\cos x-\frac{\cos 2x}{2^2}+\frac{\cos 3x}{3^2}-\text{............}\right)$$

$$-\frac{1}{\pi}\left\{\left(\frac{2}{1^3}-\frac{\pi^2}{1}\right)\sin x-\left(-\frac{\pi^2}{2}\right)\sin 2x+\left(\frac{4}{3^3}-\frac{\pi^2}{3}\right)\sin 3x-\text{.....}\right\}. \textbf{ Ans.}$$

Q.No.6.: Find the Fourier series of the following function,

$$f(x)=x^2,\qquad 0\le x\le\pi$$

$$=-x^2,\ -\pi\le x\le 0.$$

Sol.: The Fourier series is given by

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx\,. \tag{i}$$

Here $a_0=\frac{1}{\pi}\left[-\int_{-\pi}^{0}x^2dx+\int_0^{\pi}x^2dx\right]=0$;

$$a_n=\frac{1}{\pi}\left[-\int_{-\pi}^{0}x^2\cos nxdx+\int_0^{\pi}x^2\cos nxdx\right]$$

Now $\int x^2\cos nxdx=x^2\frac{\sin nx}{n}-\int 2x.\frac{\sin nx}{n}.dx=x^2\frac{\sin nx}{n}-\frac{2}{n}\int x\sin nxdx$

$$=x^2\frac{\sin nx}{n}-\frac{2}{n}\left[\frac{-x\cos nx}{n}-\int\frac{\cos nx}{n}dx\right]$$

$$=x^2\frac{\sin nx}{n}+2x\frac{\cos nx}{n^2}-\frac{2}{n^2}\int\cos nxdx$$

$$= x^2\frac{\sin nx}{n} + 2x\frac{\cos nx}{n^2} - \frac{2}{n^3}\sin nxdx\,.$$

$$-\int_{-\pi}^{0} x^2\cos nxdx = \left[-x^2\frac{\sin nx}{n} - 2x\frac{\sin nx}{n^2} + \frac{2}{n^3}\sin nx\right]_{-\pi}^{0}$$

$$= 0 - \left[-\pi^2\frac{\sin n\pi}{n} + \frac{2\pi\cos n\pi}{n^2} - \frac{2}{n^3}\sin n\pi\right]$$

$$= \frac{\pi^2\sin n\pi}{n} - \frac{2\pi\cos n\pi}{n^2} + \frac{2}{n^3}\sin n\pi$$

$$\int_{0}^{\pi} x^2\cos nxdx = \left[x^2\frac{\sin nx}{n} + 2x\frac{\cos nx}{n^2} - \frac{2}{n^3}\sin nx\right]_{0}^{\pi}$$

$$= \left(\pi^2\frac{\sin nx}{n} + 2\pi\frac{\cos n\pi}{n^2} - \frac{2}{n^3}\sin n\pi\right) - 0$$

Thus $a_n = \frac{1}{\pi}\left(\frac{2\pi^2\sin n\pi}{n}\right) = 2\pi\frac{\sin n\pi}{n} = 0$.

$\therefore\ a_1 = 2\pi\sin\pi = 0,\ a_2 = \frac{2\pi\sin 2\pi}{2} = 0,\ a_3 = \frac{2\pi\sin 3\pi}{3} = 0,$ ....etc.

Now $b_n = \frac{1}{\pi}\left[-\int_{-\pi}^{0} x^2\sin nxdx + \int_{0}^{\pi} x^2\sin nxdx\right].$

Now $\int x^2\sin nxdx = -x^2\frac{\cos nx}{n} + \frac{2}{n}\int x\cos nxdx$

$$= -x^2\frac{\cos nx}{n} + \frac{2}{n}\left[\frac{x\sin nx}{n} - \int\left(\frac{1-\sin nx}{n}\right)dx\right]$$

$$= -x^2\frac{\cos nx}{n} + \frac{2x\sin nx}{n^2} - \frac{2}{n^2}\int \sin nxdx$$

$$= -x^2\frac{\cos nx}{n} + \frac{2x\sin nx}{n^2} + \frac{2\cos nx}{n^3}$$

Now $-\int_{-\pi}^{0} x^2\sin nxdx = \left[x^2\frac{\cos nx}{n} - \frac{2}{n^2}x\sin nx - \frac{2}{n^3}\cos nx\right]_{-\pi}^{0}$

$$= -\frac{2}{n^3} - \left[\frac{\pi^2\cos n\pi}{n} - \frac{2}{n^3}\cos n\pi\right] = \frac{-2}{n^3} - \frac{\pi^2\cos n\pi}{n} + \frac{2}{n^3}\cos n\pi$$

$$\int_0^{\pi} x^2 \sin nx dx = \left[-\frac{x^2 \cos nx}{n} + \frac{2}{n^2} x \sin nx + \frac{2}{n^3} \cos nx\right]_0^{\pi}$$

$$= \frac{-\pi^2 \cos n\pi}{n} + \frac{2\pi \sin n\pi}{n^2} + \frac{2}{n^3} \cos n\pi - \frac{2}{n^3}$$

$$= -\frac{2}{n^3} - \frac{\pi^2 \cos n\pi}{n} + 0 + \frac{2}{n^3} \cos n\pi .$$

Thus $b_n = \frac{1}{\pi}\left[\frac{-4}{n^3} - \frac{2\pi^2 \cos n\pi}{n} + \frac{4}{n^3} \cos n\pi\right]$.

$\therefore\ b_1 = \frac{1}{\pi}\left[-4 - 2\pi^2 \cos \pi + 4 \cos \pi\right] = 2\left(\pi - \frac{4}{\pi}\right)$,

$b_2 = \frac{1}{\pi}\left[\frac{-4}{8} - \frac{2\pi^2 \cos 2\pi}{2} + \frac{4}{8} \cos 2\pi\right] = -\pi$,

$b_3 = \frac{1}{\pi}\left[\frac{-4}{27} - \frac{2\pi^2 \cos 3\pi}{3} + \frac{4}{27} \cos 3\pi\right] = \frac{2\pi}{3} - \frac{8\pi}{27} = \frac{2}{3}\left(\pi - \frac{4\pi}{9}\right)$,

$b_4 = \frac{1}{\pi}\left[\frac{-4}{64} - \frac{2\pi^2 \cos 4\pi}{4} + \frac{4}{64} \cos 4\pi\right] = -\frac{\pi}{2}$,............etc.

Substituting the values of a_i's and b_i's in (i), we get the required Fourier series

$f(x) = 2\left(\pi - \frac{4}{\pi}\right)\sin x - \pi \sin 2x + \frac{2}{3}\left(\pi - \frac{4\pi}{9}\right)\sin 3x - \frac{\pi}{2} \sin 4x + \ldots\ldots\ldots\ldots.$ Ans.

Q.No.7.: If $f(x) = x$ in $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ and $f(x) = 0$ in $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$, find the Fourier series of f(x).

Deduce that $\frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$.

Sol.: The Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx . \qquad \text{(i)}$$

Here $a_0 = \frac{1}{\pi}\int_{-\pi/2}^{3\pi/2} f(x) dx = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} x dx + \frac{1}{\pi}\int_{\pi/2}^{3\pi/2} 0.dx = \frac{1}{\pi}\left[\frac{x^2}{2}\right]_{-\pi/2}^{\pi/2} = \frac{1}{2\pi}\left[\frac{\pi^2}{4} - \frac{\pi^2}{4}\right] = 0$.

$$a_n = \frac{1}{\pi}\int_{-\pi/2}^{3\pi/2} f(x)\cos nx dx = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} x\cos nx dx = \frac{1}{\pi}\left[x\frac{\sin nx}{n} + \frac{1}{n}.\frac{\cos nx}{n}\right]_{-\pi/2}^{\pi/2} = 0.$$

$$b_n = \frac{1}{\pi}\int_{-\pi/2}^{3\pi/2} f(x)\sin nx dx = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} x\sin nx dx = \frac{1}{\pi}\left[x.\frac{-\cos nx}{n} + \frac{1}{n\pi}.\frac{\sin nx}{n}\right]_{-\pi/2}^{\pi/2}$$

$$b_n = \frac{-1}{n}\left[\cos n\frac{n}{2}\right] + \frac{2}{n^2\pi}\sin\frac{n\pi}{2}$$

For $n = 1$, $b_1 = \frac{2}{\pi}$, $b_2 = 0 + \frac{1}{2}$, $b_3 = -\frac{2}{9\pi}$, $b_4 = -\frac{1}{4}$, $b_5 = \frac{2}{25\pi}$ etc.

Substituting the values of a_i's and b_i's in (i), we get the required Fourier series

$$f(x) = 0 + 0 + \sum_{n=1}^{\infty}\left(\frac{-\cos n.\frac{\pi}{2}}{n} + \frac{2}{n^2\pi}\sin n\frac{\pi}{2}\right)\sin nx .$$

$$f(x) = \frac{2}{\pi}\sin x + \frac{1}{2}\sin 2x - \frac{2}{9\pi}\sin 3x - \frac{1}{4}\sin 4x + \frac{2}{25\pi}\sin 5x - ......$$

2nd Part:

At $x = \frac{\pi}{2}$, which is point of discontinuity,

$$f\left(\frac{\pi}{2}\right) = \frac{1}{2}\left[f\left(\frac{\pi}{2} - 0\right) + f\left(\frac{\pi}{2} + 0\right)\right] = \frac{1}{2}\left[\frac{\pi}{2} + 0\right] = \frac{\pi}{4}.$$

Putting $x = \frac{\pi}{2}$ in the Fourier series expansion, we get

$$\frac{\pi}{4} = \frac{2}{\pi}\sin\frac{\pi}{2} + \frac{1}{2}\sin\pi - \frac{2}{9\pi}\sin\frac{3\pi}{2} - \frac{1}{4}\sin\frac{4\pi}{2} + \frac{2}{25\pi}\sin\frac{5\pi}{2} - ......$$

$$\Rightarrow \frac{\pi}{4} = \frac{2}{\pi} + 0 + \frac{2}{9\pi} + 0 + \frac{2}{25\pi} + 0......$$

$$\Rightarrow \frac{\pi}{4} = \frac{2}{\pi}\left(1 + \frac{1}{9} + \frac{1}{25}......\right) \Rightarrow \frac{\pi^2}{8} = \left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2}......\right)$$

$$\Rightarrow \frac{\pi^2}{8} = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}.$$

Q.No.8.: Find the Fourier series to represent $f(x) = \begin{cases} -k, & -\pi < x < 0 \\ k, & 0 < x < \pi \end{cases}$.

Also deduce that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots..$

Sol.: $f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_0 \cos nx + b_n \sin nx)$ (i)

be the required series.

Here $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) = \frac{1}{\pi}\int_{-\pi}^{0} -kdx + \frac{1}{\pi}\int_{0}^{\pi} kdx$

$= \frac{1}{\pi}\left|-kx\right|_{-\pi}^{0} + \frac{1}{\pi}\left|kx\right|_{0}^{\pi} = \frac{1}{\pi}[-k\pi] + \frac{1}{\pi}[k\pi] = 0.$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx = \frac{1}{\pi}\int_{-\pi}^{0} -k\cos nxdx + \frac{1}{\pi}\int_{0}^{\pi} k\cos nxdx$

$= \frac{-k}{\pi}\left|\frac{\sin nx}{n}\right|_{-\pi}^{0} + \frac{k}{\pi}\left|\frac{\sin nx}{n}\right|_{0}^{\pi} = \frac{-k}{n\pi}(0) + \frac{k}{n\pi}(0) = 0.$ $[\sin n\pi = 0, \ n \in Z]$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nxdx = \frac{1}{\pi}\int_{-\pi}^{0} -k\sin nxdx + \frac{1}{\pi}\int_{0}^{\pi} k\sin nxdx$

$= \frac{-k}{\pi}\left|\frac{-\cos nx}{n}\right|_{-\pi}^{0} + \frac{k}{\pi}\left|\frac{-\cos nx}{n}\right|_{0}^{\pi} = \frac{k}{n\pi}\left[1-(-1)^n\right] + \frac{k}{n\pi}(-1)\left[(-1)^n - 1\right]$

$= \frac{k}{n\pi}\left[1-(-1)^n - (-1)^n + 1\right] = \frac{2k}{n\pi}\left[1-(-1)^n\right]$

$= \begin{cases} 0, & n \text{ is even} \\ \frac{4k}{n\pi}, & n \text{ is odd} \end{cases}$

Hence, from (i), we get

$f(x) = \frac{1}{2}.0 + \sum_{n=0}^{\infty} (0 + b_n \sin nx) = \sum_{\substack{n=1 \\ n \text{ is odd}}}^{\infty} \frac{4k}{n\pi}\sin x = \frac{4k}{\pi}\left[\sin x + \frac{\sin 3x}{3} + \ldots..\right].$

$f(x) = \frac{4k}{\pi}\left[\sin x + \frac{\sin 3x}{3} + \ldots..\right]$

Deduction: Put $x = \frac{\pi}{2}$, in above, we get

$$f\left(\frac{\pi}{2}\right) = k = \frac{4k}{\pi}\left[1 - \frac{1}{3} + \frac{1}{5} - \ldots\ldots\right] \qquad \left[\sin\frac{3\pi}{2} = \sin\left(\pi + \frac{\pi}{2}\right) = -\sin\frac{\pi}{2} = -1\right]$$

$$\Rightarrow \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \ldots\ldots\ldots.$$

Q.No.9.: Develop f(x) in a Fourier series in the interval $(-\pi, \pi)$ if

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ 1, & 0 < x < \pi \end{cases}.$$

Sol.: Let $f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_0 \cos nx + b_n \sin nx\right)$ (i)

be the required series.

Here $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) = \frac{1}{\pi}\int_{-\pi}^{0} 0.dx + \frac{1}{\pi}\int_{0}^{\pi} 1dx = \frac{1}{\pi}.\pi = 1$.

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx = \frac{1}{\pi}\int_{-\pi}^{0} 0dx + \frac{1}{\pi}\int_{0}^{\pi} 1.\cos nx dx$$

$$= \frac{1}{\pi}\left|\frac{\sin nx}{n}\right|_0^{\pi} = 0. \qquad [\sin n\pi = 0, \quad n \in Z]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = \frac{1}{\pi}\int_{-\pi}^{0} 0.dx + \frac{1}{\pi}\int_{0}^{\pi} \sin nx dx$$

$$= \frac{1}{\pi}\left|\frac{-\cos nx}{n}\right|_0^{\pi} = \frac{-1}{n\pi}\left[(-1)^n - 1\right] = \frac{1-(-1)^n}{n\pi}$$

$$= \begin{cases} 0, & n \text{ is even} \\ \frac{2}{n\pi}, & n \text{ is odd} \end{cases}.$$

Hence, from (i), we get

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} b_n \sin nx = \frac{1}{2} + \sum_{n=odd}^{\infty} \frac{2}{n\pi}\sin nx = \frac{1}{2} + \frac{2}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \ldots\ldots\right).$$

Q.No.10.: Find the Fourier expansion of the function defined in one period by the

relation $f(x) = \begin{cases} 1, & 0 < x < \pi \\ 2, & \pi < x < 2\pi \end{cases}$

and deduce that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots$.

Sol.: Let $f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_0 \cos nx + b_n \sin nx)$ (i)

be the required Fourier series.

Here $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx = \frac{1}{\pi}\int_0^{\pi} 1.dx + \frac{1}{\pi}\int_{\pi}^{2\pi} 2dx = \frac{1}{\pi}.\pi + \frac{2}{\pi}(2\pi - \pi) = 3$.

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx dx = \frac{1}{\pi}\int_0^{\pi} \cos nx dx + \frac{1}{\pi}\int_{\pi}^{2\pi} 2\cos nx dx$$

$$= \frac{1}{\pi}\left|\frac{\sin nx}{n}\right|_0^{\pi} + \frac{2}{\pi}\left|\frac{\sin nx}{n}\right|_0^{2\pi} = 0.$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx dx = \frac{1}{\pi}\int_0^{2\pi} \sin nx dx + \frac{2}{\pi}\int_{\pi}^{2\pi} \sin nx dx$$

$$= \frac{1}{\pi}\left|\frac{\cos nx}{n}\right|_0^{\pi} + \frac{2}{\pi}\left|\frac{-\cos nx}{n}\right|_0^{2\pi} = \frac{-1}{n\pi}\left[(-1)^n - 0\right] - \frac{-2}{n\pi}\left[1 - (-1)^n\right].$$

$$= \frac{1}{n\pi}\left[(-1)^n - 1\right] = \begin{cases} 0, & \text{n is even} \\ \frac{-2}{n\pi}, & \text{n is odd} \end{cases}$$

Hence, from (i), we get

$$f(x) = \frac{3}{2} + \sum_{n=1}^{\infty} (0 + b_n \sin nx) = \frac{3}{2} - \frac{2}{\pi}\sum_{n=odd}^{\infty} \frac{1}{n}\sin nx = \frac{3}{2} - \frac{2}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \ldots\ldots\right).$$

Deduction: Put $x = \frac{\pi}{2}$

$$f(\pi/2) = \frac{3}{2} - \frac{2}{\pi}\left(1 + \left(-\frac{1}{3}\right) + \ldots\ldots\right).$$

$$\Rightarrow \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots\ldots\ldots$$

Home Assignments

Q.No.1.: Find the Fourier expansion of the function defined in one period by the relation

$$f(x) = a\sin t, \text{ if } 0 \le t \le \pi, \quad \textbf{(Half wave rectifier)}$$
$$= 0 \quad \text{ if } \pi \le t \le 2\pi.$$

Deduce $\frac{1}{2} = \frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + ...$

Ans.: $f(x) = \frac{a}{\pi} + \frac{1}{2} a\sin x - \frac{2a}{n}\sum_{n=1}^{\infty} \frac{\cos 2nx}{4n^2 - 1}$**.**

Put $t = \pi$, then $0 = a\sin\pi = \frac{a}{\pi} + \frac{a}{2}\sin\pi - \frac{2a}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2n\pi}{4n^2 - 1}$.

Q.No.2.: Find the Fourier expansion of the **Modified saw-toothed wave form**

$$f(x) = 0 \text{ for } -\pi < x \le 0,$$
$$= x \text{ for } \quad 0 < x \le \pi.$$

Hence, deduce $\frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$.

Ans.: $f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + ...\right) + \left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + ...\right)$**.** Put $x = 0$.

Q.No.3.: Find the Fourier expansion of the function defined in one period by the relation

$$f(x) = 2x \text{ when } \quad 0 \le x \le \pi,$$
$$= x \quad \text{when } -\pi < x \le 0.$$

Ans.: $f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + ...\right) + 3\left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - ....\right)$**.**

Q.No.4.: Find the Fourier expansion of the function defined in one period by the relation

$$f(x) = -x \text{ if } -\pi < x \le 0,$$
$$= 0 \quad \text{if } \quad 0 < x \le \pi.$$

Ans.: $f(x) = \frac{\pi}{4} - \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2n+1)x}{(2n+1)^2} - \sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sin nx}{n}$**.**

Q.No.5.: Find the Fourier expansion of the function defined in one period by the relations

$$f(x) = 1 \quad \text{if } -\pi < x \le 0,$$
$$\quad = -2 \quad \text{if } \ 0 < x \le \pi.$$

Ans.: $f(x) = -\frac{1}{2} - \frac{6}{\pi}\sum_{n=0}^{\infty} \frac{\sin(2n+1)x}{(2n+1)}$.

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4^{th} Topic

Fourier Series

'Change of Interval'

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Change of Interval

In many engineering problems, the period of the function which required to be expanded is not 2π, but some other interval, say : 2c. In order to develop a Fourier series of the function of period 2c, this interval must be converted to the length 2π. This involves only a proportional change in the scale.

Consider the periodic function f(x) defined in $(\alpha,\ \alpha+2c)$.

To change the problem to period 2π, put $z=\frac{\pi x}{c}$ or $x=\frac{cz}{\pi}$ (i)

so that when $x=\alpha$, $z=\frac{\alpha\pi}{c}=\beta$ (say)

when $x=\alpha+2c$, $z=\frac{(\alpha+2c)\pi}{c}=\beta+2\pi$.

Thus, the function f(x) of period 2c in $(\alpha,\ \alpha+2c)$ is **transformed** to the function $f\left(\frac{cz}{\pi}\right)=F(z)$ (say) of period 2π in $(\beta,\ \beta+2\pi)$.

Hence, $f\left(\frac{cz}{\pi}\right)$ can be expressed as the Fourier series

$$f\left(\frac{cz}{\pi}\right)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos nz+\sum_{n=1}^{\infty} b_n \sin nz, \qquad \text{(ii)}$$

where

$$\left.\begin{aligned} a_0 &= \frac{1}{\pi}\int_{\beta}^{\beta+2\pi} f\left(\frac{cz}{\pi}\right)dz, \\ a_n &= \frac{1}{\pi}\int_{\beta}^{\beta+2\pi} f\left(\frac{cz}{\pi}\right)\cos nz dz, \\ b_n &= \frac{1}{\pi}\int_{\beta}^{\beta+2\pi} f\left(\frac{cz}{\pi}\right)\sin nz dz \end{aligned}\right] \qquad \text{(iii)}$$

Making the **inverse substitution** $z=\frac{\pi x}{c},\ dz=\left(\frac{\pi}{c}\right)dx$ in (ii) and (iii).

Then the Fourier expansion of f(x) in the interval $(\alpha,\ \alpha+2c)$ is given by

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{c}+\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{c},$$

where

$$\left.\begin{aligned} a_0 &= \frac{1}{c}\int_{\alpha}^{\alpha+2c} f(x)dx, \\ a_n &= \frac{1}{c}\int_{\alpha}^{\alpha+2c} f(x)\cos\frac{n\pi x}{c}dx, \\ b_n &= \frac{1}{c}\int_{\alpha}^{\alpha+2c} f(x)\sin\frac{n\pi x}{c}dx \end{aligned}\right] \qquad \text{(iv)}$$

Remarks:

Putting $\alpha=0$ in (iv), we get the results for the interval (0, 2c) and putting $\alpha=-c$ in (iv), we get result for the interval $(-c, c)$.

Now let us develop a Fourier series of some functions of period 2c:

Q.No.1.: Expand $f(x)=e^{-x}$ as a Fourier series in the interval $(-\ell,\ \ell)$.

Sol.: The required Fourier series is of the form

$$e^{-x}=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{\ell}+\sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{\ell}. \qquad \text{(i)}$$

Here $a_0=\frac{1}{\ell}\int_{-\ell}^{\ell} e^{-x}dx=\frac{1}{\ell}\left[-e^{-x}\right]_{-\ell}^{\ell}=\frac{1}{\ell}\left(e^{\ell}-e^{-\ell}\right)=\frac{2\sinh\ell}{\ell}$.

$$a_n=\frac{1}{\ell}\int_{-\ell}^{\ell} e^{x}\cos\frac{n\pi x}{\ell}dx \qquad \left[\because \int e^{ax}\cos bx dx=\frac{e^{ax}}{a^2+b^2}\left(a\cos bx+b\sin bx\right)\right]$$

$$=\frac{1}{\ell}\left|\frac{e^{-x}}{1+\left(\frac{n\pi}{\ell}\right)^2}\left(-\cos\frac{n\pi x}{\ell}+\frac{n\pi}{\ell}\sin\frac{n\pi x}{\ell}\right)\right|_{-\ell}^{\ell}=\frac{2\ell(-1)^n\sinh\ell}{\ell^2+(n\pi)^2}. \qquad \left[\because \cos n\pi=(-1)^n\right]$$

$$a_1=\frac{-2\ell\sinh\ell}{\ell^2+\pi^2},\quad a_2=\frac{2\ell\sinh\ell}{\ell^2+2^2\pi^2},\quad a_3=\frac{2\ell\sinh\ell}{\ell^2+3^2\pi^2},\ \ldots\ldots\ \text{etc.}$$

Finally, $b_n=\frac{1}{\ell}\int_{-\ell}^{\ell} e^{-x}\sin\frac{n\pi x}{\ell}dx \qquad \left[\because \int e^{ax}\sin bx dx=\frac{e^{ax}}{a^2+b^2}\left(a\sin bx-b\cos bx\right)\right]$

$$=\frac{1}{\ell}\left|\frac{e^{-x}}{1+\left(\frac{n\pi}{\ell}\right)^2}\left(-\sin\frac{n\pi x}{\ell}-\frac{n\pi}{\ell}\cos\frac{n\pi x}{\ell}\right)\right|_{-\ell}^{\ell}=\frac{2n\pi(-1)^n\sinh\ell}{\ell^2+(n\pi)^2}.$$

$$\therefore b_1=\frac{-2\pi\sinh\ell}{\ell^2+\pi^2},\quad b_2=\frac{4\pi\sinh\ell}{\ell^2+2^2\pi^2},\quad b_3=\frac{-6\pi\sinh\ell}{\ell^2+3^2\pi^2},\ \ldots\ldots\ldots\ \text{etc.}$$

Substituting the values of a_i's and b_i's in (i), we get the required Fourier series

$$e^{-x}=\sinh\ell\left\{\begin{array}{l}\frac{1}{\ell}-2\ell\left(\frac{1}{\ell^2+\pi^2}\cos\frac{\pi x}{\ell}-\frac{1}{\ell^2+2^2\pi^2}\cos\frac{2\pi x}{\ell}+\frac{1}{\ell^2+3^2\pi^2}\cos\frac{3\pi x}{\ell}-\ldots\ldots\right)\\ -2\pi\left(\frac{1}{\ell^2+\pi^2}\sin\frac{\pi x}{\ell}-\frac{2}{\ell^2+2^2\pi^2}\sin\frac{2\pi x}{\ell}+\frac{3}{\ell^2+3^2\pi^2}\sin\frac{3\pi x}{\ell}-\ldots\ldots\right)\end{array}\right\}.$$

Q.No.2.: Obtain the Fourier series for $f(x)=\pi x$ in $0\le x\le 2$.

Sol.: The required Fourier series for the function $f(x)$ defined in (0, 2), is in the form,

$$\pi x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{1} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{1}. \qquad \text{(i)}$$

Here $a_0 = \frac{1}{1}\int_0^2 \pi x dx = \pi\int_0^2 x dx = \pi\left[\frac{x^2}{2}\right]_0^2 = \pi\left[\frac{4}{2} - 0\right] = 2\pi$.

$$a_n = \frac{1}{1}\int_0^2 \pi x \cos n\pi x dx = \pi\int_0^2 x \cos n\pi x dx = \pi\left\{\left[\frac{x \sin n\pi x}{n\pi}\right]_0^2 - \int_0^2 \frac{\sin n\pi x}{n\pi}.dx\right\}$$

$$= \pi\left\{\left[\frac{x \sin n\pi x}{n\pi}\right]_0^2 + \left[\frac{1}{(n\pi)^2}\cos n\pi x\right]_0^2\right\} = \frac{\pi}{n\pi}\left[x \sin n\pi x + \frac{1}{n\pi}\cos n\pi x\right]_0^2$$

$$a_n = \frac{1}{n}\left[0 + \frac{1}{n\pi}(1-1)\right] = 0.$$

$\therefore a_1 = a_2 = a_3 = \ldots\ldots\ldots\ldots\ldots = 0$.

Finally, $b_n = \frac{1}{1}\int_0^2 \pi x \sin(n\pi x) dx = \pi\left\{\left[\frac{-x \cos n\pi x}{n\pi}\right]_0^2 + \int_0^2 \frac{\cos n\pi x}{n\pi} dx\right\}$

$$= \frac{\pi}{n\pi}\left[-x \cos n\pi x + \frac{\sin n\pi x}{n\pi}\right]_0^2 = \frac{\pi}{n\pi}\left[-2\cos 2n\pi + 0 + \frac{\sin 2n\pi - 0}{n\pi}\right].$$

$$b_n = \frac{1}{n}[-2] = \frac{-2}{n}.$$

$\therefore b_1 = \frac{-2}{1},\ b_2 = \frac{-2}{2},\ b_3 = \frac{-2}{3}, \ldots\ldots\ldots\ldots$ etc.

Substituting the values of a_i's and b_i's in (i), we get the required Fourier series

$$f(x) = \pi - \frac{2}{1}\sin \pi x - \frac{2}{2}\sin 2\pi x - \frac{2}{3}\sin 3\pi x - \ldots\ldots\ldots\ldots\ldots$$

$$= \pi + \sum_{n=1}^{\infty}\left(\frac{-2}{n}\right)\sin n\pi x = \pi - 2\sum_{n=1}^{\infty}\frac{\sin(n\pi x)}{n}.$$

Thus $\pi x = \pi - 2\sum_{n=1}^{\infty}\frac{\sin(n\pi x)}{n}$.

Q.No.3.: Find the Fourier series for $f(t) = 1 - t^2$, when $-1 \le t \le 1$.

Sol.: The required Fourier series for the function $f(t)$ defined in $(-1,\ 1)$, is in the form,

$$1-t^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{1} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi t}{1}. \qquad \text{(i)}$$

Here $a_0 = \frac{1}{1}\int_{-1}^{1} f(t)dt = \int_{-1}^{1}(1-t^2)dt = \left[t - \frac{t^3}{3}\right]_{-1}^{1} = 1 - \frac{1}{3} + 1 - \frac{1}{3} = 2 - \frac{2}{3} = \frac{4}{3}$.

$$a_n = \frac{1}{1}\int_{-1}^{1} f(t)\cos n\pi t dt = \int_{-1}^{1}(1-t^2)\cos n\pi t dt = \left[(1-t^2)\frac{\sin n\pi t}{n\pi}\right]_{-1}^{1} - \int_{-1}^{1}(-2t)\frac{\sin n\pi t}{n\pi}dt$$

$$= \left[(1-1)\frac{\sin n\pi}{n\pi} - (1-1)\frac{(-\sin(n\pi))}{n\pi}\right] + \frac{2}{n\pi}\int_{-1}^{1} t\sin(n\pi t)dt$$

$$= 0 + 0 + \frac{2}{n\pi}\left\{\left[\frac{-t\cos n\pi t}{n\pi}\right]_{-1}^{1} + \int_{-1}^{1}\frac{\cos n\pi t}{n\pi}dt\right\}$$

$$= \frac{2}{n\pi}\left[-\frac{1}{n\pi}\cos n\pi - \frac{1}{n\pi}\cos n\pi\right] + \left[\frac{\sin n\pi t}{(n\pi)^2}\right]_{-1}^{1}$$

$$= \frac{2}{n\pi}\left[-\frac{2}{n\pi}\cos n\pi\right] + \frac{1}{(n\pi)^2}[\sin n\pi - \sin n\pi] = \frac{-4}{n^2\pi^2}\cos n\pi = \frac{-4}{n^2\pi^2}(-1)^n.$$

Finally, $b_n = \frac{1}{1}\int_{-1}^{1} f(t)\sin n\pi t dt = \int_{-1}^{1}(1-t^2)\sin n\pi t dt$

$$= -\left[(1-t^2)\frac{\cos n\pi t}{n\pi}\right]_{-1}^{1} - \int_{-1}^{1}\frac{2.t\cos n\pi t}{n\pi}dt$$

$$= 0 - \frac{2}{n\pi}\left\{\left[\frac{t.\sin n\pi t}{n\pi}\right]_{-1}^{1} - \int_{-1}^{1}\frac{\sin n\pi t}{n\pi}dt\right\}$$

$$= \frac{-2}{n\pi}\left[0 + \frac{\cos n\pi t}{(n\pi)^2}\right]_{-1}^{1} = \frac{-2}{(n\pi)}[\cos n\pi - \cos n\pi]$$

$\Rightarrow b_n = 0$.

Also $b_1 = \frac{1}{1}\int_{-1}^{1}(1-t^2)\sin \pi t dt = 0$.

Substituting the values of a_i's and b_i's in (i), we get the required Fourier series

$$f(t) = \frac{2}{3} - \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2\pi^2}\cos n\pi t$$

$$= \frac{2}{3} + \frac{4}{\pi^2}\left[\frac{\cos \pi t}{1^2} - \frac{\cos 2\pi t}{2^2} + \frac{\cos 3\pi t}{3^2} - \dots\dots\dots\dots\dots\dots\right].$$

Q.No.4.: Develop f(x) in a Fourier series in the interval $(-2, 2)$, if

$$f(x) = 0, \quad -2 < x < 0,$$

$$= 1, \qquad 0 < x < 2.$$

Sol.: The Fourier series for the function f(x) defined in the interval $(-2, 2)$ can be written as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2}\ . \qquad [\because \ell = 2] \qquad \text{(i)}$$

Here, $a_0 = \frac{1}{2}\int_{-2}^{2} f(x)dx = \frac{1}{2}\left[\int_{-2}^{0} f(x)dx + \int_{0}^{2} f(x)dx\right] = \frac{1}{2}\int_{0}^{2} 1dx = \frac{1}{2}.[x]_0^2 = \frac{1}{2}.2 = 1.$

$$a_n = \int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}dx = \frac{1}{2}\left[\int_{-2}^{0} f(x)\cos\frac{n\pi x}{2}dx + \int_{0}^{2} f(x)\cos\frac{n\pi x}{2}dx\right]$$

$$= \frac{1}{2}\int_{0}^{2} 1.\cos\frac{n\pi x}{2}dx = \frac{1}{2}\left[\sin\frac{n\pi x}{2}.\frac{2}{n\pi}\right]_0^2 = \frac{1}{n\pi}\left[\sin\frac{n\pi x}{2}\right]_0^2 = \frac{1}{n\pi}\left[\sin\frac{2n\pi}{2} - \sin 0\right] = 0.$$

$$b_n = \int_{-2}^{2} f(x)\sin\frac{n\pi x}{2}dx = \frac{1}{2}\left[\int_{-2}^{0} f(x)\sin\frac{n\pi x}{2}dx + \int_{0}^{2} f(x)\sin\frac{n\pi x}{2}dx\right]$$

$$= \frac{1}{2}\int_{0}^{2} 1.\sin\frac{n\pi x}{2}dx = \frac{1}{2} - \left[\cos\frac{n\pi x}{2}.\frac{2}{n\pi}\right]_0^2 = -\frac{1}{n\pi}\left[\cos\frac{n\pi x}{2}\right]_0^2$$

$$= -\frac{1}{n\pi}\left[\cos\frac{2n\pi}{2} - 1\right] = -\frac{1}{n\pi}\left[(-1)^n - 1\right] = \frac{\left(1-(-1)^n\right)}{n\pi}.$$

Substituting the values of a_i's and b_i's in (i), we get the required Fourier series

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{1-(-1)^n}{n\pi}\right)\sin\frac{n\pi x}{2}$$

$$\Rightarrow f(x)=\frac{1}{2}+\frac{2}{\pi}\left[\sin\frac{\pi x}{2}+\frac{\sin\frac{3\pi x}{2}}{3}+\frac{\sin\frac{5\pi x}{2}}{5}+\dots\dots\dots\dots\right].$$

Q.No.5.: If $f(x)=\pi x$, $\quad 0\le x\le 1$,

$$=\pi(2-x),\quad 1\le x\le 2;$$

show that in the interval (0, 2)

$$f(x)=\frac{\pi}{2}-\frac{4}{\pi}\left[\frac{\cos\pi x}{1^2}+\frac{\cos 3\pi x}{3^2}+\frac{\cos 5\pi x}{5^2}+\dots\dots\dots\right],$$

and hence deduce that $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\dots\dots\dots=\frac{\pi^2}{8}$.

Sol.: The Fourier series for the function f(x) defined in (0, 2) can be written as

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{1}+\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{1}. \qquad \text{(i)}$$

Here $a_0=\int_0^2 f(x)dx=\int_0^1 f(x)dx+\int_1^2 f(x)dx=\int_0^1 \pi x dx+\int_1^2 \pi(2-x)dx$

$$=\pi\left\{\left[\frac{x^2}{2}\right]_0^1+\left[2x-\frac{x^2}{2}\right]_1^2\right\}=\pi\left[\frac{1}{2}+\left(4-\frac{4}{2}-2+\frac{1}{2}\right)\right]=\pi\left(\frac{1}{2}+\frac{1}{2}\right)=\pi.$$

$$a_n=\int_0^2 f(x)\cos n\pi x dx=\int_0^1 \pi x\cos n\pi x dx+\int_1^2 \pi(2-x)\cos n\pi x dx$$

$$=\pi\left\{\left[\frac{x\sin n\pi x}{n\pi}\right]_0^1-\int_0^1\frac{\sin n\pi x}{n\pi}dx+\left[\frac{(2-x)\sin n\pi x}{n\pi}\right]_1^2-\int_1^2\frac{(-1)\sin n\pi x}{n\pi}dx\right\}$$

$$=\pi\left\{\left[\frac{x\sin n\pi x}{n\pi}\right]_0^1+\left[\frac{\cos n\pi x}{(n\pi)^2}\right]_0^1+\left[\frac{(2-x)\sin n\pi x}{n\pi}\right]_1^2-\left[\frac{\cos n\pi x}{(n\pi)^2}\right]_1^2\right\}$$

$$=\pi\left\{\left[\frac{x\sin n\pi x}{n\pi}+\frac{\cos n\pi x}{(n\pi)^2}\right]_0^1+\left[\frac{(2-x)\sin n\pi x}{n\pi}-\frac{\cos n\pi x}{(n\pi)^2}\right]_1^2\right\}$$

$$= \pi\left\{\frac{(-1)^n}{(n\pi)^2} - \frac{1}{(n\pi)^2} - \frac{1}{(n\pi^2)}\left[1-(-1)^n\right]\right\} = 2\pi\left[\frac{(-1)^n}{(n\pi)^2} - \frac{1}{(n\pi)^2}\right]$$

$$= \frac{2\pi}{n^2.\pi^2}\left[(-1)^n - 1\right].$$

Hence $a_n = \dfrac{2}{n^2\pi}\left[(-1)^n - 1\right]$.

$$b_n = \pi\int_0^1 x\sin n\pi x dx + \int_1^2 \pi(2-x)\sin n\pi x dx$$

$$= \pi\left\{\left[\frac{-x\cos n\pi x}{n\pi} + \frac{\sin n\pi x}{(n\pi)^2}\right]_0^1 - \left[\frac{(2-x)\cos n\pi x}{n\pi}\right]_1^2\right\}$$

$$= \pi\left[\frac{(-1)(-1)^n}{n\pi} + \frac{(2-1)\cos n\pi}{n\pi}\right] + \left[\frac{\sin n\pi x}{(n\pi)^2}\right]_1^2 = \pi\left[-\frac{(-1)^n}{n\pi} + \frac{(-1)^n}{n\pi}\right] = 0$$

Substituting the values of a_i's and b_i's in (i), we get the required Fourier series

$$f(x) = \frac{\pi}{2} - \sum_{n=1}^{\infty} \frac{2}{n^2\pi}\left[(-1)^n - 1\right]\cos n\pi x$$

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos\pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + \dots\dots\dots\dots\right].$$

2nd Part:

When $x = 0$, $f(x) = 0$.

$$\Rightarrow 0 = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots\dots\dots\right]$$

$$\Rightarrow \frac{\pi}{2} = \frac{4}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots\dots\dots\right]$$

$$\Rightarrow \frac{\pi^2}{8} = \left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots\dots\dots\right].$$

Q.No.6.: A sinusoidal voltage $E\sin\omega t$ is passed through a half-wave rectifier which clips the negative portion of the wave. Develop the resulting periodic function,

$$U(t) = 0, \qquad \text{when } -\frac{T}{2} < t < 0,$$

$$= E\sin\omega t, \text{ when } \quad 0 < t < \frac{T}{2},$$

and $T = \frac{2\pi}{\omega}$, in a Fourier series.

Sol.: The Fourier series for the function U(t) defined in $\left(-\frac{T}{2}, \frac{T}{2}\right)$ can be written as

$$U(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{\frac{T}{2}} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi t}{\frac{T}{2}}. \qquad \text{(i)}$$

Here $a_0 = \frac{1}{\ell}\int_{-T/2}^{T/2} U(t)dt = \frac{1}{T/2}\left[\int_{-T/2}^{0} U(t)dt + \int_{0}^{T/2} U(t)dt\right]$

$$= 0 + \frac{2}{T}\int_0^{T/2} E\sin\omega t dt = \frac{2}{T}.E\left[\frac{-\cos\omega t}{\omega}\right]_0^{T/2}$$

$$= \frac{2\omega E}{2\pi}\left[-\cos\left(\frac{2\pi}{T}\times\frac{T}{2}\right) + \cos 0\right] = \frac{2E}{\pi}.$$

$$a_n = \frac{1}{T/2}\int_0^{T/2} E\sin\omega t\cos\frac{n\pi t}{T/2}dt = \frac{\omega E}{\pi}\int_0^{T/2}\frac{2}{2}\sin\omega t\cos n\omega t dt \left[\because \cos\frac{n\pi t}{\frac{T}{2}} = \cos\frac{n\pi t}{\frac{\pi}{\omega}} = \cos n\omega t\right]$$

$$= \frac{\omega E}{2\pi}\int_0^{T/2}\left[\sin(1+n)\omega t + \sin(1-n)\omega t\right]dt = \frac{E\omega}{2\pi}\left[\frac{-\cos(1+n)\omega t}{(1+n)\omega} - \frac{\cos(1-n)\omega t}{(1-n)\omega}\right]_0^{T/2}$$

$$= \frac{E\omega}{2\pi\omega}\left[\frac{n-1-1-n}{n^2-1}\right] + \frac{E\omega}{2\pi}\left[\frac{(-1)(-1)^n}{\omega(1-n)} - \frac{(-1)(-1)^n}{\omega(1+n)}\right]$$

$$= \frac{E\omega}{2\pi\omega}\left[\frac{-2}{n^2-1}\right] + \frac{E\omega}{2\pi\omega}\left[\frac{-(-1)^n}{(1-n)} + \frac{(-1)^n}{(1+n)}\right] = \frac{-2E}{2\pi(n^2-1)} + \frac{E(-1)^n}{2\pi}\left[\frac{-(1+n)+(1-n)}{(1-n^2)}\right]$$

$$= \frac{-2E}{2\pi(n^2-1)} + \frac{2E(-1)^n}{2\pi(n^2-1)} = \frac{-E\left[(-1)^n - 1\right]}{\pi(n^2-1)}.$$

Finally, $b_n = \frac{1}{T/2}\int_0^{T/2}\frac{2E}{2}\sin\omega t\sin n\omega t dt = \frac{\omega E}{2\pi}\int_0^{T/2}\left[\cos(n+1)\omega t - \cos(n-1)\omega t\right]dt$

$$= \frac{\omega E}{2\pi}\left[\frac{\sin(n+1)\omega t}{(n+1)\omega}\right]_0^{T/2} - \frac{\omega E}{2\pi}\left[\frac{\sin(n-1)\omega t}{(n-1)\omega}\right]_0^{T/2} = \frac{\omega E}{2\pi}[0] - \frac{E}{T}[0] = 0.$$

$\Rightarrow b_n = 0$, when $n > 1$.

When $n = 1$, then

$$b_1 = \frac{E}{T/2}\int_0^{T/2} \sin\omega t.\sin\omega t dt = \frac{E}{T/2}\int_0^{T/2} \sin^2\omega t dt$$

$$= \frac{2E\omega}{2\pi}\int_0^{T/2}\left(\frac{1-\cos 2\omega t}{2}\right)dt = \frac{E\omega}{2\pi}\left[t - \frac{\sin 2\omega t}{2}\right]_0^{T/2} = \frac{E\omega}{2\pi}\left[\frac{T}{2}\right] = \frac{E\omega}{2\pi}\frac{2\pi}{2\omega} = \frac{E}{2}.$$

Substituting the values of a_i's and b_i's in (i), we get the required Fourier series

$$U(t) = \frac{E}{\pi} + \frac{E}{2}\sin\omega t + \sum_{n=1}^{\infty}\frac{-E\left[(-1)^n - 1\right]}{\pi\left[n^2 - 1\right]}\cos n\omega t$$

$$\Rightarrow U(t) = \frac{E}{\pi} + \frac{E}{2}\sin\omega t - \frac{2E}{\pi}\left[\frac{\cos 2\omega t}{3} + \frac{\cos 4\omega t}{15} + \frac{1}{35}\cos 6\omega t + \dots\dots\dots\dots\right].$$ **Ans.**

Q.No.7.: Find the Fourier series of the function

$$f(x) = \begin{cases} 0 & \text{when } -2 < x < -1 \\ k & \text{when } -1 < x < 1 \\ 0 & \text{when } 1 < x < 2 \end{cases}$$

Sol.: The function f(x) is given in the interval $-2 < x < 2$.

Comparing with $\alpha < x < \alpha + 2c$, we have $\alpha = -2$ and $c = 2$

For this, Fourier series is written as,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{c} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{c}. \qquad \text{(i)}$$

Here, $a_0 = \frac{1}{c}\int_{\alpha}^{\alpha+2c} f(x)dx = \frac{1}{2}\left[\int_{-2}^{-1} 0dx + \int_{-1}^{1} kdx + \int_{1}^{2} 0dx\right] = \frac{k}{2}[x]_{-1}^{1} = \frac{k}{2}[1+1] = k.$

$$a_n = \frac{1}{2}\left[\int_{-2}^{-1} 0\cos\frac{n\pi x}{2}dx + \int_{-1}^{1} k\cos\frac{n\pi x}{2}dx + \int_{1}^{2} 0\cos\frac{n\pi x}{2}dx\right]$$

$$= \frac{k}{2}\int_{-1}^{1}\cos\frac{n\pi x}{2}dx = \frac{k}{2}\left[\sin\frac{n\pi x}{2}.\frac{2}{n\pi}\right]_{-1}^{1} = \frac{k}{n\pi}\left[\sin\frac{n\pi x}{2}\right]_{-1}^{1}$$

$$=\frac{k}{n\pi}\left[\sin\frac{n\pi}{2}+\sin\frac{n\pi}{2}\right]=\frac{2k}{n\pi}\left(\sin\frac{n\pi}{2}\right).$$

So, $a_1=\frac{2k}{n\pi}$, $a_2=0$, $a_3=\frac{-2k}{3\pi}$, $a_4=0$ ....etc.

Now, $b_n=\frac{1}{2}\left[\int_{-2}^{-1}0\sin\frac{n\pi x}{2}dx+\int_{-1}^{1}k\sin\frac{n\pi x}{2}dx+\int_{1}^{2}0\sin\frac{n\pi x}{2}dx\right]$

$$=\frac{k}{2}\int_{-1}^{1}\sin\frac{n\pi x}{2}dx=\frac{k}{2}\left[-\cos\frac{n\pi x}{2}.\frac{2}{n\pi}\right]_{-1}^{1}=\frac{k}{n\pi}\left[-\cos\frac{n\pi}{2}+\cos\frac{n\pi}{2}\right]=0$$

Substituting the values of a_i's and b_i's in (i), we get the required Fourier series

$$f(x)=\frac{k}{2}+\frac{2k}{\pi}\left(\cos\frac{\pi x}{2}-\frac{1}{3}\cos\frac{3\pi x}{2}+\frac{1}{5}\cos\frac{5\pi x}{2}......................\right).$$ **Ans.**

$$\Rightarrow f(x)=\frac{k}{2}+\frac{2k}{\pi}\sum_{n=1}^{\infty}\frac{\sin\left(\frac{n\pi}{2}\right)}{n}\cos\frac{n\pi x}{2}.$$

Q.No.8.: Obtain the Fourier series expansion of $f(x)=\frac{(\pi-x)}{2}$ in $0<x<2$.

Sol.: Here the length of interval is 2L = 2 (i.e., L = 1).

The Fourier series for the function f(x) defined in $(0,\ 2)$ can be written as

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}an\cos\frac{n\pi x}{1}+\sum_{n=1}^{\infty}an\sin\frac{n\pi x}{1}. \qquad \text{(i)}$$

Here $a_0=\frac{1}{1}\int_0^2 f(x)dx=\int_0^2\frac{\pi-x}{2}dx=\frac{1}{2}\left[\pi x-\frac{x^2}{2}\right]_0^2=(\pi-1).$

$$a_n=\frac{1}{1}\int_0^2 f(x)\cos\frac{n\pi x}{1}dx=\int_0^2\left(\frac{\pi-x}{2}\right)\cos n\pi x dx$$

$$=\frac{1}{2}\left[(\pi-x)\left(\frac{1}{n\pi}\right)\sin n\pi x-(-1)\times\frac{1}{n\pi}.(-1)\frac{\cos n\pi x}{n\pi}\right]_0^2$$

$$=-\frac{1}{2n^2\pi^2}[\cos 2n\pi-\cos 0]=-\frac{1}{2n^2\pi^2}[1-1]=0.$$

$$b_n = \frac{1}{1}\int_0^2 f(x)\sin\frac{n\pi x}{1}dx \quad = \int_0^2 \left(\frac{\pi - x}{2}\right)\sin n\pi x dx$$

$$= \frac{1}{2}\left[(\pi - x)\left(\frac{-\cos n\pi x}{n\pi}\right) - (-1).[-1]\frac{\sin n\pi x}{n^2\pi^2}\right]_0^2.$$

$$= \frac{-1}{2n\pi}\left[(\pi - 2)\cos 2n\pi - \pi.\cos 0\right] = \frac{1}{n\pi}.$$

Substituting the values of a_i's and b_i's in (i), we get the required Fourier series

$$f(x) = \frac{\pi - x}{2} = \frac{(\pi - 1)}{2} + 0 + \sum_{n=1}^{\infty} \frac{1}{n\pi}.\sin n\pi x.$$

Q.No.9.: Find the Fourier expansion for the function $f(x) = x - x^2$, $-1 < x < 1$.

Sol.: Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x$.

Here $a_0 = \int_{-1}^{1} (x - x^2)dx = \int_{-1}^{1} xdx - \int_{-1}^{1} x^2dx = 0 - 2\int_{-1}^{1} x^2dx = -2\left[\frac{x^3}{3}\right]_0^1 = \frac{2}{3}$.

$$a_n = \int_{-1}^{1} (x - x^2)\cos n\pi x dx = \int_{-1}^{1} x\cos n\pi x dx - \int_{-1}^{1} x^2 \cos n\pi x dx$$

$$= 0 - 2\int_0^1 x^2 \cos n\pi x dx = -2\left[x^2.\frac{\sin n\pi x}{n\pi} - 2x\left(-\frac{\cos n\pi x}{n^2\pi^2}\right) + 2\left(-\frac{\sin n\pi x}{n^3\pi^3}\right)\right]_0^1$$

$$= -\left[\frac{2\cos n\pi}{n^2\pi^2}\right] = \frac{-4(-1)^n}{n^2\pi^2} = \frac{4(-1)^{n+1}}{n^2\pi^2}.$$

$$b_n = \int_{-1}^{1} (x - x^2)\sin n\pi x dx = \int_{-1}^{1} x\sin n\pi x dx - \int_{-1}^{1} x^2 \sin n\pi x dx$$

$$= 2\int_0^1 x\sin n\pi x dx - 0 = 2\left[x\left(-\frac{\cos n\pi x}{n\pi}\right) - 1\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right]_0^1$$

$$= -2\left[-\frac{\cos n\pi}{n\pi}\right] = \frac{-2(-1)^n}{n\pi} = \frac{2(-1)^{n+1}}{n\pi}.$$

Substituting the values of a_i's and b_i's in (i), we get the required Fourier series

$$x - x^2 = -\frac{1}{3} + \frac{4}{\pi^2}\left(\frac{\cos \pi x}{1^2} - \frac{\cos 2\pi x}{2^2} + \frac{\cos 3\pi x}{3^2} - \ldots\ldots\ldots\ldots\right)$$
$$+\frac{2}{\pi}\left(\frac{\sin \pi x}{1} - \frac{\sin 2\pi x}{2} + \frac{\sin 3\pi x}{3} - \ldots\ldots\ldots\ldots\right)$$

Q.No.10.: Find the Fourier series to represent $f(x) = x^2 - 2$, when $-2 \le x \le 2$

Sol.: Since f(x) is an even function, $b_n = 0$.

Let $f(x) = x^2 - 2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2}$.

Here $a_0 = \frac{2}{2}\int_0^2 \left(x^2 - 2\right)dx = \left[\frac{x^3}{3} - 2x\right]_0^2 = \frac{8}{4} - 4 = -\frac{4}{3}$.

$$a_n = \frac{2}{2}\int_0^2 \left(x^2 - 2\right)\cos\frac{n\pi x}{2}dx = \left[\left(x^2 - 2\right)\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}} - 2x\left(-\frac{\cos\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right) + 2\left(-\frac{\sin\frac{n\pi x}{2}}{\frac{n^3\pi^3}{8}}\right)\right]_0^2$$

$$= \frac{16\cos n\pi}{n^2\pi^2} = \frac{16(-1)^n}{n^2\pi^2}.$$

Substituting the values of a_i's and b_i's in (i), we get the required Fourier series

$$x^2 - 2 = -\frac{2}{3} - \frac{16}{\pi^2}\left(\cos\frac{n\pi}{2} - \frac{1}{4}\cos \pi x + \frac{1}{9}\cos\frac{3\pi x}{2} - \ldots\ldots\right).$$

Q.No.11.: Develop f(x) in a Fourier series in the interval (0, 2) if

$$f(x) = \begin{cases} x, & 0 < x < 1 \\ 0, & 1 < x < 2 \end{cases}.$$

Sol.: Here the interval is (0, 2). Its length $= 2 - 0 = 2 = 2\ell, \quad \ell = 1$.

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{1} + b_n \sin\frac{n\pi x}{1}\right)$ (i)

be the required Fourier series.

Here $a_0 = \frac{1}{1}\int_0^2 f(x)dx = \int_0^1 x dx + \int_1^2 0 dx = \frac{1}{2}$.

$$a_n = \frac{1}{1}\int_0^2 f(x)\cos\frac{n\pi x}{1}dx = \int_0^1 \underset{\mathrm{I}}{x}\cos\underset{\mathrm{II}}{\frac{n\pi x}{1}}dx + \int_1^2 0dx$$

Integrating by parts, we get

$$= \left[(x)\left(\frac{\sin n\pi x}{n\pi}\right) - (1)\left(\frac{-\cos n\pi x}{n^2\pi^2}\right)\right]_0^1 = \frac{1}{n\pi}\sin n\pi + \frac{1}{n^2\pi^2}(\cos n\pi - 1)$$

$$= \frac{1}{n^2\pi^2}\left[(-1)^n - 1\right].$$

$$b_n = \frac{1}{1}\int_0^2 f(x)\sin\frac{n\pi x}{1}dx = \int_0^1 x\sin\frac{n\pi x}{1}dx + \int_1^2 0dx = \int_0^1 \underset{\mathrm{I}}{x}\sin\underset{\mathrm{II}}{\frac{n\pi x}{1}}dx$$

Integrating by parts, we get

$$= \left[(x)\left(\frac{-\cos n\pi x}{n\pi}\right) - (1)\left(\frac{-\sin n\pi x}{n^2\pi^2}\right)\right]_0^1 = \frac{-1}{n\pi}\cos n\pi + \frac{1}{n^2\pi^2}\sin n\pi - 0$$

$$= \frac{-1}{n\pi}\cos n\pi + \frac{1}{n^2\pi^2}\sin n\pi = -\frac{(-1)^n}{n\pi}.$$

Hence, from (i), we get

$$f(x) = \frac{1}{4} + \sum_{n=1}^{\infty}(a_n\cos n\pi x) + \sum_{n=1}^{\infty}(b_n\sin n\pi x)$$

$$= \frac{1}{4} - \frac{2}{\pi^2}\left((\cos\pi x) + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + \ldots\ldots\right)$$

$$+ \frac{1}{\pi}\left((\sin\pi x) - \frac{\sin 2\pi x}{2} + \frac{\sin 3\pi x}{3} - \ldots\ldots\right),$$

is the required Fourier series.

Q.No.12.: Find the Fourier expansion for the function $f(x) = \pi x$ from $x = -c$ to $x = c$.

Sol.: Let $f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{c}$ (i)

be the required Fourier series.

$$b_n = \frac{1}{c}\int_{-c}^{c} f(x)\sin\frac{n\pi x}{c}dx = \frac{1}{c}\int_{-c}^{c}\pi x\sin\frac{n\pi x}{c}dx = \frac{2\pi}{c}\int_0^c x\sin\frac{n\pi x}{c}dx$$

$$= \frac{2\pi}{c}\left[(x)\left(\frac{-\cos\frac{n\pi x}{c}}{\frac{n\pi}{c}}\right) - (1)\left(\frac{-\sin\frac{n\pi x}{2}}{\frac{n\pi}{c}}\right)\right]_0^c$$

$$= \frac{2\pi}{c}\left(\frac{-c^2}{n\pi}(-1)^n + 0 - (0+0)\right) = \frac{2c}{n}(-1)^{n+1}.$$

Hence, from (i), we get

$$f(x) = \sum_{n=1}^{\infty} \frac{2c}{n}(-1)^{n+1}\sin\frac{n\pi x}{c} = 2c\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{c}$$

$$= 2c\left(\sin\frac{nx}{c} - \frac{1}{2}\sin\frac{2\pi x}{c} + ...\right),$$

is the required Fourier series.

Q.No.13.: Find the Fourier expansion for the function $f(x) = x - x^3$ in the interval $-1 < x < 1$.

Sol.: Here the interval is $(-1,\ 1)$. Its length $= 1(-1) = 2 = 2\ell,\ \ \ell = 1$

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{1} + b_n\sin\frac{n\pi x}{1}\right)$ (i)

be the required Fourier series.

Here $f(-x) = -x + x^3 = -(x - x^3) = -f(x)$.

$\Rightarrow$ f(x) is odd function. $\therefore a_0 = 0,\ \ a_n \forall n$.

Now $b_n = \frac{1}{1}\int_{-1}^{1} f(x)\sin n\pi x dx = \int_{-1}^{1}(x - x^3)\sin n\pi x dx = 2\int_0^1 \underset{\text{I}}{(x - x^3)}\underset{\text{II}}{\sin n\pi x}\, dx$

$$\left[(x - x^3)\sin n\pi x \text{ is even function}\right]$$

$$= \left[(x - x^3)\left(\frac{-\cos n\pi x}{n\pi}\right) - (1 - 3x^2)\left(\frac{-\sin n\pi x}{n^2\pi^2}\right) + (-6x)\left(\frac{\cos n\pi x}{n^3\pi^3}\right) - (-6)\left(\frac{\sin n\pi x}{n^4\pi^4}\right)\right]_0^1$$

$$= 2\left[\frac{-6(-1)^n}{n^3\pi^3} - 0\right] = \frac{12(-1)^{n+1}}{n^3\pi^3}. \qquad \left[\begin{array}{l}\sin n\pi = 0,\ \ n \in Z \\ \cos n\pi = (-1)^n, n \in Z\end{array}\right]$$

Hence, from (i), we get

$$f(x) = x - x^3 = \frac{12}{\pi^3}\left[\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3} \sin n\pi x\right] = \frac{12}{\pi^3}\left[\frac{\sin \pi x}{1^3} - \frac{\sin 2\pi x}{2^3} + ...\right]$$

is the required Fourier series.

Q.No.14.: Find the Fourier series for the function given by $f(t) = \begin{cases} t, & -0 < t < 1 \\ 1 - t, & 1 < t < 2 \end{cases}$.

Sol.: Here interval is (0, 2). Its length $= 2 - 0 = 2 = 2\ell, \quad \ell = 1$.

Let $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{1} + b_n \sin\frac{n\pi x}{1}\right) \tag{i}$$

be the required Fourier series.

$$a_0 = \frac{1}{1}\int_0^2 f(t)dt = \int_0^1 tdt + \int_1^2 (1-t)dt = \left|\frac{t^2}{2}\right|_0^1 + \left|t - \frac{t^2}{2}\right|_1^2$$

$$= \frac{1}{2} + \left((2-2) - \left(1 - \frac{1}{2}\right)\right) = \frac{1}{2} + 0 - \frac{1}{2} = 0.$$

$$a_n = \frac{1}{1}\int_0^2 f(t)\cos\frac{n\pi t}{1}dt = \int_0^1 \underset{I}{t}.\underset{II}{\cos n\pi t}dt + \int_1^2 \underset{I}{(1-t)}.\underset{II}{\cos n\pi t}dt$$

Integrating by parts, we get

$$= \left[(t)\left(\frac{\sin n\pi t}{n\pi}\right) - (1)\left(\frac{-\cos n\pi t}{n^2\pi^2}\right)\right]_0^1 + \left[(1-t)\left(\frac{\sin n\pi t}{n\pi}\right) - (-1)\left(\frac{-\cos n\pi t}{n^2\pi^2}\right)\right]_0^2$$

$$= \left(0 + \frac{(-1)^n}{n^2\pi^2} - \frac{1}{n^2\pi^2}\right) + \left(0 - \frac{1}{n^2\pi^2} - \left(0 - \frac{(-1)^n}{n^2\pi^2}\right)\right)$$

$$= \frac{2}{n^2\pi^2}\left[(-1)^n - 1\right] = \begin{cases} 0, & \text{if } n \text{ is even} \\ \dfrac{-4}{n^2\pi^2}, & n \text{ is odd} \end{cases}.$$

$$b_n = \frac{1}{1}\int_0^2 f(t)\sin n\pi t dt = \int_0^1 \underset{I}{t}.\underset{II}{\sin n\pi t}dt + \int_1^2 \underset{I}{(1-t)}.\underset{II}{\sin n\pi t}dt$$

Integrating by parts, we get

$$= \left[(t)\left(\frac{-\cos n\pi t}{n\pi}\right) - (1)\left(\frac{-\sin n\pi t}{n^2\pi^2}\right)\right]_0^1 + \left[(1-t)\left(\frac{-\cos n\pi t}{n\pi}\right) - (-1)\left(\frac{-\sin n\pi t}{n^2\pi^2}\right)\right]_0^2$$

$$= -\frac{1}{n\pi}(-1)^n + \frac{1}{n\pi}$$

$$= \frac{1}{n\pi}\left[1-(-1)^n\right] = \begin{cases} 0, & \text{n is even} \\ \dfrac{2}{n\pi}, & \text{n is odd} \end{cases}$$

Hence, from (i), we get

$$f(t) = 0 + \sum_{n=odd}^{\infty} -\frac{4}{n^2\pi^2}\cos n\pi t + \sum_{n=odd}^{\infty} \frac{2}{n\pi}\sin n\pi t$$

$$= \frac{-4}{\pi^2}\left(\cos \pi t + \frac{\cos 3\pi t}{3^2} + \frac{\cos 5\pi t}{5^2} + \ldots\ldots\ldots\right) + \frac{2}{\pi}\left(\sin \pi t + \frac{\sin 3\pi t}{3} + \frac{\sin 5\pi t}{5} + \ldots\ldots.\right).$$

Q.No.11.: Sum of functions: Suppose the Fourier series of

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right) \text{ and}$$

$$f(g) = \frac{c_0}{2} + \sum_{n=1}^{\infty}\left(c_n \cos\frac{n\pi x}{L} + d_n \sin\frac{n\pi x}{L}\right) \text{ in the interval e to e + 2L,}$$

then find the Fourier series of $h(x) = \alpha f(x) + \beta g(x)$.

Sol.: $h(x) = \alpha\left[\frac{a_0}{2} + \sum a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right] + \beta\left[\frac{c_0}{2} + \sum c_n \cos\frac{n\pi x}{L} + d_n \sin\frac{n\pi x}{L}\right]$

$$h(x) = \left(\alpha\frac{a_0}{2} + \beta\frac{c_0}{2}\right) + \sum (\alpha a_n + \beta c_n)\cos\frac{n\pi x}{L} + \sum (\alpha b_n + \beta d_n)\sin\frac{n\pi x}{L}$$

$$h(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty}\left[A_n.\cos\left(\frac{n\pi x}{L}\right) + B_n.\cos\left(\frac{n\pi x}{L}\right)\right]$$

where $A_0 = \alpha a_0 + \beta c_0$, $A_n = \alpha a_n + \beta c_n$, $B_n = \alpha b_n + \beta d_n$.

Home Assignments

Q.No.1.: Find the Fourier series of the following function f(x) in the indicated interval.

f(x) is periodic with period 2L.

$f(x) = e^x$ in $(-L, L)$.

Ans.: $f(x) = \sinh L\left[\frac{1}{L} + 2L\sum_{n=1}^{\infty}\frac{(-1)^n\cos\left(\frac{n\pi x}{L}\right)}{L^2+n^2\pi^2} - 2\pi\sum_{n=1}^{\infty}\frac{(-1)^n\sin\left(\frac{n\pi x}{L}\right)}{L^2+n^2\pi^2}\right]$

Q.No.2.: Find the Fourier series of the following function f(x) in the indicated interval.

f(x) is periodic with period 2.

$f(x) = x \qquad \text{for } -1 < x \le 0$

$\quad = x + 2 \qquad \text{for } \ 0 < x \le 1$

Deduce that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ...$

Ans.: $f(x) = 1 + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[1 - 2(-1)^n\right]\sin n\pi x$

Put $x = \frac{1}{2}$ on both sides to deduce the result.

Q.No.3.: Find the Fourier series of the following function f(x) in the indicated interval.

f(x) is periodic with period 10.

$f(x) = 0 \qquad \text{when } -5 < x < 0$

$\quad = 3 \qquad \text{when } \ 0 < x < 5.$

Ans.: $f(x) = \frac{3}{2} + \sum_{n=1}^{\infty}\frac{3(1-\cos n\pi)}{n\pi}\sin\left(\frac{n\pi x}{5}\right)$**.**

Q.No.4.: Find the Fourier series of the following function f(x) in the indicated interval.

f(x) is periodic with period 2.

$$f(x) = \begin{cases} 1 & \text{if } \ 0 < x < 1 \\ 2 & \text{if } \ 1 < x < 2 \end{cases}$$

Ans.: $f(x) = \frac{3}{2} - \sum_{n=1}^{\infty}\frac{2}{(2n-1)\pi}.\sin(2n-1)\pi x$.

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5^{th} Topic

Fourier Series

'Expansions of even and odd functions'

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Even and odd functions

A function f(x) is said to be **even** if $f(-x) = f(x)$. e.g. $\cos x,\ \sec x,\ x^2$ are all even functions. Graphically an even function is symmetrical about the y-axis.

A function f(x) is said to be **odd** if $f(-x) = -f(x)$. e.g. $\sin x,\ \tan x,\ x^3$ are all odd functions. Graphically an odd function is symmetrical about the origin.

We shall be using the following property of definite integrals:

$$\int_{-c}^{c} f(x)dx = 2\int_{0}^{c} f(x)dx, \text{ when f(x) is an even function.}$$

$$= 0, \qquad \text{when f(x) is an odd function.}$$

Expansions of even or odd functions:

We know that a periodic function f(x) defined in $(-c,\ c)$ can be represented by the Fourier series

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{c}+\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{c},$$

where $a_0=\frac{1}{c}\int_{-c}^{c} f(x)dx$, $a_n=\frac{1}{c}\int_{-c}^{c} f(x)\cos\frac{n\pi x}{c}dx$, $b_n=\frac{1}{c}\int_{-c}^{c} f(x)\sin\frac{n\pi x}{c}dx$.

Case I: When f(x) is an **even** function, then $a_0=\frac{1}{c}\int_{-c}^{c} f(x)dx=\frac{2}{c}\int_{0}^{c} f(x)dx$.

Since $\cos\frac{n\pi x}{c}$ is an even function $\Rightarrow f(x)\cos\frac{n\pi x}{c}$ is also an even function,

$$\therefore a_n=\frac{1}{c}\int_{-c}^{c} f(x)\cos\frac{n\pi x}{c}dx=\frac{2}{c}\int_{0}^{c} f(x)\cos\frac{n\pi x}{c}dx.$$

Again, since $\sin\frac{n\pi x}{c}$ is an odd function $\Rightarrow f(x)\sin\frac{n\pi x}{c}$ is an odd function.

$$\therefore b_n=\frac{1}{c}\int_{-c}^{c} f(x)\sin\frac{n\pi x}{c}dx=0.$$

Hence, if a periodic function f(x) is even, then it's Fourier expansion contains only cosine terms, and

$$a_0=\frac{2}{c}\int_{0}^{c} f(x)dx, \qquad a_n=\frac{2}{c}\int_{0}^{c} f(x)\cos\frac{n\pi x}{c}dx. \qquad \text{(i)}$$

Case II: When f(x) is an **odd** function, then $a_0=\frac{1}{c}\int_{-c}^{c} f(x)dx=0$.

Since $\cos\frac{n\pi x}{c}$ is an even function $\Rightarrow f(x)\cos\frac{n\pi x}{c}$ is an odd function.

$$\therefore a_n=\frac{1}{c}\int_{-c}^{c} f(x)\cos\frac{n\pi x}{c}dx=0.$$

Again, since $\sin\frac{n\pi x}{c}$ is an odd function $\Rightarrow f(x)\sin\frac{n\pi x}{c}$ is an even function.

$$\therefore b_n = \frac{1}{c}\int_{-c}^{c} f(x)\sin\frac{n\pi x}{c}dx = \frac{2}{c}\int_{0}^{c} f(x)\sin\frac{n\pi x}{c}dx.$$

Thus, if a periodic function f(x) is odd, then its Fourier expansion contains only sine terms and

$$b_n = \frac{2}{c}\int_0^c f(x)\sin\frac{n\pi x}{c}dx. \qquad \text{(ii)}$$

Now let us develop the Fourier series of some even and odd functions:

Q.No.1.: Express $f(x) = x$ as a Fourier series in the interval $-\pi < x < \pi$.

Saw-toothed waveform

Sol.: Since $f(-x) = -x = -f(x)$.

$\therefore f(x)$ is an **odd function** and hence $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$.

Here $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx dx = \frac{2}{\pi}\int_0^{\pi} x\sin nx dx$

$$= \frac{2}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - \left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi} = -\frac{2\cos n\pi}{n} = -\frac{2(-1)^n}{n}.$$

$\therefore b_1 = \frac{2}{1},\ b_2 = \frac{-2}{2},\ b_3 = \frac{2}{3},\ b_4 = \frac{-2}{4}$, etc.

Hence, the required Fourier series is

$$x = -\sum_{n=1}^{\infty}\frac{2(-1)^n}{n}\sin nx$$

$$\Rightarrow x = 2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \ldots\ldots\ldots\right). \text{ Ans.}$$

Q.No.2.: Find a Fourier series to represent x^2 in the interval $(-\ell, \ell)$.

Sol.: Since $f(x) = x^2$ is an **even function** in $(-\ell, \ell)$,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{\ell}. \qquad \text{(i)}$$

Then $a_0 = \frac{2}{\ell}\int_0^{\ell} x^2 dx = \frac{2}{\ell}\left|\frac{x^3}{3}\right|_0^{\ell} = \frac{2\ell^2}{3}$.

$$a_n = \frac{2}{\ell}\int_0^{\ell} x^2\cos\frac{n\pi x}{\ell}dx = \frac{2}{\ell}\left[x^2\left(\frac{\sin\frac{n\pi x}{\ell}}{\frac{n\pi}{\ell}}\right) - 2x\left(\frac{\cos\frac{n\pi x}{\ell}}{\frac{n^2\pi^2}{\ell^2}}\right) + 2\left(\frac{\sin\frac{n\pi x}{\ell}}{\frac{n^3\pi^3}{\ell^3}}\right)\right]_0^{\ell}$$

$$= \frac{4\ell^2(-1)^n}{n^2\pi^2}. \qquad \left[\because \cos n\pi = (-1)^n\right]$$

$$\therefore a_1 = \frac{-4\ell^2}{\pi^2},\ a_2 = \frac{4\ell^2}{2^2\pi^2},\ a_3 = \frac{-4\ell^2}{3^2\pi^2},\ a_4 = \frac{4\ell^2}{4^2\pi^2} \text{ etc.}$$

Substitute these values in (i), we get the required Fourier series

$$x^2 = \frac{\ell^2}{3} - \frac{4\ell^2}{\pi^2}\left(\frac{\cos\frac{\pi x}{\ell}}{1^2} - \frac{\cos\frac{2\pi x}{\ell}}{2^2} + \frac{\cos\frac{3\pi x}{\ell}}{3^2} - \frac{\cos\frac{4\pi x}{\ell}}{4^2} + \ldots\ldots\ldots\ldots\right).$$

Q.No.3.: If $f(x) = |\cos x|$, expand $f(x)$ as a Fourier series in the interval $(-\pi, \pi)$.

Sol.: As $f(-x) = |\cos(-x)| = |\cos x| = f(x)$, $|\cos x|$ is an **even function**.

$$\therefore f(x) = \frac{a_0}{2} + \sum a_n \cos nx .$$

Here $a_0 = \frac{2}{\pi}\int_0^{\pi} |\cos x| dx = \frac{2}{\pi}\int_0^{\pi/2} \cos x dx + \frac{2}{\pi}\int_{\pi/2}^{\pi} (-\cos x) dx$

$$\left[\because \cos x \text{ is negative, when } -\frac{\pi}{2} < x < \pi\right]$$

$$= \frac{2}{\pi}\left\{|\sin x|_0^{\pi/2} - |\sin x|_{\pi/2}^{\pi}\right\} = \frac{2}{\pi}[(1-0)-(0-1)] = \frac{4}{\pi}.$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} |\cos x|\cos nx dx = \frac{2}{\pi}\left[\int_0^{\pi/2} \cos x \cos nx dx + \int_{\pi/2}^{\pi} (-\cos x)\cos nx dx\right]$$

$$= \frac{1}{\pi}\left\{\int_0^{\pi/2} [\cos(n+1)x + \cos(n-1)x]dx - \int_{\pi/2}^{\pi} [\cos(n+1)x + \cos(n-1)x]dx\right\}$$

$$= \frac{1}{\pi}\left\{\left|\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1}\right|_0^{\pi/2} - \left|\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1}\right|_{\pi/2}^{\pi}\right\}$$

$$= \frac{1}{\pi}\left[\left\{\frac{\sin(n+1)\frac{\pi}{2}}{n+1} + \frac{\sin(n-1)\frac{\pi}{2}}{n-1}\right\} + \left\{\frac{\sin(n+1)\frac{\pi}{2}}{n+1} + \frac{\sin(n-1)\frac{\pi}{2}}{n-1}\right\}\right]$$

$$= \frac{2}{\pi}\left(\frac{\cos\frac{n\pi}{2}}{n+1} - \frac{\cos\frac{n\pi}{2}}{n-1}\right) = \frac{-4\cos\frac{n\pi}{2}}{\pi(n^2-1)}. \qquad (n \neq 1)$$

In particular, $a_1 = \frac{2}{\pi}\left[\int_0^{\pi/2} \cos^2 x dx - \int_{\pi/2}^{\pi} \cos^2 x dx\right] = 0$.

Substitute these values in (i), we get the required Fourier series

$$|\cos x| = \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{-4\cos\frac{n\pi}{2}}{\pi(n^2-1)} \cos nx$$

$$\Rightarrow |\cos x| = \frac{2}{\pi} + \frac{4}{\pi}\left\{\frac{1}{3}\cos 2x - \frac{1}{15}\cos 4x + \dots\dots\dots\dots\right\}.$$ **Ans.**

Q.No.4.: Obtain Fourier series for the function f(x) given by

$$f(x) = 1 + \frac{2x}{\pi}, \quad -\pi \le x \le 0,$$

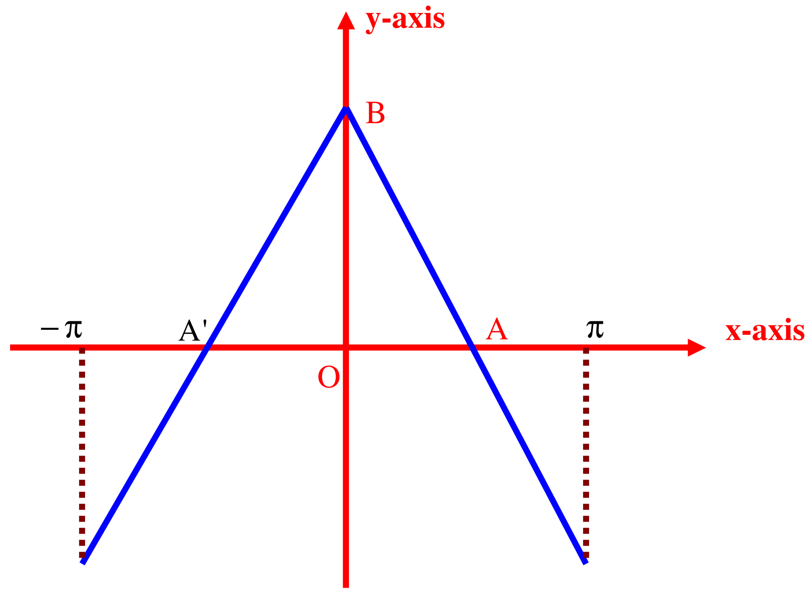
$$= 1 - \frac{2x}{\pi}, \quad 0 \le x \le \pi,$$

and hence deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots\dots\dots = \frac{\pi^2}{8}$.

Sol.: Since $f(-x) = 1 - \frac{2x}{\pi}$ in $(-\pi, 0) = f(x)$ in $(0, \pi)$,

and $f(-x) = 1 + \frac{2x}{\pi}$ in $(0, \pi) = f(x)$ in $(-\pi, 0)$.

$\therefore f(x)$ is an **even function** in $(-\pi, \pi)$.



This is also clear from its graph $A'BA$ which is symmetrical about the y-axis.

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx. \qquad \text{(i)}$$

Here $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx = \frac{2}{\pi}\int_0^{\pi}\left(1-\frac{2x}{\pi}\right)dx = \frac{2}{\pi}\left(x-\frac{x^2}{\pi}\right)_0^{\pi} = 0.$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx dx = \frac{2}{\pi}\int_0^{\pi}\left(1-\frac{2x}{\pi}\right)\cos nx dx$$

$$= \frac{2}{\pi}\left|\left(1-\frac{2x}{\pi}\right)\frac{\sin nx}{n} - \left(-\frac{2}{\pi}\right)\left(-\frac{\cos nx}{n^2}\right)\right|_0^{\pi}$$

$$= \frac{2}{\pi}\left(-\frac{2\cos n\pi}{\pi n^2} + \frac{2}{\pi n^2}\right) = \frac{4}{n^2\pi^2}\left[1-(-1)^n\right]$$

$\therefore a_1 = \frac{8}{\pi^2},\ a_3 = \frac{8}{3^2\pi^2},\ a_5 = \frac{8}{5^2\pi^2}, ...$ and $a_2 = a_4 = a_6 = ... = 0.$

Thus substituting the values of a_i's in (i), we get the required Fourier expansion

$$f(x) = \frac{8}{\pi^2}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + ...\right]. \qquad \text{(ii)}$$

2nd Part: Deduce that $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\ldots\ldots\ldots=\frac{\pi^2}{8}$.

Putting x = 0 in (ii), we get

$$1=f(x)=\frac{8}{\pi^2}\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\ldots\ldots\ldots\right)$$

$$\Rightarrow \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\ldots\ldots\ldots=\frac{\pi^2}{8}.$$

which is the required result.

Q.No.5.: Show that, for $-\pi<x<\pi$,

$$\sin ax=\frac{2\sin a\pi}{\pi}\left[\frac{\sin x}{1^2-a^2}-\frac{2\sin 2x}{2^2-a^2}+\frac{3\sin 3x}{3^2-a^2}-\ldots\ldots\ldots\ldots\right]$$

Sol.: Here $f(x)=\sin ax$.

As $f(-x)=-f(x)$ in $(-\pi,\pi)$. It is an **odd function**. $\Rightarrow a_0=a_n=0$.

$$\therefore f(x)=\sum_{n=1}^{\infty} b_n\sin nx. \qquad \text{(i)}$$

Here $b_n=\frac{2}{\pi}\int_0^{\pi}\sin ax\sin nx dx=\frac{1}{\pi}\int_0^{\pi}[\cos(a-n)x-\cos(a+n)x]dx$

$$=\frac{1}{\pi}\left|\frac{\sin(a-n)x}{a-n}-\frac{\sin(a+n)x}{a+n}\right|_0^{\pi}=\frac{1}{\pi}\left[\frac{\sin(a-n)\pi}{a-n}-\frac{\sin(a+n)\pi}{a+n}\right]$$

$$=\frac{1}{\pi}\left[\frac{\sin a\pi\cos n\pi-\cos a\pi\sin n\pi}{a-n}-\frac{\sin a\pi\cos n\pi+\cos a\pi\sin n\pi}{a+n}\right]$$

$$=\frac{1}{\pi}\left[\frac{(-1)^n\sin a\pi}{a-n}-\frac{(-1)^n\sin a\pi}{a+n}\right]=\frac{(-1)^n}{\pi}\sin a\pi\left[\frac{a+n-a+n}{a^2-n^2}\right]=\frac{2n(-1)^{n+1}}{\pi(n^2-a^2)}\sin(a\pi)$$

$$\therefore b_1=\frac{2\sin a\pi}{\pi(1^2-a^2)},\quad b_2=\frac{-2.2\sin a\pi}{\pi(2^2-a^2)},\quad b_3=\frac{2.3\sin a\pi}{\pi(3^2-a^2)},\ldots\ldots\ \text{etc.}$$

Thus substituting the values of b_i's in (i), we get the required Fourier expansion

$$f(x)=\frac{2\sin a\pi}{\pi}\left[\frac{\sin x}{1^2-a^2}-\frac{2\sin 2x}{2^2-a^2}+\frac{3\sin 3x}{3^2-a^2}-\ldots\ldots\ldots\ldots\right].$$

Q.No.6.: Expand the function $f(x)=x\sin x$ as a Fourier series in the interval $-\pi\le x\le\pi$, and hence deduce that $\frac{1}{1.3}-\frac{1}{3.5}+\frac{1}{5.7}-\frac{1}{7.9}+.................=\frac{1}{4}(\pi-2)$.

Sol.: Here $f(x)=x\sin x$.

As $f(-x)=f(x)$, $\therefore$ it is an **even function** $\Rightarrow b_n=0$.

$$\therefore f(x)=\frac{a_0}{2}+\sum a_n\cos nx . \qquad \text{(i)}$$

Here $a_0=\frac{2}{\pi}\int_0^{\pi} x\sin x dx=\frac{2}{\pi}\left[(-x\cos x)_0^{\pi}+\int_0^{\pi}\cos x dx\right]$

$$=\frac{2}{\pi}\left|-x\cos x+\sin x dx\right|_0^{\pi} \quad =\frac{2}{\pi}[\pi]=2 .$$

$$a_n=\frac{2}{\pi}\int_0^{\pi} x\sin x\cos nx dx=\frac{1}{\pi}\int_0^{\pi} x[\sin(1+n)x+\sin(1-n)x]dx$$

$$=\frac{1}{\pi}\left[\begin{array}{l}\left(-\frac{x\cos(1+x)x}{1+n}\right)_0^{\pi}+\frac{1}{1+n}\int_0^{\pi}\cos(1+n)xdx\\ -\left(\frac{x\cos(1-x)x}{1-n}\right)_0^{\pi}+\frac{1}{1-n}\int_0^{\pi}\cos(1-n)xdx\end{array}\right]$$

$$=\frac{1}{\pi}\left[-\frac{x\cos(1+x)x}{1+n}+\frac{\sin(1+n)x}{(1+n)^2}-\frac{x\cos(1-x)x}{1-n}+\frac{\sin(1-n)x}{(1-n)^2}\right]_0^{\pi}$$

$$=\frac{1}{\pi}\left[-\frac{\pi(-1)^{1+n}}{1+n}-\frac{\pi(-1)^{1+n}}{1-n}\right]=-\frac{\pi(-1)^{1+n}}{\pi}\left[\frac{1-n+1+n}{1-n^2}\right].$$

$$\Rightarrow a_n=\frac{2}{n^2-1}(-1)^{1+n} . \qquad [n>1]$$

When n = 1, we get

$$a_1=\frac{2}{\pi}\int_0^{\pi} x\sin x\cos x dx=\frac{1}{\pi}\int_0^{\pi} x\sin 2x dx=\frac{1}{\pi}\left[\left(-\frac{x\cos 2x}{2}\right)_0^{\pi}+\frac{1}{2}\int_0^{\pi}\cos 2x dx\right]$$

$$=\frac{1}{\pi}\left[-\frac{x\cos 2x}{2}+\frac{\sin 2x}{4}\right]_0^{\pi}=\frac{1}{\pi}\left[-\frac{\pi}{2}\right]=-\frac{1}{2}.$$

Substituting the values of a_i's in (i), we get the required Fourier expansion

$$\therefore a_1 = -\frac{1}{2},\ \ a_2 = -\frac{2}{1.3},\ \ a_3 = \frac{2}{8} = \frac{2}{2.4}$$

$$\therefore f(x) = x\sin x = 1 - \frac{1}{2}\cos x - \frac{2}{1.3}\cos 2x + \frac{2}{2.4}\cos 3x - \frac{2}{3.5}\cos 4x + \ldots\ldots\ldots\ldots$$

2nd Part: Deduce that $\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \frac{1}{7.9} + \ldots\ldots\ldots\ldots\ldots\ldots = \frac{1}{4}(\pi - 2)$.

Put $x = \frac{\pi}{2}$, we get

$$\frac{\pi}{2} = 1 + \frac{2}{1.3} - \frac{2}{3.5} + \frac{2}{5.7} - \frac{2}{7.9} + \ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{\pi}{2} - 1 = 2\left[\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \frac{1}{7.9} + \ldots\ldots\ldots\ldots\ldots\ldots\right]$$

$$\frac{1}{4}(\pi - 1) = \left[\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \frac{1}{7.9} + \ldots\ldots\ldots\ldots\ldots\ldots\right].\ \text{Ans.}$$

Q.No.7.: Prove that in the interval $-\pi < x < \pi$, $x\cos x = -\frac{1}{2}\sin x + 2\sum_{n=2}^{\infty} \frac{(-1)^n}{n^2 - 1}\sin nx$.

Sol. Here $f(x) = x\cos x$.

As $f(-x) = -f(x) \Rightarrow$ it is an **odd function** $\Rightarrow a_0 = a_1 = 0$.

$$\therefore f(x) = \sum_{n=1}^{\infty} b_n \sin nx. \qquad \text{(i)}$$

Here $b_n = \frac{2}{\pi}\int_0^{\pi} x\cos x \sin nx dx = \frac{1}{\pi}\int_0^{\pi} x[\sin(1+n)x + \sin(1-n)x]dx$

$$= \frac{1}{\pi}\left[\begin{array}{l}\left(\frac{-\cos(1+n)x}{1+n}.x\right)_0^{\pi} + \frac{1}{1+n}\int_0^{\pi}\cos(1+n)x dx \\ -\left(\frac{\cos(n-1)x}{n-1}.x\right)_0^{\pi} + \frac{1}{n-1}\int_0^{\pi}\cos(n-1)x dx\end{array}\right]$$

$$= \frac{1}{\pi}\left[\frac{-\cos(1+n)x}{1+n}.x + \frac{\sin(1+n)x}{(1+n)^2} - \frac{\cos(n-1)x}{n-1}.x + \frac{\cos(n-1)x}{(n-1)^2}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-\pi\cos(1+n)\pi}{1+n} - \frac{\cos(n-1)x}{n-1}\right] = -\frac{\pi}{\pi}\left[\frac{(-1)^{1+n}}{1+n} + \frac{(-1)^{1+n}}{n-1}\right]$$

$$= -(-1)^{1+n}\frac{2}{n^2-1} = (-1)^n\frac{2}{n^2-1}. \quad [n>1]$$

When n = 1, we get

$$b_1 = \frac{2}{\pi}\int_0^{\pi} x\cos x\sin x dx = \frac{1}{\pi}\int_0^{\pi} x\sin 2x dx = \frac{1}{\pi}\left[\left(-\frac{x}{2}\cos 2x\right)_0^{\pi} + \frac{1}{2}\int_0^{\pi} \cos 2x dx\right]$$

$$= \frac{1}{\pi}\left|-\frac{x}{2}\cos 2x + \frac{\sin 2x}{4}\right|_0^{\pi} = \frac{1}{\pi}\left[-\frac{\pi}{2}\right] = -\frac{1}{2}.$$

Substituting the values of b_i's in (i), we get the required Fourier series expansion

$$\therefore f(x) = x\cos x = -\frac{1}{2}\sin x + 2\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2-1}\sin nx \text{ . Ans.}$$

Q.No.8.: For a function f(x) defined by $f(x) = |x|, \; -\pi < x < \pi$, obtain a Fourier series,

and hence deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = \frac{\pi^2}{8}$.

Sol.: Here $f(x) = |x|, \; -\pi < x < \pi$.

As $f(-x) = f(x)$, $\therefore$ it is an **even function** $\Rightarrow b_n = 0$.

$$\therefore f(x) = \frac{a_0}{2} + \sum a_n \cos nx . \qquad \text{(i)}$$

Here $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx = \frac{2}{\pi}\int_0^{\pi} x dx = \frac{2}{\pi}\left|\frac{x^2}{2}\right|_0^{\pi} = \pi$.

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx dx = \frac{2}{\pi}\int_0^{\pi} x\cos nx dx = \frac{2}{\pi}\left|\left(\frac{x\sin nx}{n}\right)_0^{\pi} - \frac{1}{n}\int_0^{\pi} \sin nx dx\right|$$

$$= \frac{2}{\pi}\left|\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right|_0^{\pi} = \frac{2}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right].$$

$$\therefore a_1 = \frac{2}{1^2\pi}(-2),\; a_2 = 0,\; a_3 = \frac{2}{3^2\pi}(-2),\; a_4 = 0,\; a_5 = \frac{2}{5^2\pi}(-2), \ldots\ldots\ldots\text{etc.}$$

Substituting the values of a_i's in (i), we get the required Fourier series expansion

$$\therefore f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left[\cos x + \frac{\cos 5x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\ldots\ldots\ldots\ldots\right].$$

2nd Part: Deduce that $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}\ldots\ldots\ldots\ldots\ldots = \frac{\pi^2}{8}$.

Put $x = 0$, then $f(x) = 0$.

$$\therefore 0 = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}\ldots\ldots\ldots\ldots\right]$$

$$\Rightarrow \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}\ldots\ldots\ldots\ldots\ldots = \frac{\pi^2}{8}.$$

Q.No.9.: Find the Fourier series to represent the function $f(x) = |\sin x|$, $-\pi < x < \pi$

Sol.: $f(-x) = \sin x = f(x)$, $\therefore$ it is an **even function** $\Rightarrow b_n = 0$.

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx . \qquad (i)$$

Here $a_0 = \frac{2}{\pi}\int_0^{\pi} \sin x dx = \frac{2}{\pi}[-\cos x]_0^{\pi} = \frac{2}{\pi}[1+1] = \frac{4}{\pi}$.

$$a_n = \frac{2}{\pi}\int_0^{\pi} \sin x \cos nx dx = \frac{1}{\pi}\int_0^{\pi} [\sin(n+1)x - \sin(n-1)x]dx$$

$$= \frac{1}{\pi}\left[\frac{-\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right]_0^{\pi} = \frac{1}{\pi}\left[\frac{-\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$= \frac{1}{\pi}\left[\frac{1}{n+1} - \frac{1}{n-1} - \frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1}\right] = \frac{1}{\pi}\left[\frac{n-1-n-1}{n^2-1} - \frac{(-1)^n(-n+1+n+1)}{n^2-1}\right]$$

$$= \frac{-2}{\pi}\left[\frac{1}{n^2-1} + \frac{(-1)^n}{n^2-1}\right] = \frac{-2}{\pi(n^2-1)}\left[(-1)^n + 1\right]$$

$$= \frac{2}{\pi(n^2-1)}\left[(-1)^{n+1} - 1\right].$$

Now $a_1 = \frac{2}{\pi}\int_0^{\pi} \sin x \cos x dx = 0$.

Also $a_2 = \frac{2}{3\pi}(-2)$, $a_3 = 0$, $a_4 = \frac{2}{15\pi}(-2)$,......etc

Substituting the values of a_i's in (i), we get the required Fourier expansion

$$\therefore f(x) = \frac{2}{\pi} - \frac{4}{\pi}\left(\frac{\cos 2x}{3} + \frac{\cos 4x}{15} + \dots\dots\dots\dots\right). \text{ Ans.}$$

Q.No.10.: Given $f(x) = -x + 1$ for $-\pi \le x \le 0$

and $= x + 1$ for $0 \le x \le \pi$.

Is the function even or odd? Find the Fourier series for f(x) and deduce the value of $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots\dots\dots\dots$

Sol.: Given $f(x) = -x + 1$ for $-\pi \le x \le 0$

and $= x + 1$ for $0 \le x \le \pi$

$\therefore f(-x) = x + 1$ in $(-\pi, 0) = f(x)$ in $(0, \pi)$

$f(-x) = -x + 1$ in $(0, \pi) = f(x)$ in $(-\pi, 0)$

$\therefore$ f(x) is an **even function** in $(-\pi, \pi)$ and is symmetrical about y-axis $\Rightarrow b_n = 0$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx. \qquad \text{(i)}$$

Here $a_0 = \frac{2}{\pi}\int_0^{\pi}(x+1)dx = \frac{2}{\pi}\left|\frac{x^2}{2} + x\right|_0^{\pi} = \frac{2}{\pi}\left[\frac{\pi^2}{2} + \pi\right] = \pi + 2$.

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx dx = \frac{2}{\pi}\int_0^{\pi} (x+1)\cos nx dx = \frac{2}{\pi}\left|(x+1)\frac{\sin nx}{n} + \frac{\cos nx}{n^2}\right|_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right] = \frac{2}{n^2\pi}\left[(-1)^n - 1\right].$$

$$\therefore a_1 = -\frac{4}{1^2.\pi},\ a_2 = 0,\ a_3 = -\frac{4}{3^2.\pi},\ a_4 = 0,\ a_5 = -\frac{4}{5^2.\pi}\ \dots\dots\text{etc}$$

Substituting the values of a_i's in (i), we get the required Fourier series expansion

$$f(x) = \frac{\pi + 2}{2} - \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots\dots\dots\dots\right]$$

2nd Part: Deduce the value of $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots\dots\dots\dots$

Put x = 0, we get

$$f(0)=1=\frac{\pi}{2}+1-\frac{4}{\pi}\left[\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\ldots\ldots\ldots\ldots\right]$$

$$\Rightarrow \frac{\pi}{2}=\frac{4}{\pi}\left[\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\ldots\ldots\ldots\ldots\right]$$

$$\Rightarrow \frac{\pi^2}{8}=\left[\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\ldots\ldots\ldots\ldots\right]. \text{ Ans.}$$

This is the required expression.

Q.No.11.: Find the Fourier series of the periodic function f(x):

$f(x)=-k$ when $-\pi<x<0$,

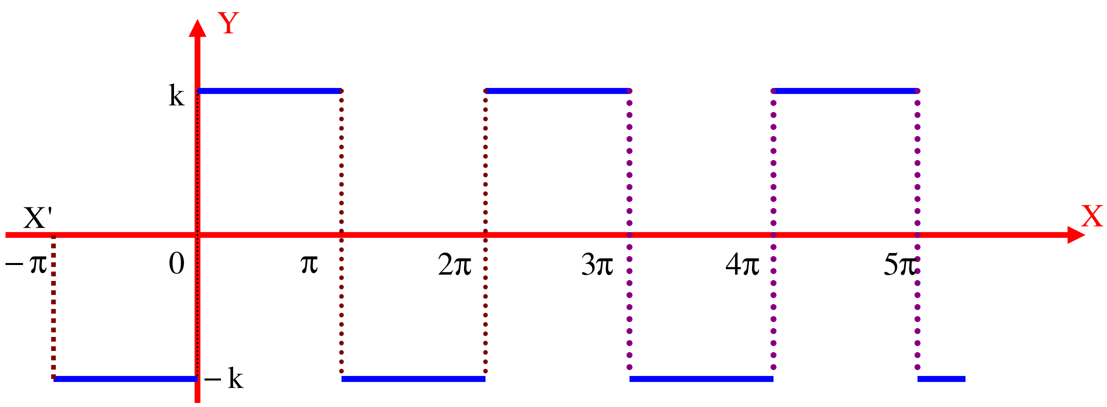
and $f(x)=k$ when $0<x<\pi$,

and $f(x+2\pi)=f(x)$.

Sketch the graph of two partial sums. Also deduce that $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1}=\frac{\pi}{4}$.

Sol.: As $f(x)=-k$ in $(-\pi,0)$

and $f(x)=k$ in $(0,\pi)$,



$\therefore$ $f(x)$ is an **odd function** $\Rightarrow a_0=a_n=0$.

$$\therefore f(x)=\sum_{n=1}^{\infty} b_n \sin nx . \tag{i}$$

Here $b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx=\frac{1}{\pi}\left[\int_{-\pi}^{0} -k\sin nx dx+\int_{-\pi}^{0} k\sin nx dx\right]$

$$= \frac{1}{\pi}\left[\left|\frac{k\cos nx}{n}\right|_{-\pi}^{0} - \left|\frac{k\cos nx}{n}\right|_{\pi}^{0}\right] = \frac{1}{\pi}\left[\frac{k}{n} - \frac{k(-1)^n}{n} - \frac{(-1)^n k}{n} + \frac{k}{n}\right]$$

$$= \frac{2k}{n\pi}\left[1-(-1)^n\right].$$

$$\therefore\ b_1 = \frac{4k}{\pi},\ b_2 = 0,\ b_3 = \frac{4k}{3\pi},\quad b_4 = 0,\ b_5 = \frac{4k}{5\pi}\ ....\text{etc.}$$

Substituting the values of b_i's in (i), we get the required Fourier expansion

$$f(x) = \sum_{n=1}^{\infty} \frac{2k}{n\pi}\left[1-(-1)^n\right]\sin nx$$

$$\Rightarrow f(x) = \frac{4k}{\pi}\left[\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + ...\right].$$

2nd Part: Deduce that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} = \frac{\pi}{4}$.

Put $x = \frac{\pi}{2}$, we get

$$k = \frac{4k}{\pi}\left[\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - ....\right] \qquad \left[\text{at } x = \frac{\pi}{2},\ f(x) = k\right]$$

$$\Rightarrow \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - ....$$

$$\Rightarrow \frac{\pi}{4} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}.\ \text{Ans.}$$

which is the required result.

Q.No.12.: A function is defined as $f(x) = -x$ when $-\pi < x \le 0$

$= x$ when $0 < x < \pi$.

Show that $f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{1}{1^2}\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + ...\right]$,

and hence deduce that $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$.

Sol.: Since $f(-x) = x$ in $(-\pi, 0) = f(x)$ in $(0, \pi)$

$f(-x) = -x$ in $(0, \pi) = f(x)$ in $(-\pi, 0)$, $\therefore f(x)$ is an **even function** $\Rightarrow b_n = 0$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx. \qquad \text{(i)}$$

Here $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} -xdx + \int_{0}^{\pi} xdx\right] = \frac{1}{\pi}\left[\left|\frac{-x^2}{2}\right|_{-\pi}^{0} + \left|\frac{x^2}{2}\right|_{0}^{\pi}\right]$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2} + \frac{\pi^2}{2}\right] = \pi.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx = \frac{1}{\pi}\left[\int_{-\pi}^{0} -x\cos nxdx + \int_{0}^{\pi} x\cos nxdx\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{-\sin nx}{n}.x\right)_{-\pi}^{0} + \frac{1}{n}\int_{-\pi}^{0} \sin nxdx\right] + \left[\left(\frac{x.\sin x}{n}\right)_{0}^{\pi} - \frac{1}{n}\int_{0}^{\pi} \sin nxdx\right]$$

$$= \frac{1}{\pi}\left[\left|\frac{-x\sin nx}{n} - \frac{\cos nx}{n^2}\right|_{-\pi}^{0} + \left|\frac{x.\sin nx}{n} + \frac{\cos nx}{n^2}\right|_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{-1}{n^2} + \frac{(-1)^n}{n^2} + \frac{(-1)^n}{n^2} - \frac{1}{n^2}\right] = \frac{2}{\pi}\left[\frac{(-1)^n - 1}{n^2}\right].$$

In particular, $a_1 = \frac{2}{1^2.\pi}(-2)$, $a_2 = 0$, $a_3 = \frac{2}{3^2.\pi}(-2)$, $a_4 = 0$, $a_5 = \frac{2}{5^2.\pi}(-2)$,….

Substituting the values of a_i's in (i), we get the required Fourier series expansion

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots\dots\dots\right].$$

2nd Part: Deduce that $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$.

Put x = 0, we get

$$0 = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots\dots\dots\right]$$

$$\Rightarrow \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots\dots\dots$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8},$$

which is the required expression.

Q.No.13.: Expand the function $f(x)=\frac{x^2}{12}-\frac{x^2}{4}$ in Fourier series in the interval $(-\pi,\ \pi)$.

Sol.: Here f(x) is an **even function**, since $f(-x)=f(x)$.

The Fourier series reduces to Fourier cosine series given by

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_0 \cos nx. \tag{i}$$

Here $a_0=\frac{2}{\pi}\int_0^{\pi} f(x)dx=\frac{2}{\pi}\int_0^{\pi}\left(\frac{\pi^2}{12}-\frac{x^2}{4}\right)dx=\frac{2}{\pi}\left[\frac{\pi^2}{12}x-\frac{x^3}{12}\right]_0^{\pi}=0.$

$$a_n=\frac{2}{\pi}\int_0^{\pi}\left(\frac{x^2}{12}-\frac{x^2}{4}\right)\cos nxdx$$

$$=\frac{2}{\pi}\left[\left(\frac{\pi^2}{12}-\frac{x^2}{4}\right)\frac{\sin nx}{n}-\left(\frac{-2x}{4}\right)\times\left(\frac{-\cos nx}{n^2}\right)+\left(-\frac{1}{2}\right)\left(\frac{-\sin nx}{n^3}\right)\right].$$

$$a_n=\frac{2}{\pi}\left(-\frac{1}{n^2}\right)\frac{\pi}{2}.\cos n\pi=\frac{(-1)^{n+1}}{n^2}.$$

Substituting these values in (i), we get the required Fourier series expansion

$$f(x)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}.\cos nx=\cos x-\frac{\cos 2x}{2^2}+\frac{\cos 3x}{3^2}-\frac{\cos 4x}{4^2}+.........$$

Q.No.14.: Find the Fourier series $f(x)=x^3$ in $(-\pi,\ \pi)$

Sol.: Since f(x) is odd function (i.e., $f(-x)=f(x)$).

The series reduces to Fourier sine series given by

$$f(x)=\sum_{n=1}^{\infty} b_n \sin nx. \tag{i}$$

Here $b_n=\frac{2}{\pi}\int_0^{\pi} f(x)\sin nxdx=\frac{2}{\pi}\int_0^{\pi}\ \sin nxdx$

$$=\frac{2}{\pi}\left[x^3\left(\frac{-\cos nx}{n}\right)-3x^2\left(\frac{-\sin nx}{n^2}\right)+6x(-1)(-)\frac{\cos nx}{n^3}\frac{-6}{n^4}\sin nx\right]_0^{\pi}.$$

$$b_n = \frac{2}{\pi}\left[\frac{-\pi^3}{n}\cos n\pi + \frac{6\pi}{n^3}\cos n\pi\right] = 2(-1)^n\left[\frac{-\pi^2}{n} + \frac{6}{n^3}\right].$$

Substituting these values in (i), we get the required Fourier series expansion

$$f(x) = 2\sum_{n=1}^{\infty}\left(\frac{6}{n^3} - \frac{\pi^2}{n}\right)(-1)^n \sin nx = 2\left[\left(\frac{\pi^2}{1} - \frac{6}{1^3}\right)\sin x - \left(\frac{\pi^2}{2} - \frac{6}{2^3}\right)\sin 2x + ...\right].$$

Q.No.15.: Find the Fourier series of f(x) defined

$$f(x) = 0 \qquad \text{when } -c < x < 0$$
$$= 1 \qquad \text{when } 0 < x < c$$

Find the value of Fourier series at the point of discontinuity,

Sol.: The given interval of length 2c. The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{c} + b_n \sin\frac{n\pi x}{c}\right).$$

Here $a_0 = \frac{1}{c}\int_{-c}^{c} f(x)dx = \frac{1}{c}\int_{-c}^{0} 0.dx + \frac{1}{c}\int_{0}^{c} 1.dx = \frac{1}{c}.x\Big|_0^c = 1.$

$$a_0 = \frac{1}{c}\int_{-c}^{c} f(x)\cos\frac{n\pi x}{c}dx = \frac{1}{c}\int_{0}^{c} \cos\frac{n\pi x}{c}dx.$$

$$a_n = \frac{1}{c}\left(\frac{c}{n\pi}\right)\sin\left(\frac{n\pi x}{c}\right)\Bigg|_0^c = \frac{1}{n\pi}[\sin n\pi - \sin 0] = 0$$

$$b_n = \frac{1}{c}\int_{-c}^{c} f(x)\sin\frac{n\pi x}{c}dx = \frac{1}{c}\int_{0}^{c} \sin\frac{n\pi x}{c}dx = \frac{1}{c}\left(\frac{c}{n\pi}\right)\left(\frac{-\cos n\pi x}{c}\right)\Bigg|_0^c = -\frac{1}{n\pi}[\cos n\pi - 1]$$

$$= \frac{1}{n\pi}\left[1 - (-1)^n\right].$$

Substituting these values in (i), we get the required Fourier series expansion

$$f(x) = \frac{1}{2} + \frac{1}{n}\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{n}\right]\sin\left(\frac{n\pi x}{c}\right).$$

2nd Part:

The sum of the series at $x_0 = 0$ is obtained by putting $x = 0$ in the above series i.e., $f(0)=\frac{1}{2}+\frac{1}{n}.0=\frac{1}{2}$. At a point of discontinuity $x_0 = 0$,

$$f(0)=\frac{1}{2}\left[f(0+0)+f(0-0)\right]=\frac{1}{2}[1+0]=\frac{1}{2}.$$

Q.No.16.: Find the Fourier series of $f(x)=\begin{cases} 0 & \text{if } -2\le t\le -1 \\ 1+t & \text{if } -1\le t\le 0 \\ 1-t & \text{if } 0\le t\le 1 \\ 0 & \text{if } 1\le t\le 2. \end{cases}$

Sol.: Here f(t) is defined in the interval $(-2,\ 2)$ of length 2L = 4 (i.e., L = 2).

Observe that f(t) is an **even function** (symmetric about y-axis).

Fourier series reduces Fourier cosine series (with all $b_n = 0$).

Thus $f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n.\cos\frac{n\pi t}{2}$.

Here $a_0=\frac{2}{L}\int_0^L f(t)dt=\frac{2}{L}\int_0^2 f(t)dt=\int_0^1 (1-t)dt+\int_1^2 0.dt=\left(\frac{t-t^2}{2}\right)\Bigg|_0^1=\frac{1}{2}$.

$$a_n=\frac{2}{L}\int_0^L f(t)\cos\left(\frac{n\pi t}{2}\right)dt=\frac{2}{2}\int_0^L f(t)\cos\left(\frac{n\pi t}{2}\right)dt=\int_0^1 f(1-t)\cos\left(\frac{n\pi t}{2}\right)dt+0$$

$$=\left[(1-t).\frac{2}{n\pi}.\sin\left(\frac{n\pi t}{2}\right)-(-1)\frac{2}{n\pi}.\frac{2}{n\pi}.[-1]\cos\left(\frac{n\pi t}{2}\right)\right]_0^1.$$

$$a_n=\frac{-4}{n^2\pi^2}\left[\cos\frac{n\pi}{2}-1\right]$$

$= 0$ when $n = 4, 8, 12,\ldots..$

$=\frac{8}{n^2\pi^2}$ when $n = 2, 6, 10,\ldots\ldots$

$=\frac{4}{n^2\pi^2}$ when $n = 1, 3, 5,\ldots..$

Substituting these values in (i), we get the required Fourier series expansion

$$f(t)=\frac{1}{4}+\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1-\cos\left(\frac{n\pi}{2}\right)}{n^2}.\cos\left(\frac{n\pi t}{2}\right).$$

Q.No.17.: Obtain Fourier series expansion of $f(x) = x.\cos\left(\frac{\pi x}{L}\right)$ in the interval

$-L \le x \le L$.

Sol.: Here f(x) is an odd functions (since 0. e = 0) in the interval $(-L, L)$ of length 2L.

So a_0 and a_n's are zero.

The Fourier series is $f(x) = \sum_{n=1}^{\infty} b_n.\sin\left(\frac{n\pi x}{L}\right)$.

For n =1,

$$b_1 = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{\pi x}{L}\right)dx . = \frac{2}{L}\int_0^L x.\cos\left(\frac{\pi x}{L}\right).\sin\left(\frac{\pi x}{L}\right)dx = \frac{2}{L}.\frac{L}{x}.\int_0^L x.\sin\left(\frac{\pi x}{L}\right)d\left(\frac{\sin \pi x}{L}\right)$$

$$= \frac{2}{\pi}\int_0^L \frac{x}{2}.d\left(\frac{\sin^2 \pi x}{L}\right) = \frac{1}{\pi}\left[x.\sin^2\frac{\pi x}{2}\Big|_0^L - \sin^2\frac{\pi x}{L}dx\right] = -\frac{1}{\pi}\int_0^L \sin^2\frac{\pi x}{L}dx$$

$$= -\frac{1}{\pi}\int_0^L \frac{1}{2}\left(1 - \cos\frac{2\pi x}{L}\right)dx = -\frac{1}{2\pi}\left[x - \frac{L}{2\pi}.\sin\frac{2\pi x}{L}\right]_0^L$$

So $b_1 = -\frac{L}{2\pi}$.

For $n \ne 1$

$$b_n = \frac{2}{L}\int_0^L \left(x.\cos\frac{\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\int_0^L x\frac{1}{2}\left[\sin\frac{(n-1)\pi x}{L} + \sin\frac{(n+1)\pi x}{L}\right]dx$$

$$= I_1 + I_2.$$

Here $I_1 = \frac{1}{L}\int_0^L x.\sin\left(\frac{(n-1)\pi x}{L}\right)dx$

$$= \frac{1}{L}\left[x.\frac{L}{(n-1)\pi}\left(-\cos\frac{(n-1)\pi x}{L}\right) - 1.\frac{L^2}{(n-1)^2\pi^2}(-1).\sin\frac{(n-1)\pi x}{L}\right]_0^L$$

$$I_1 = \frac{L}{\pi(n-1)}(-1)(-1)^{n-1}.$$

Similarly,

$$I_2 = \frac{1}{L}\int_0^L x.\sin\left(\frac{(n+1)\pi x}{L}\right)dx$$

$$= \frac{1}{L}\left[x.\frac{L}{(n+1)\pi}\left(-\cos\frac{(n+1)\pi x}{L}\right) - \frac{L^2}{(n+1)^2\pi^2}(-1).\sin\frac{(n+1)\pi x}{L}\right]_0^L$$

$$I_1 = \frac{L}{\pi(n+1)}(-1)(-1)^{n+1}.$$

Thus $b_n = I_1 + I_2 = \frac{L(-1)^n}{n}\left[\frac{1}{n-1} + \frac{1}{n+1}\right] \quad = \frac{2Ln}{\pi}\frac{(-1)^n}{(n^2-1)}.$

Substituting these values in (i), we get the required Fourier series expansion

$$f(x) = -\frac{L}{2\pi}\sin\left(\frac{\pi x}{L}\right) + \frac{2L}{\pi}\sum_{n=2}^{\infty}\frac{(-1)^2 n}{n^2-1}\sin\left(\frac{n\pi x}{L}\right).$$

Home Assignments

Q.No.1.: Find the Fourier series of the following function f(x) in the indicated interval.

f(x) is periodic with period 2π.

$$f(x) = -\frac{(\pi + x)}{2} \quad \text{for} \quad -\pi \le x < 0$$

$$= \frac{(\pi - x)}{2} \quad \text{for} \quad 0 \le x < \pi.$$

Ans.: **Odd function**, $f(x) = \sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \ldots\ldots\ldots\ldots$

Q.No.2.: Find the Fourier series of the following function f(x) in the indicated interval.

f(x) is periodic with period 2π.

$f(x) = \sin ax$ in $(-\pi, \ \pi)$, where a is not an integer.

Ans.: **Odd function**, $f(x) = \frac{2\sin a\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n n}{a^2 - n^2}\sin nx$.

Q.No.3.: Find the Fourier series of the following function f(x) in the indicated interval.

f(x) is periodic with period 2c.

$$f(x) = -a \quad \text{when} \quad -c < x < 0$$

$$= a \quad \text{when} \quad 0 < x < c.$$

Ans.: Odd function, $f(x)=\frac{2a}{\pi}\sum_{n=1}^{\infty}\left(\frac{1-\cos n\pi}{n}\right).\sin\left(\frac{n\pi x}{c}\right)$.

Q.No.4.: Find the Fourier series of the following function f(x) in the indicated interval.

f(x) is periodic with period 2L.

$f(x)=\sin ax$ in $(-L, L)$.

Ans.: Odd function, $f(x)=2\pi\sin aL\sum_{n=1}^{\infty}\frac{n(-1)^{n+1}}{n^2\pi^2-a^2L^2}\sin\left(\frac{n\pi x}{L}\right)$.

Q.No.5.: Find the Fourier series of the following function f(x) in the indicated interval.

f(x) is periodic with period 2π.

$f(x)=\cos ax$ in $(-\pi,\ \pi)$, where a is not an integer.

Ans.: Even function, $f(x)=\frac{\sin a\pi}{a\pi}+\frac{2a\sin a\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{a^2-n^2}\cos nx$.

Q.No.6.: Find the Fourier series of the following function f(x) in the indicated interval.

f(x) is periodic with period 2π.

$f(x)=\sqrt{1-\cos x}$ in $(-\pi,\ \pi)$,

$-\sqrt{2}\sin\frac{x}{2}$ in $(-\pi,\ 0)$ and

$\sqrt{2}\sin\left(\frac{x}{2}\right)$ in $(0, \pi)$

Ans.: Even function, $f(x)=\frac{2\sqrt{2}}{\pi}-\frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{\cos nx}{(4n^2-1)}$.

Q.No.7.: Find the Fourier series of the following function f(x) in the indicated interval.

f(x) is periodic with period 4.

$f(t)=4-t^2$ in $(-2,\ 2)$.

Ans.: Even function, $f(t)=\frac{8}{3}+\frac{16}{\pi^2}\sum\frac{(-1)^{n+1}}{n^2}\cos\frac{n\pi t}{2}$.

Q.No.8.: Find the Fourier series of the following function f(x) in the indicated interval.

f(x) is periodic with period 2.

$f(x)=\frac{1}{2}+x$ when $-1\le x\le 0$

$$= \frac{1}{2} - x \qquad \text{when} \qquad 0 \le x \le 1$$

Ans.: **Even function**, $f(x) = \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{\cos[(2n-1)\pi x]}{(2n-1)^2}$.

Q.No.9.: Find the Fourier series of the following function f(x) in the indicated interval.

f(x) is periodic with period 2L.

$f(x) = |x|$ in $(-L, L)$.

Ans.: **Even function**, $f(x) = \frac{L}{2} - \frac{4L}{\pi^2} \sum_{n=\text{odd}} \frac{\cos\left(\frac{n\pi}{L}\right)}{n^2}$.

Q.No.10.: Find the Fourier series of the following function f(x) in the indicated interval.

f(x) is periodic with period 2L.

$f(x) = x^2$ in $(-L, L)$.

Ans.: **Even function**, $f(x) = \frac{L^2}{3} + \sum_{n=1}^{\infty} \frac{(-1)^n 4L^2}{n^2 \pi^2} \cos\left(\frac{n\pi x}{L}\right)$.

Q.No.11.: Find the Fourier series of the following function f(x) in the indicated interval.

f(x) is periodic with period 2.

$f(x) = x + x^2$ in $(-1, \ 1)$.

Hint: f(x) may be defined as the sum of an **odd function** g(x) = x and an **even function** $h(x) = x^2$. Fourier series f(x) is obtained by adding the Fourier series of g(x) and h(x).

Ans.: $f(x) = \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos n\pi x + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin n\pi x$.

Q.No.12.: Show that the familiar identity $\sin^3 x = \frac{3}{4} \sin x - \frac{1}{4} \sin 3x$ can be interpreted as a Fourier series expansion.

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6^{th} Topic

Fourier Series

'Half Range Series'

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Introduction:

Sometime it is required to obtain a Fourier expansion of a function f(x) for the range $(0,\ c)$, which is half the period of the Fourier series. As it is immaterial whatever the function may outside the range $0 < x < c$, we extend the function to cover the range $-c < x < c$ so that the new function may be even or odd. Therefore, the Fourier expansion of such a function of half the period consists of sine or cosine terms only. In such cases the graphs of the values of x in $(0,\ c)$ are the same, but outside $(0,\ c)$ are different for odd or even functions.

How can we develop the half range sine and cosine series?

Half-range sine series:

If it be required to expand f(x) as a sine series in $0 < x < c$; then we extend the function reflecting it in the origin, so that $f(-x) = -f(x)$.

Then, the extended function is odd in $(-c,\ c)$ and the expansion will give the desired Fourier sine series:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{c}, \qquad \text{(i)}$$

where $b_n = \frac{2}{c}\int_0^c f(x)\sin\frac{n\pi x}{c}dx$.

Half-range cosine series:

If it be required to expand f(x) as a cosine series in $0 < x < c$; then we extend the function reflecting it in the y-axis, so that $f(-x) = f(x)$.

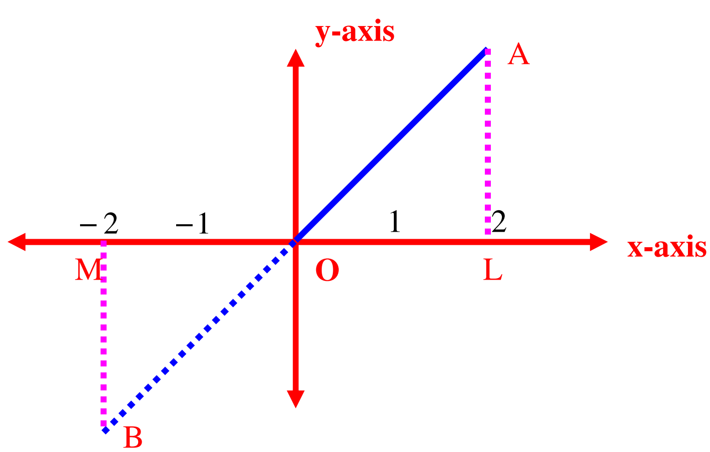
Then, the extended function is even in $(-c, c)$ and the expansion will give the desired Fourier cosine series:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{c}, \qquad \text{(ii)}$$

where $a_0 = \frac{2}{c}\int_0^c f(x)dx$, and $a_n = \frac{2}{c}\int_0^c f(x)\cos\frac{n\pi x}{c}dx$.

Q.No.1.: Express $f(x) = x$ as a **half range sine series** in $0 < x < 2$.

Sol.: The graph of $f(x) = x$ in $0 < x < 2$ is the line OA. Let us extend the function f(x) in the interval $-2 < x < 0$ (shown by the line BO), so that the **new function** is symmetrical about the origin and, therefore, represents an **odd function** in $(-2, 2)$ [see figure]



Hence, the Fourier series for f(x) over the full period $(-2, 2)$ will contain only sine terms given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2},$$

where $b_n = \frac{2}{2}\int_0^2 f(x)\sin\frac{n\pi x}{2}dx = \int_0^2 x\sin\frac{n\pi x}{2}dx$

$$= \left| -\frac{2x}{n\pi}\cos\frac{n\pi x}{2} + \frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2} \right|_0^2 = -\frac{4(-1)^n}{n\pi}.$$

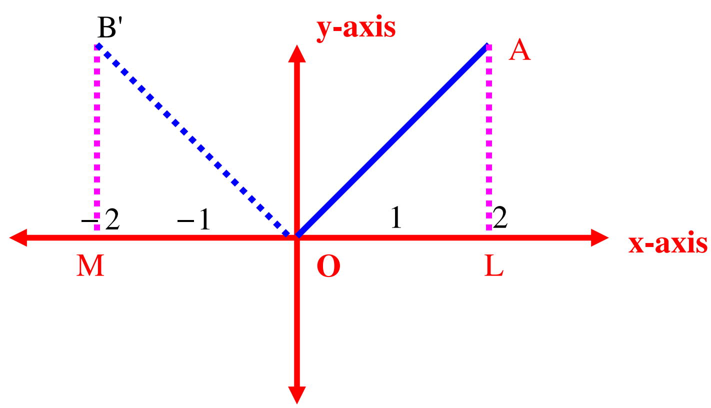
Thus $b_1 = \frac{4}{\pi}$, $b_2 = -\frac{4}{2\pi}$, $b_3 = \frac{4}{3\pi}$, $b_4 = -\frac{4}{4\pi}$ ....etc.

Hence, the Fourier sine series for f(x) over the half-range (0, 2) is

$$f(x) = \frac{4}{\pi}\left(\sin\frac{\pi x}{2} - \frac{1}{2}\sin\frac{2\pi x}{2} + \frac{1}{3}\sin\frac{3\pi x}{2} - \frac{1}{4}\sin\frac{4\pi x}{2} + ... \right).$$

Q.No.2.: Express $f(x) = x$ as a **half-range cosine series** in $0 < x < 2$.

Sol.: The graph of $f(x) = x$ in (0, 2) is the line OA. Let us extend the function f(x) in the interval $(-2, 0)$ shown by the line OB' so that the **new function** in symmetrical about the y-axis and, therefore, represents an **even function** in $(-2, 2)$ [see figure].



Hence, the Fourier series for f(x) over the full period $(-2, 2)$ will contain only cosine terms given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2},$$

where $a_0 = \frac{2}{2}\int_0^2 f(x)dx = \int_0^2 x dx = \left[\frac{x^2}{2}\right]_0^2 = 2,$

and $a_n = \frac{2}{2}\int_0^2 f(x)\cos\frac{n\pi x}{2}dx = \int_0^2 x\cos\frac{n\pi x}{2}dx = \left|\frac{2x}{n\pi}\sin\frac{n\pi x}{2} + \frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right|_0^2$

$$= \frac{4}{n^2\pi^2}\left[(-1)^n - 1\right].$$

Thus $a_1 = -\frac{8}{\pi^2},\ a_2 = 0,\ a_3 = -\frac{8}{3^2\pi^2},\ a_4 = 0,\ a_5 = -\frac{8}{5^2\pi^2}$ etc.

Hence, the Fourier cosine series for f(x) over the half-range (0, 2) is

$$f(x) = 1 - \frac{8}{\pi^2}\left[\frac{\cos\frac{\pi x}{2}}{1^2} + \frac{\cos\frac{3\pi x}{2}}{3^2} + \frac{\cos\frac{5\pi x}{2}}{5^2} + \ldots\ldots\ldots\ldots\right].$$

Q.No.3.: Expand $f(x) = \frac{1}{4} - x$ if $0 < x < \frac{1}{2}$

$= x - \frac{3}{4}$ if $\frac{1}{2} < x < 1$, as the Fourier series of **sine terms.**

Sol.: Let us extend the function f(x) in the interval $-1 < x < 0$, so that the **new function** is symmetrical about the origin and, therefore, represents an **odd function** in $(-1,\ 1)$.

Thus $f(x) = \sum_{n=1}^{\infty} b_n \sin n\pi x$.

Here $b_n = \frac{2}{1}\int_0^1 f(x)\sin n\pi x dx = 2\left[\int_0^{1/2}\left(\frac{1}{4} - x\right)\sin n\pi x dx + \int_{1/2}^1 \left(x - \frac{3}{4}\right)\sin n\pi x dx\right]$

$$= 2\left|-\left(\frac{1}{4} - x\right)\frac{\cos n\pi x}{n\pi} - \frac{\sin n\pi x}{n^2\pi^2}\right|_0^{1/2} + 2\left|-\left(x - \frac{3}{4}\right)\frac{\cos n\pi x}{n\pi} + \frac{\sin n\pi x}{n^2\pi^2}\right|_{1/2}^1$$

$$= 2\left[\frac{1}{4n\pi}\cos\frac{n\pi}{2} + \frac{1}{4n\pi} - \frac{\sin\frac{n\pi}{2}}{n^2\pi^2}\right] + 2\left[-\frac{1}{4n\pi}\cos n\pi - \frac{1}{4n\pi}\cos\frac{n\pi}{2} - \frac{\sin\frac{n\pi}{2}}{n^2\pi^2}\right]$$

$$= \frac{1}{2n\pi}\left[1 - (-1)^n\right] - \frac{4\sin\frac{n\pi}{2}}{n^2\pi^2}.$$

Thus $b_1 = \frac{1}{\pi} - \frac{4}{\pi^2},\ b_2 = 0,\ b_3 = \frac{1}{3\pi} + \frac{4}{3^2\pi^2},\ b_4 = 0,\ b_5 = \frac{1}{5\pi} - \frac{4}{5^2\pi^2},\ b_6 = 0$,...etc.

Hence, the desired Fourier series of sine terms is

$$f(x)=\left(\frac{1}{\pi}-\frac{4}{\pi^2}\right)\sin\pi x+\left(\frac{1}{3\pi}+\frac{4}{3^2\pi^2}\right)\sin3\pi x+\left(\frac{1}{5\pi}+\frac{4}{5^2\pi^2}\right)\sin5\pi x+........ \ .$$

Q.No.4.: Show that the constant c can be expanded in a infinite series

$$\frac{4c}{\pi}\left\{\sin x+\frac{\sin3x}{3}+\frac{\sin5x}{5}+...........\right\} \text{ in the range } 0<x<\pi .$$

Sol.: Let us extend the function f(x) = c in the interval $-\pi<x<0$, so that the **new function** is symmetrical about the origin and, therefore, represents an **odd function** in $(-\pi,\ \pi)$.

Thus $f(x)=\sum_{n=1}^{\infty} b_n \sin nx$,

where $b_n=\frac{2}{\pi}\int_0^{\pi} f(x)\sin nxdx=\frac{2c}{\pi}\int_0^{\pi}\sin nxdx=\frac{-2c}{n\pi}[\cos nx]_0^{\pi}=\frac{2c}{n\pi}\left[(-1)^n-1\right].$

Thus $b_1=\frac{4c}{1.\pi},\ b_2=0,\ b_3=\frac{4c}{3.\pi},\ b_4=0$,..... etc.

Hence, the required Fourier series expansion is

$$\therefore f(x)=\frac{4c}{\pi}\left[\sin x+\frac{\sin3x}{3}+\frac{\sin5x}{5}+............\right]. \text{ Ans.}$$

Q.No.5.: Obtain **cosine and sine series** for $f(x)=x$ in the interval $0\le x\le\pi$.

Hence show that $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+..........=\frac{\pi^2}{8}$.

or

Q.No.5a.: Expand f(x) = x in $(0.\pi)$ by **Fourier sine series**

Ans.: $f(x)=x$ in $(0,\pi)$

$= + x$ in $(-\pi,0)$

$$f(x)=x=2\left[\frac{\sin x}{1}-\frac{\sin2x}{3}+\frac{\sin3x}{3}.........\right]=\sum_{n=1}^{\infty}\frac{2}{n}(-1)^{n+1}\sin nx .$$

Q.No.5b.: Expand f(x) = x in $(0.\pi)$ by **Fourier cosine series.**

Ans.: $f(x)=|x|$ in $(-\pi,\pi)$

$$f(x) = x = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos x}{1} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots\dots\dots\right] = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^2}.$$

Sol.: Let us extend the function f(x) in the range $-\pi \le x \le 0$.

1st Part: To produce cosine series:

Suppose the function f(x) in the range $-\pi \le x \le 0$ represent an even function, then the Fourier series of function f(x) will contain cosine terms only given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx,$$

where $a_0 = \frac{2}{\pi}\int_0^{\pi} x dx = \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = \pi$

and $a_n = \frac{2}{\pi}\int_0^{\pi} x \cos nx dx = \frac{2}{\pi}\left[\frac{x}{n}\sin x + \frac{1}{n^2}\cos nx\right]_0^{\pi} = \frac{2}{n^2\pi}[\cos n\pi - 1] = \frac{2}{n^2\pi}\left[(-1)^n - 1\right].$

Thus $a_1 = -\frac{4}{\pi}$, $a_2 = 0$, $a_3 = -\frac{4}{\pi.3^2}$, $a_4 = 0$,.....etc.

Hence, the required Fourier series expansion is

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots\dots\dots\dots\dots\right]$$

Now put x = 0, we get

$$0 = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots\dots\dots\dots\dots\right]$$

$$\Rightarrow \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots\dots\dots,$$

which is the required result.

2nd Part: To produce sine series:

Now again suppose in the range $-\pi \le x \le 0$ function represent an odd function.

Then the Fourier series for f(x) will contain only sine terms given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx,$$

where $b_n = \frac{2}{\pi}\int_0^{\pi} x \sin nx dx = \frac{2}{\pi}\left[\frac{-x\cos nx}{n} - \frac{\sin nx}{n^2}\right]_0^{\pi} = \frac{-2\pi}{n}\cos n\pi.$

Thus $b_1 = \frac{2}{1}, \; b_2 = \frac{-2}{2}, \; b_3 = \frac{2}{3}, \; b_4 = \frac{-2}{4}$,.......etc.

Hence, the required Fourier series expansion is

$$f(x) = 2\left[\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + ...\right]. \text{ Ans.}$$

Q.No.6.: Find the **half-range cosine series** for $f(x) = x^2$ in the range $0 \le x \le \pi$.

Sol.: Let us extend the function f(x) in the interval $-\pi < x < 0$, so that the **new function** is symmetrical about the y-axis and, therefore, represents an **even function** in $(-\pi, \; \pi)$.

Thus $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$,

where $a_0 = \frac{2}{\pi}\int_0^{\pi} x^2 dx = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{2\pi^2}{3}$,

and $a_n = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx dx = \frac{2}{\pi}\left[\left|x^2 \frac{\sin n\pi}{n}\right|_0^{\pi} - \int_0^{\pi} 2x \frac{\sin nx}{n} dx\right]$

$$= \frac{2}{\pi}\left[\left(x^2 \frac{\sin n\pi}{n}\right)_0^{\pi} - 2x\left(\frac{-\cos nx}{n}\right)\Big)_0^{\pi} \int 2\left(\frac{-\cos nx}{n^2}\right) dx\right]$$

$$= \frac{2}{\pi}\left[x^2 \frac{\sin nx}{n} - 2x\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi} = \frac{2}{\pi}\left[0 + \frac{2\pi}{n^2}(-1)^n\right] = \frac{4}{n^2}(-1)^n .$$

$\therefore a_1 = -4, \; a_2 = \frac{4}{2^2}, \; a_3 = -\frac{4}{3^2}$,.......etc.

Hence, the Fourier cosine series for f(x) over the half-range $(0, \pi)$ is

$$f(x) = \frac{2\pi^2}{2.3} - 4\cos x + \frac{4}{2^2}\cos 2x - \frac{1}{3^2}4\cos 3x..................$$

$$\Rightarrow f(x) = \frac{\pi^2}{3} - 4\left[\cos x - \frac{1}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x + ...\right]. \text{ Ans.}$$

Q.No.7.: Find the **half-range cosine series** for the function $f(x) = (x-1)^2$ in the interval $0 < x \le 1$.

Hence, show that (i) $1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots\ldots=\frac{\pi^2}{6}$,

(ii) $1-\frac{1}{2^2}+\frac{1}{3^2}-\ldots\ldots=\frac{\pi^2}{12}$,

(iii) $\frac{1}{1^2}+\frac{1}{3^2}+\ldots\ldots\ldots=\frac{\pi^2}{8}$.

Sol.: Let us extend the function f(x) in the interval $(-1,0)$. New function is symmetrical about y-axis and therefore represent an **even function** in interval $(-1,1)$.

Hence the Fourier series for f(x) over the full period $(-1,1)$ will contain only cosine terms given by

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \frac{\cos n\pi x}{1}, \qquad \text{(i)}$$

where $a_0=\frac{2}{1}\int_0^1 f(x)dx=\frac{2}{1}\int_0^1 (x-1)^2 dx=2\int_0^1 (x^2+1-2x)dx=2\left[\frac{x^3}{3}+x-x^2\right]_0^1$

$$=2\left[\frac{1}{3}+1-1\right]=\frac{2}{3},$$

and $a_n=\frac{2}{1}\int_0^1 (x-1)^2\frac{\cos n\pi x}{1}dx=2\left[\left|(x-1)^2\frac{\sin n\pi x}{n\pi}\right|_0^1-\int_0^1 2(x-1)\frac{\sin n\pi x}{n\pi}dx\right]$

$$=2\left[\left|(x-1)^2\frac{\sin n\pi x}{n\pi}\right|_0^1-\left|2(x-1)\left(-\frac{\cos n\pi x}{n\pi}\right)\right|_0^1-\int_0^1\frac{\cos n\pi x}{n^2\pi^2}dx\right]$$

$$=2\left[0+0+0+\frac{2}{n^2\pi^2}-\left|\frac{2\sin n\pi x}{n^3\pi^3}\right|_0^1\right]=\frac{4}{n^2\pi^2}.$$

Hence, the Fourier cosine series for f(x) over the half-range $(0,1)$ is

$$f(x)=\frac{1}{3}+\sum_{n=1}^{\infty}\frac{4}{n^2\pi^2}\cos n\pi x$$

$$\Rightarrow (x-1)^2=\frac{1}{3}+\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos n\pi x.$$

2nd Part: Show that **(i)** $1+\frac{1}{2^2}+\frac{1}{3^2}+......=\frac{\pi^2}{6}$,

(ii) $1-\frac{1}{2^2}+\frac{1}{3^2}-......=\frac{\pi^2}{12}$,

(iii) $\frac{1}{1^2}+\frac{1}{3^2}+..........=\frac{\pi^2}{8}$.

Now since $(x-1)^2=\frac{1}{3}+\frac{4}{\pi^2}\left[\frac{\cos\pi x}{1^2}+\frac{\cos 2\pi x}{2^2}+\frac{\cos 3\pi x}{3^2}+........\right]$.

Put $x=0$, we get $f(0)=\frac{1}{3}+\frac{4}{\pi^2}\left(1+\frac{1}{2^2}+\frac{1}{3^2}+........\right)$

$\Rightarrow(0-1)^2=\frac{1}{3}+\frac{4}{\pi^2}\left[\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+........\right]\Rightarrow 1=\frac{1}{3}+\frac{4}{\pi^2}\left[\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+........\right]$.

$\Rightarrow 1+\frac{1}{2^2}+\frac{1}{3^2}+......=\frac{\pi^2}{6}$.

Again put $x=1$, we get $f(1)=\frac{1}{3}+\frac{4}{\pi^2}\left(-1+\frac{1}{2^2}-\frac{1}{3^2}+........\right)$

$\Rightarrow(1-1)^2=\frac{1}{3}+\frac{4}{\pi^2}\left[-\frac{1}{1^2}+\frac{1}{2^2}-\frac{1}{3^2}+........\right]\Rightarrow 0=\frac{1}{3}+\frac{4}{\pi^2}\left[-\frac{1}{1^2}+\frac{1}{2^2}-\frac{1}{3^2}+........\right]$.

$\Rightarrow 1-\frac{1}{2^2}+\frac{1}{3^2}-......=\frac{\pi^2}{12}$.

Now adding, we get

$$2\left[1+\frac{1}{3^2}+\frac{1}{5^2}+........\infty\right]=\frac{\pi^2}{6}+\frac{\pi^2}{12}\Rightarrow\left[\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+..........\right]=\frac{\pi^2}{8}.$$

Q.No.8.: Find the **half-range sine series** for the function $f(t)=(t-t^2),\ 0<t<1$.

Sol.: Let us extend the function f(x) in the range $-\pi<x<0$.

Suppose the function f(t) in the range $-1<x<0$ represent an **odd function**, then the Fourier series of function f(t) will contains sine terms only given by

$$f(t)=\sum_{n=1}^{\infty} b_n\frac{\sin n\pi t}{1}, \tag{i}$$

where $b_n=\frac{2}{1}\int_0^1 (t-t^2)\frac{\sin n\pi t}{1}dt=2\left[-\left[(t-t^2)\frac{\cos n\pi t}{n\pi}\right]_0^1+\int_0^1 (1-2t)\frac{\cos n\pi t}{n\pi}dt\right]$

$$= 2\left[0 + \left[(1-2t)\frac{\sin n\pi t}{n^2\pi^2}\right]_0^1 - \int_0^1 (-2)\frac{\sin n\pi t}{n^2\pi^2}dt\right] 2\left[0 + 2\left(\frac{-\cos n\pi t}{n^3\pi^3}\right)\right]$$

$$= -4\left[\frac{(-1)^n}{n^3\pi^3} - \frac{1}{n^3\pi^3}\right] = \frac{-4}{n^3\pi^3}\left[(-1)^n - 1\right].$$

Thus $b_1 = \frac{8}{\pi^3}$, $b_2 = 0$, $b_3 = \frac{8}{3^3\pi^3}$,..............etc.

Hence, the Fourier sine series for f(t) over the half-range (0, 1) is

$$f(x) = \sum_{n=1}^{\infty} \frac{-4}{n^3\pi^3}\left[(-1)^n - 1\right]\sin n\pi t = \frac{8}{\pi^3}\left[\frac{\sin \pi t}{1^3} + \frac{\sin 3\pi t}{3^3} + \frac{\sin 5\pi t}{5^3} + \ldots\ldots\ldots\ldots\right]. \text{ Ans.}$$

Q.No.9.: Obtain the **half-range sine series** for e^x in $0 < x < 1$.

Sol.: Let us extend the function f(x) in the interval $(-1, 0)$. New function is symmetrical about the origin and represent an **odd function** in $(-1, 1)$.

Hence, the Fourier series for f(x) over the full period $(-1, 1)$ will contain only sine term, given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{1},$$

where $b_n = \frac{2}{1}\int_0^1 e^x \sin\frac{n\pi x}{1}dx = \frac{2}{1}\left[\int_0^1 e^x \sin n\pi x dx\right]$

$$= 2\left[\frac{e^x}{1+n^2\pi^2}\{\sin n\pi x - n\pi\cos n\pi x\}\right]_0^1$$

$$= 2\left[\frac{e}{1+n^2\pi^2}\left\{0 - n\pi\cos n\pi - \left((-1)\frac{n\pi}{e}\right)\right\}\right]$$

$$= 2\left[\frac{e}{1+n^2\pi^2}\left\{\frac{n\pi}{e} - n\pi\cos n\pi\right\}\right] = \frac{2n\pi}{1+n^2\pi^2}(1 - e\cos n\pi).$$

Hence, the Fourier sine series for f(x) over the half-range (0, 1) is

$$f(x) = \sum_{n=1}^{\infty} \frac{2n\pi}{1+n^2\pi^2}(1 - e\cos n\pi)\sin n\pi x. \text{ Ans.}$$

Q.No.10.: Express $\sin x$ as a cosine series in $0 < x < \pi$.

Sol.: Here interval is $(0,\pi)$ and hence $\ell=\pi$.

Let $f(x)=\dfrac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos\dfrac{n\pi x}{\pi}=\dfrac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos nx.$ (i)

$$a_0=\frac{2}{\pi}\int_0^{\pi} f(x)dx=\frac{2}{\pi}\int_0^{\pi} \sin xdx=-\frac{2}{\pi}\left|\cos x\right|_0^{\pi}=\frac{4}{\pi}.$$

$$a_n=\frac{2}{\pi}\int_0^{\pi} f(x)\cos nxdx=\frac{2}{\pi}\int_0^{\pi} \sin x\cos nxdx$$

$$=\frac{1}{\pi}\int_0^{\pi} 2\sin x\cos nxdx=\frac{1}{\pi}\int_0^{\pi} 2\cos nx\sin xdx$$

$$=\frac{1}{\pi}\int_0^{\pi} [\sin(n+1)x-\sin(n-1)x]dx \qquad [2\cos A\sin B=\sin(A+B)-\sin(A-B)]$$

$$=\frac{1}{\pi}\left|\frac{-\cos(n+1)x}{n+1}+\frac{\cos(n-1)x}{n-1}\right|_0^{\pi}, \qquad n\neq 1$$

$$=\frac{1}{\pi}\left[-\frac{(-1)^{n+1}}{n+1}+\frac{(-1)^{n-1}}{n-1}-\left(\frac{-1}{n+1}+\frac{1}{n-1}\right)\right]$$

$$=\frac{1}{\pi}\begin{cases}-\dfrac{1}{n+1}+\dfrac{1}{n-1}+\dfrac{1}{n+1}-\dfrac{1}{n-1}; & \text{if n is odd}\\ \dfrac{1}{n+1}-\dfrac{1}{n-1}+\dfrac{1}{n+1}-\dfrac{1}{n-1}; & \text{if n is even}\end{cases}$$

$$=\frac{1}{\pi}\begin{cases}0; & \text{if n is odd}\\ \dfrac{2}{n+1}-\dfrac{2}{n-1}; & \text{if n is even}\end{cases}$$

$$=\begin{cases}0; & \text{if n is odd}\\ -\dfrac{4}{(n^2-)\pi}; & \text{if n is even}\end{cases}$$

Hence, from (i), we get

$$f(x)=\sin x=\frac{1}{2}.\frac{4}{\pi}+\sum_{n=2,4,6,\ldots\ldots}^{\infty} a_n\cos nx=\frac{2}{\pi}-\frac{4}{\pi}\left(\frac{\cos 2x}{3}+\frac{\cos 4x}{15}+\ldots\ldots\right).$$

Q.No.11.: Obtain the Fourier expansion of $x\sin x$ as a **cosine series** in $(0,\pi)$ and hence show that $1+\frac{2}{1.3}-\frac{2}{3.5}+\frac{2}{5.7}-............... = \frac{\pi}{2}$.

Sol.: Let us extend the function f(x) in the interval $(-\pi, 0)$. New function is symmetrical about y-axis and therefore represent an **even function** in interval $(-\pi, \pi)$.

Hence, the Fourier series for f(x) over the full period $(-\pi, \pi)$ will contain only cosine terms given by

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos n\pi x \ , \qquad \text{(i)}$$

where $a_0 = \frac{2}{\pi}\int_0^{\pi} x \sin dx = \frac{2}{\pi}[-x\cos x+\sin x]_0^{\pi} = \frac{2}{\pi}[-\pi\cos\pi] = \frac{2}{\pi}\times\pi = 2$,

and $a_n = \frac{2}{\pi}\int_0^{\pi} x\sin x\cos nx dx = \frac{1}{\pi}\int_0^{\pi} x\{\sin(n+1)x-\sin(n-1)x\}dx$

$$=\frac{1}{\pi}\left[\left|x\left(-\frac{\cos(n+1)x}{n+1}\right)\right|_0^{\pi}+\int_0^{\pi}\frac{\cos(n+1)x}{n+1}\right]-\left[\left|x\left(-\frac{\cos(n-1)x}{n-1}\right)\right|_0^{\pi}+\int_0^{\pi}\frac{\cos(n-1)x}{n-1}dx\right]$$

$$=\frac{1}{\pi}\left[-\frac{\pi}{(n+1)}(-1)^{n+1}+0-\left\{-\frac{\pi}{(n-1)}(-1)^{n-1}+0\right\}\right]=\left[\frac{(-1)^{n-1}}{n-1}-\frac{(-1)^{n+1}}{n+1}\right]$$

$$=(-1)^{n+1}\left[\frac{n+1-n+1}{(n+1)(n-1)}\right]=\frac{2(-1)^{n-1}}{(n-1)(n+1)}.$$

Thus $a_2=\frac{-2}{1.3}$, $a_3=\frac{2}{2.4}$, $a_4=\frac{2}{3.5}$.....................etc.

When n =1, we get

$$a_1=\frac{2}{\pi}\int_0^{\pi}\sin x\cos x dx=\frac{1}{\pi}\int_0^{\pi}\sin 2x dx=\frac{1}{\pi}\left[\frac{-x\cos 2x}{2}+\frac{\sin 2x}{4}\right]_0^{\pi}=\frac{1}{\pi}\left[-\frac{\pi}{2}\right]=-\frac{1}{2}.$$

Hence, the Fourier cosine series for f(x) over the half-range $(0, \pi)$ is

$$f(x)=1-\frac{1}{2}\cos x-\frac{2}{1.3}\cos 2x+\frac{2}{2.4}\cos 3x-\frac{2}{3.5}\cos 4x+.................. .$$

2nd Part: Show that $1+\frac{2}{1.3}-\frac{2}{3.5}+\frac{2}{5.7}-............... = \frac{\pi}{2}$.

When $x = \frac{\pi}{2}$. Then $\frac{\pi}{2} = 1 + \frac{2}{1.3} - \frac{2}{3.5} + \frac{2}{5.7} - ....$.

$$\Rightarrow \frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{4.7} - .... = \frac{\pi - 2}{4},$$

which is the required result.

Q.No.12.: If $f(x) = x, \quad 0 < x < \frac{\pi}{2}$

$$= \pi - x, \quad \frac{\pi}{2} < x < \pi,$$

show that **(i)** $f(x) = \frac{4}{\pi}\left[\sin x - \frac{1}{3^2}\sin 3x + \frac{1}{5^2}\sin 5x - ....\right]$

(ii) $f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left[\frac{1}{1^2}\cos 2x + \frac{1}{3^2}\cos 6x + \frac{1}{5^2}\cos 10x + ...\right]$

Sol.: (i) Let f(x) is an **odd function** in $(-\pi, \pi)$. Hence, the Fourier series for f(x) over the period $(-\pi, \pi)$ will contain only sine terms given by:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx, \tag{i}$$

where $b_n = \frac{2}{\pi}\left[\int_0^{\pi/2} x \sin nx dx + \int_{\pi/2}^{\pi} (\pi - x)\sin nx dx\right]$

$$= \frac{2}{\pi}\left[\left|\frac{x(-\cos nx)}{n}\right|_0^{\pi/2} + \frac{1}{n}\int_0^{\pi/2} \cos nx dx\right] + \left[\left|(\pi - x)\left(\frac{-\cos nx}{n}\right)\right|_{\pi/2}^{0} - \frac{1}{n}\int_{\pi/2}^{\pi} \cos nx dx\right]$$

$$= \frac{2}{\pi}\left[\frac{-1}{n}\frac{\pi}{2}\cos\left(\frac{n\pi}{2}\right) - 0 + \frac{1}{n^2}\left(\sin\frac{n\pi}{2}\right) - 0 + \frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2}\right]$$

$$= \frac{4}{\pi n^2}\sin\left(\frac{n\pi}{2}\right) = \frac{4}{\pi}\left[\frac{1}{n^2}\sin\frac{n\pi}{2}\right].$$

Hence, the Fourier sine series for f(x) over the half range $(0, \pi)$ is

$$f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{n^2}\sin\frac{n\pi}{2}\sin nx$$

$$\Rightarrow f(x) = \frac{4}{\pi}\left[\sin x - \frac{1}{3^2}\sin 3x + \frac{1}{5^2}\sin 5x - ....\right]. \text{ Ans.}$$

(ii): Let us extend the function f(x) in the interval $(-\pi, 0)$. New function is symmetrical about y-axis and therefore represent an **even function** in interval $(-\pi, \pi)$.

Hence, the Fourier series for f(x) over the full period $(-\pi, \pi)$ will contain only cosine terms given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x \ , \qquad \text{(ii)}$$

where $a_0 = \frac{2}{\pi}\left[\int_0^{\pi/2} x dx + \int_{\pi/2}^{\pi} (\pi - x) dx\right] = \frac{2}{\pi}\left[\frac{1}{2}\left[x^2\right]_0^{\pi/2} + \left[\pi x - \frac{x^2}{2}\right]_{\pi/2}^{\pi}\right]$

$$= \frac{2}{\pi}\left(\frac{1}{2}\frac{\pi^2}{4}\right) + \left[\left(\pi^2 - \frac{\pi^2}{2}\right) - \left(\frac{\pi^2}{2} - \frac{\pi^2}{8}\right)\right] = \frac{2}{\pi}\left[\frac{\pi^2}{8} + \frac{\pi^2}{2} - \frac{\pi^2}{2} + \frac{\pi^2}{8}\right]$$

$$= \frac{2}{\pi} \times \frac{2\pi^2}{8} = \frac{\pi}{2},$$

and $a_n = \frac{2}{\pi}\left[\int_0^{\pi/2} x \cos nx dx + \int_{\pi/2}^{\pi} (\pi - x) \cos nx dx\right]$

$$= \frac{2}{\pi}\left[\left\{\frac{x}{n}\sin nx + \frac{1}{n^2}\cos nx\right\}_0^{\pi/2} + \left\{(\pi - x)\frac{\sin nx}{n} - \frac{\cos nx}{n^2}\right\}_{\pi/2}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\left(\frac{\pi}{2n}\sin\frac{n\pi}{2} + \frac{\cos\frac{n\pi}{2}}{n^2} - \frac{1}{n^2}\right) + \left(\frac{-(-1)^n}{n^2} - \frac{\pi}{2n}\sin\frac{n\pi}{2} + \frac{\cos\frac{n\pi}{2}}{n^2}\right)\right]$$

$$= \frac{2}{\pi}\left[\frac{2}{n^2}\cos\frac{n\pi}{2} - \frac{1}{n^2}\left(1 + (-1)^n\right)\right].$$

Thus $a_1 = 0, \ \ a_2 = \frac{2}{\pi}\left[\frac{-2}{2^2} - \frac{2}{2^2}\right] = \frac{-2}{\pi}, \ \ a_3 = 0, \ \ a_4 = \frac{2}{\pi}\left[\frac{2}{4^2} - \frac{2}{4^2}\right] = 0$

$a_5 = 0, \ \ a_6 = \frac{2}{\pi}\left[\frac{-2}{6^2} - \frac{2}{6^2}\right] = \frac{-2}{\pi}\left[\frac{4}{6^2}\right] = \frac{-2}{\pi}.\frac{1}{3^2}$, ...etc.

Hence, the Fourier cosine series for f(x) over the half range (0, π) is

$f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left[\frac{\cos 2x}{1^2} + \frac{\cos 6x}{3^2} + \text{........................}\right]$. Ans.

Q.No.13.: If $f(x)=\begin{cases} \frac{\pi}{3}, & 0\le x\le \frac{\pi}{3} \\ 0, & \frac{\pi}{3}\le x\le \frac{2\pi}{3} \\ -\frac{\pi}{3}, & \frac{2\pi}{3}\le x\le \pi \end{cases}$,

then show that $f(x)=\frac{2}{\sqrt{3}}\left[\cos x-\frac{\cos 5x}{5}+\frac{\cos 7x}{7}-........\right]$.

Sol.: Here the interval is $(0,\pi)$ and we need to find Fourier half-range cosine series.

Let $f(x)=a_0+\sum_{n=1}^{\pi} a_n\cos nx$ (i)

be the required Fourier Half-range cosine series.

$$a_0=\frac{2}{\pi}\int_0^{\pi} f(x)dx=\frac{2}{\pi}\left[\int_0^{\pi/3} f(x)dx+\int_{\pi/3}^{2\pi/3} 0dx+\int_{2\pi/3}^{\pi} f(x)dx\right]$$

$$=\frac{2}{\pi}\int_0^{\pi/3}\frac{\pi}{3}dx+\frac{2}{\pi}\int_{2\pi/3}^{\pi}-\frac{\pi}{3}dx=\frac{2}{\pi}\frac{\pi}{3}\left(\frac{\pi}{3}-0\right)-\frac{2}{3}.\left(\pi-\frac{2\pi}{3}\right)=\frac{2\pi}{3}.\frac{1}{3}-\frac{2\pi}{3}.\frac{1}{3}=0.$$

$$a_n=\frac{2}{\pi}\int_0^{\pi} f(x)\cos nxdx=\frac{2}{\pi}\int_0^{\pi/3} f(x)\cos nxdx+\frac{2}{\pi}\int_{\pi/3}^{2\pi/3} 0dx+\frac{2}{\pi}\int_{2\pi/3}^{\pi} f(x)\cos nxdx$$

$$=\frac{2}{\pi}\int_0^{\pi/3}\frac{\pi}{3}\cos nxdx+\frac{2}{\pi}\int_{2\pi/3}^{\pi}-\frac{\pi}{3}\cos nxdx$$

$$=\frac{2}{3}\left|\frac{\sin nx}{n}\right|_0^{\pi/3}-\frac{2}{3}\left|\frac{\sin nx}{n}\right|_{2\pi/3}^{\pi}=\frac{2}{3n}\left[\sin\frac{n\pi}{3}-\left(0-\sin\frac{2n\pi}{3}\right)\right]$$

$[\sin n\pi=0,\quad n\in Z]$

$$=\frac{2}{3n}\left[\sin\frac{n\pi}{3}+\sin\left(\frac{3n\pi-n\pi}{3}\right)\right]=\frac{2}{3n}\left[\sin\frac{n\pi}{3}+\sin\left(n\pi-\frac{n\pi}{3}\right)\right]$$

$$=\frac{2}{3n}\left[\sin\frac{n\pi}{3}+(-1)^n\sin\left(-\frac{n\pi}{3}\right)\right]$$ $[\sin(n\pi+\theta)=(-1)^n\sin\theta]$

$$=\frac{2}{3n}\sin\frac{n\pi}{3}\left[1-(-1)^n\right]$$ $[\sin(-\theta)=-\sin\theta]$

$$= \begin{cases} 0, & \text{if } n \text{ is even} \\ \dfrac{4}{3\pi}\sin\dfrac{n\pi}{3}, & \text{if } n \text{ is odd} \end{cases}.$$

Therefore, from (i), we get

$$f(x) = 0 + \sum_{n=1}^{\infty} a_n \cos nx = \sum_{n=\text{odd}}^{\infty} \frac{4}{3n}\sin\frac{n\pi}{3}\cos nx$$

$$= \frac{4}{3}\left(\sin\frac{\pi}{3}\cos x + \frac{\cos 3x}{3}.0 + \frac{1}{5}\sin\frac{5\pi}{3}\cos 5x + \frac{1}{7}\sin\frac{7\pi}{3}\cos 7x + ...\right)$$

$$\left[\because \sin\frac{5\pi}{3} = \sin\left(2\pi - \frac{\pi}{5}\right) = -\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2}\right]$$

$$= \frac{4}{3}\left(\frac{\sqrt{3}}{2}\cos x - \frac{\sqrt{3}}{2}.\frac{1}{5}\cos 5x + \frac{\sqrt{3}}{2}.\frac{1}{7}\cos 7x + ...\right)$$

$$= \frac{2}{\sqrt{3}}\left[\cos x - \frac{\cos 5x}{5} + \frac{\cos 7x}{7}...........\right].$$

Q.No.14.: Obtain a **half-range cosine series** for

$$f(x) = kx \quad \text{for } 0 \le x \le \frac{\ell}{2}$$

$$f(x) = k(\ell - x) \text{ for } \frac{\ell}{2} \le x \le \ell,$$

and hence, deduce the sum of the series $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ...$.

Sol.: Let us extend the function f(x) in the interval $(-\ell, 0)$. New function is symmetrical about y-axis and therefore represent an **even function** in interval $(-\ell, \ell)$.

Hence, the Fourier series for f(x) over the full period $(-\ell, \ell)$ will contain only cosine terms given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \frac{\cos n\pi x}{\ell}, \qquad \text{(i)}$$

where $a_0 = \frac{2}{\ell}\left[\int_0^{\ell/2} kx dx + \int_{\ell/2}^{\ell} k(\ell - x)dx\right] = \frac{2k}{\ell}\left[\frac{\ell^2}{8} + \frac{\ell^2}{2} - \left(\ell^2 - \frac{\ell^2}{4}\right)\right]$

$$= \frac{2k}{\ell}\left[\frac{\ell^2}{8} + \frac{\ell^2}{2} - \frac{3\ell^2}{4}\right] = \frac{2k}{8\ell}\left[\ell^2 + 4\ell^2 - 3\ell^2\right] = \frac{4k\ell}{8},$$

and $a_n = \frac{2}{\ell}\left[\int_0^{\ell/2} kx\cos\frac{n\pi x}{\ell}dx + \int_{\ell/2}^{\ell} k(\ell - x)\cos\frac{n\pi x}{\ell}\right]$

$$= \frac{2k}{\ell}\left[\int_0^{\ell/2} x\cos\frac{n\pi x}{\ell}dx + \int_{\ell/2}^{\ell} (\ell - x)\cos\frac{n\pi x}{\ell}\right]$$

$$= \frac{2k}{\ell}\left[\left|x\left(\frac{\ell}{n\pi}\right)\sin\frac{n\pi x}{\ell}\right|_0^{\ell/2} - \left(\frac{\ell}{n\pi}\right)\int_0^{\ell/2}\sin\frac{n\pi x}{\ell}dx\right]$$

$$+ \left\{\left|\frac{(\ell - x)}{n\pi}\left(\frac{\ell}{n\pi}\right)\sin\frac{n\pi x}{\ell}\right|_{\ell/2}^{\ell} + \int_{\ell/2}^{\ell}\left(\frac{\ell}{n\pi}\right)\sin\frac{n\pi x}{\ell}dx\right\}$$

$$= \frac{2k}{\ell}\left(\frac{\ell}{n\pi}\right)\left[\begin{array}{l}\frac{\ell}{2}\sin\frac{n\pi}{2} - 0 + \frac{\ell}{n\pi}\left(\cos\frac{n\pi}{2} - 1\right)\\ + \left\{\left|(\ell - x)\sin\frac{n\pi x}{\ell}\right|_{\ell/2}^{\ell} - \left(\frac{\ell}{n\pi}\right)\left(\cos n\pi - \cos\frac{n\pi}{2}\right)\right\}\end{array}\right]$$

$$= \frac{2k}{n\pi}\left[\frac{\ell}{n\pi}\left\{\cos\frac{n\pi}{2} - 1 - \cos n\pi + \cos\frac{n\pi}{2}\right\}\right] = \frac{2k\ell}{n^2\pi^2}\left\{2\cos\frac{n\pi}{2} - \cos n\pi - 1\right\}$$

$$= \frac{2k\ell}{\pi^2}.\frac{1}{n^2}\left[2\cos\frac{n\pi}{2} - \left\{1 + (-1)^n\right\}\right]$$

Thus $a_1 = 0,\ a_2 = \frac{2k\ell}{\pi^2}.\frac{1}{2^2}\{-2 - 2\} = \frac{-8k\ell}{\pi^2.2^2},\ \ a_3 = 0,\ \ a_4 = 0,\ \ a_5 = 0,$

$a_6 = \frac{2k\ell}{\pi^2}.\frac{1}{6^2}[-2 - 2] = \frac{-8k\ell}{\pi^2 6^2}$ .... etc.

Hence, the Fourier cosine series for f(x) over the half range $(0, \ell)$ is

$$f(x) = \frac{k\ell}{4} - \frac{8k\ell}{\pi^2}\left[\frac{1}{2^2}\cos\frac{2\pi x}{\ell} + \frac{1}{6^2}\cos\frac{6\pi x}{\ell} + ...\right]$$

2nd part: Deduce the sum of the series $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ...$.

Now put $x = 0$, we get

$$0=\frac{k\ell}{4}-\frac{8k\ell}{\pi^2}\left[\frac{1}{4}+\frac{1}{36}+\frac{1}{100}+\ldots\ldots\ldots\ldots\right]\Rightarrow\frac{k\ell}{4}=\frac{8k\ell}{\pi^2}\left[\frac{1}{4}+\frac{1}{36}+\frac{1}{100}+\ldots\ldots\ldots\right]$$

$$\Rightarrow\frac{\pi^2}{8}=\left[\frac{1}{1}+\frac{1}{9}+\frac{1}{25}+\ldots\ldots\ldots\right]\Rightarrow\frac{\pi^2}{8}=\left[\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\ldots\ldots\right]$$

$$\Rightarrow\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\ldots\ldots=\frac{\pi^2}{8}.$$

Q.No.15.: If $f(x)=\begin{cases} mx, & 0\le x\le\frac{\pi}{2} \\ m(\pi-x), & \frac{\pi}{2}\le x\le\pi \end{cases}$.

Then, show that $f(x)=\frac{4}{\pi}\left[\frac{\sin x}{i^2}-\frac{\sin 3x}{3^2}+\frac{\cos 5x}{5^2}-\ldots\ldots\right]$.

Sol.: Let $f(x)=\sum_{n=1}^{\infty} b_n \sin nx.$ (i)

$$b_n=\frac{2}{\pi}\int_0^{\pi} f(x)\sin nxdx=\frac{2}{\pi}\left[\int_0^{\pi/2} mx\sin nxdx+\int_{\pi/2}^{0} m(\pi-x)\sin nxdx\right]$$

$$=\frac{2m}{\pi}\int_0^{\pi/2}\underset{\mathrm{I}}{x}\,\underset{\mathrm{II}}{\sin nx}\,dx+\frac{2m}{\pi}\int_{\pi/2}^{\pi}\underset{\mathrm{I}}{(\pi-x)}\underset{\mathrm{II}}{\sin nx}\,dx$$

Integrating by parts, we get

$$=\frac{2m}{\pi}\left[\left|x\left(\frac{-\cos nx}{n}\right)-(1)\left(\frac{-\sin nx}{n^2}\right)\right|_0^{\pi/2}\right]+\frac{2m}{\pi}\left[\left|(\pi-x)\left(\frac{-\cos nx}{n}\right)-(-1)\left(\frac{-\sin nx}{n}\right)\right|_{\pi/2}^{\pi}\right]$$

$$=\frac{2m}{\pi}\left[\frac{-\pi}{2n}\cos\frac{n\pi}{2}+\frac{1}{n^2}\sin\frac{n\pi}{2}\right]+\frac{2m}{\pi}\left[0-\left(\frac{-\pi}{2n}\cos\frac{n\pi}{2}+\frac{1}{n^2}\sin\frac{n\pi}{2}\right)\right]$$

$$=\frac{2m}{\pi}\left[\frac{2}{n^2}.\sin\frac{n\pi}{2}\right]=\frac{4m}{n^2\pi}.\sin\frac{n\pi}{2}.$$

Therefore, from (i), we get

$$f(x)=\frac{4m}{\pi}\sum_{n=1}^{\infty}\sin\frac{n\pi}{2}\frac{\sin nx}{n^2}=\frac{4m}{\pi}\left[\sin x-\frac{\sin 3x}{3^2}+\frac{\sin 5x}{5^2}-\ldots\ldots\right].$$

Q.No.16.: Find the Half-range sine series for $f(x)=\begin{cases}\frac{1}{4}-x, & 0<x<\frac{1}{2}\\ x-\frac{3}{4}, & \frac{1}{2}<x<1\end{cases}$.

Sol.: Here the interval is (0, 1) and hence $\ell=1$.

Let $f(x)=\sum_{n=1}^{\infty} b_n \frac{\sin nx}{1}=\sum_{n=1}^{\infty} b_n \sin nx.$ (i)

Now $b_n=\frac{2}{1}\int_0^1 f(x)\sin n\pi x dx=2\int_0^{1/2}\left(\frac{1}{4}-x\right)\sin n\pi x dx+2\int_{1/2}^{1}\left(x-\frac{3}{4}\right)\sin n\pi x dx$

$$=2\left[\left(\frac{1}{4}-x\right)\left(\frac{-\cos n\pi x}{n\pi}\right)-(-1)\left(\frac{-\sin n\pi x}{n^2\pi^2}\right)\right]_0^{1/2}$$

$$+2\left[\left(x-\frac{3}{4}\right)\left(\frac{-\cos n\pi x}{n\pi}\right)-(1)\left(\frac{-\sin n\pi x}{n^2\pi^2}\right)\right]_{1/2}^{1}$$

$$=2\left[\left(\frac{1}{4n\pi}\cos\frac{n\pi}{2}-\frac{\sin\frac{n\pi}{2}}{n^2\pi^2}\right)-\left(\frac{-1}{4n\pi}\right)\right] \qquad [\sin n\pi=0,\ \ n\in Z]$$

$$+2\left[-\frac{1}{4n\pi}(-1)^n+0-\left(\frac{1}{4n\pi}\cos\frac{n\pi}{2}+\frac{\sin\frac{n\pi}{2}}{n^2\pi^2}\right)\right]$$

$$\Rightarrow b_n=\frac{-4\sin\frac{n\pi}{2}}{n^2\pi^2}+\frac{1}{2n\pi}\left[1-(-1)^n\right].$$

Put n = 1, 2, 3, ……, we get

$b_1=\frac{-4}{\pi^2}+\frac{1}{2\pi}.2=\frac{1}{\pi}-\frac{4}{\pi^2}$, $b_2=0$, $b_3=\frac{1}{3\pi}+\frac{4}{9\pi^2}$,..............

Therefore, from (i), we get

$$f(x)=\sum_{n=1}^{\infty} b_n \sin n\pi x=\left(\frac{1}{\pi}-\frac{4}{\pi^2}\right)\sin\pi x+\left(\frac{1}{3\pi}+\frac{4}{9\pi^2}\right)\sin 3\pi+.......$$

Q.No.17.: Represent the function by Fourier sine series $f(x)=\begin{cases}1, & 0<x<\frac{\ell}{2}\\ 0, & \frac{\ell}{2}<x<\ell\end{cases}$.

Sol.: Let $f(x)=\sum_{n=1}^{\infty} b_n \sin\frac{\pi nx}{\ell}$ (i)

be the required Fourier series.

where $b_n=\frac{2}{\ell}\int_0^{\ell} f(x)\sin n\frac{\pi x}{\ell}dx=\frac{2}{\ell}\left[\int_0^{\ell/2} 1.\sin\frac{n\pi x}{\ell}dx+\int_{\ell/2}^{\ell} 0dx\right]=\frac{2}{\ell}\int_0^{\ell/2}\sin\frac{n\pi x}{\ell}dx$

$$=\frac{2}{\ell}\left|\frac{-\cos\frac{n\pi x}{\ell}}{\frac{n\pi}{\ell}}\right|_0^{\ell/2}=\frac{-2}{n\pi}\left[\cos\frac{n\pi}{\ell}.\frac{\ell}{2}-1\right]=\frac{-2}{n\pi}\left(\cos\frac{n\pi}{2}-1\right),$$

$[n\neq 0]$

i.e., $b_1=\frac{-2}{\pi}\left(\cos\frac{\pi}{2}-1\right)=\frac{2}{\pi}$, $b_2=\frac{-2}{2\pi}(\cos\pi-1)=\frac{2}{\pi}$, $b_3=\frac{-2}{3\pi}\left(\cos\frac{3\pi}{2}-1\right)=\frac{2}{3\pi}$,.........

Therefore, from (i), and using the above values of b_1, b_2, b_3, ……, we get

$$f(x)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{\ell}=b_1\sin\frac{\pi x}{\ell}+b_2\sin\frac{2\pi x}{\ell}+b_3\sin\frac{3\pi x}{\ell}+......$$

$$=\frac{2}{\pi}\left[\sin\frac{\pi x}{\ell}+\sin\frac{2\pi x}{\ell}+\frac{1}{3}\sin\frac{3\pi x}{\ell}+........\right].$$

Q.No.18.: If $f(x)=\sin x$ for $0\le x\le\frac{\pi}{4}$,

$=\cos x$ for $\frac{\pi}{4}\le x\le\frac{\pi}{2}$, expand f(x) in a series for **sines**.

Sol.: Let us extend the function f(x) in the interval $\left(-\frac{\pi}{2},\ 0\right)$. New function is symmetrical about the origin and represent an **odd function** in $\left(-\frac{\pi}{2},\ \frac{\pi}{2}\right)$.

Hence, the Fourier series for f(x) over the full period $\left(-\frac{\pi}{2},\ \frac{\pi}{2}\right)$ will contain only sine term, given by

$$f(x)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{\pi/2}, \qquad \text{(i)}$$

$$\text{where } b_n=\frac{2}{\frac{\pi}{2}}\int_0^{\pi/2} f(x)\sin\frac{n\pi x}{\frac{\pi}{2}}=\frac{4}{\pi}\left[\int_0^{\pi/4}\sin x\sin 2nx dx+\int_{\pi/4}^{\pi/2}\cos x\sin 2nxd\right]$$

$$=\frac{2}{\pi}\int_0^{\pi/4}\left[\cos(2n-1)x-\cos(2n+1)x\right]dx+\frac{2}{\pi}\int_{\pi/4}^{\pi/2}\left[\sin(2n+1)x+\sin(2n-1)x\right]dx$$

$$=\frac{2}{\pi}\left[\frac{\sin(2n-1)x}{2n-1}-\frac{\sin(2n+1)x}{2n+1}\right]_0^{\pi/4}-\frac{2}{\pi}\left[\frac{\cos(2n+1)x}{2n+1}+\frac{\cos(2n-1)x}{2n-1}\right]_{\pi/4}^{\pi/2}$$

$$=\frac{2}{\pi}\left[\begin{array}{l}\dfrac{\sin(2n-1)\frac{\pi}{4}}{2n-1}-\dfrac{\sin(2n+1)\frac{\pi}{4}}{2n+1}-\dfrac{\cos(2n+1)\frac{\pi}{2}}{2n+1}-\dfrac{\cos(2n-1)\frac{\pi}{2}}{2n-1}\\ \qquad +\dfrac{\cos(2n+1)\frac{\pi}{4}}{2n+1}+\dfrac{\cos(2n-1)\frac{\pi}{4}}{2n-1}\end{array}\right]$$

$$=\frac{2}{\pi}\left[\frac{\sin(2n-1)\frac{\pi}{4}+\cos(2n-1)\frac{\pi}{4}}{(2n-1)}-\frac{\sin(2n+1)\frac{\pi}{4}-\cos(2n+1)\frac{\pi}{4}}{(2n+1)}\right]$$

$$\text{Thus } b_1=\frac{2}{\pi}\left[\frac{\sin\frac{\pi}{4}+\cos\frac{\pi}{4}}{1}-\frac{\sin\frac{3\pi}{4}-\cos\frac{3\pi}{4}}{3}\right]=\frac{2}{\pi}\left[\frac{2}{\sqrt{2}}-\frac{\frac{2}{\sqrt{2}}}{3}\right]=\frac{2}{\pi}\left[\sqrt{2}-\frac{\sqrt{2}}{3}\right]$$

$$=\frac{2}{\pi}\times 2\times\frac{\sqrt{2}}{3}=\frac{4\sqrt{2}}{3\pi}=\frac{8}{3\pi}\cos\frac{\pi}{4}=\frac{8}{\pi}\cos\frac{\pi}{4}.\frac{1}{1.3},$$

$$b_2=\frac{2}{\pi}\left[\frac{\sin\left(\pi-\frac{\pi}{4}\right)+\cos\left(\pi-\frac{\pi}{4}\right)}{3}-\frac{\sin\left(\pi+\frac{\pi}{4}\right)+\cos\left(\pi+\frac{\pi}{4}\right)}{5}\right]=\frac{2}{\pi}\left[\frac{\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}}{3}-\frac{-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}}{5}\right]=0,$$

$$b_3=\frac{2}{\pi}\left[\frac{\sin\left(\pi+\frac{\pi}{4}\right)+\cos\left(\pi+\frac{\pi}{4}\right)}{5}-\frac{\sin\left(2\pi-\frac{\pi}{4}\right)-\cos\left(2\pi-\frac{\pi}{4}\right)}{7}\right]$$

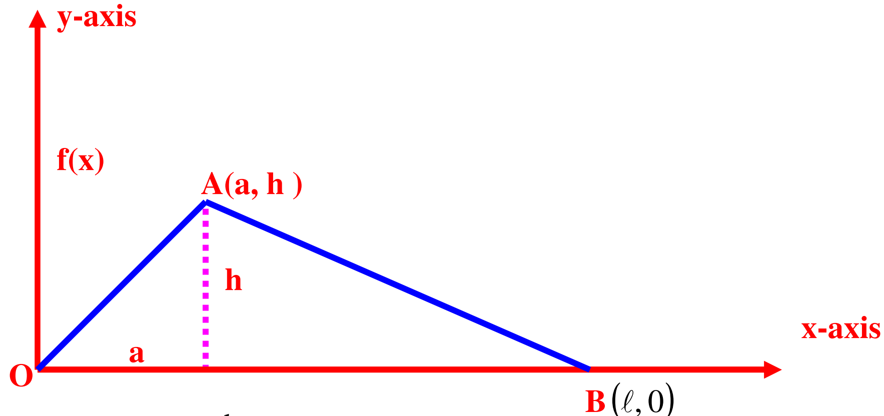
$$= \frac{2}{\pi}\left[\frac{-\frac{2}{\sqrt{2}}}{5} - \frac{-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}}{7}\right] = \frac{2}{\pi}\left[\frac{-\sqrt{2}}{5} + \frac{\sqrt{2}}{7}\right] = \frac{2}{\pi 5.7}\left[-2\sqrt{2}\right]$$

$$= \frac{-8}{\pi\sqrt{2}}.\frac{1}{5.7} = \frac{-8}{\pi}\cos\frac{\pi}{4}\times\frac{1}{5.7}, \text{........................etc.}$$

Hence, the Fourier sine series for f(x) over the half range $\left(0, \frac{\pi}{2}\right)$ is

$$f(x) = \frac{8}{\pi}\cos\frac{\pi}{4}\left[\frac{\sin 2x}{1.3} - \frac{\sin 6x}{5.7} + \frac{\sin 10x}{9.11} + \text{..........................}\right]. \text{ Ans.}$$

Q.No.19.: For the function defined by the graph OAB in figure, find the half range Fourier **sine** series.



Sol.: Here the function is $f(x) = \frac{h}{a}x, \quad 0 \le x \le a$

$$= \frac{h}{a-\ell}(x-\ell), \qquad a \le x \le \ell$$

Let us extend the function f(x) in the interval $(-\ell, 0)$. New function is symmetrical about the origin and represent an **odd function** in $(-\ell, \ell)$.

Hence, the Fourier series for f(x) over the full period $(-\ell, \ell)$ will contain only sine term, given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{\ell}dx,$$

where $b_n = \frac{2}{\ell}\int_0^{\ell} f(x)\sin\frac{n\pi x}{\ell}dx = \frac{2}{\ell}\int_0^{a}\left(\frac{h}{a}x\right)\frac{\sin n\pi x}{\ell}dx + \frac{2h}{\ell(a-\ell)}\int_a^{\ell}(x-\ell)\frac{\sin n\pi x}{\ell}dx$

$$=\frac{2}{\ell}\frac{h}{a}\left[\frac{-x\cos\frac{n\pi x}{\ell}}{\frac{n\pi}{\ell}}+\frac{\sin\frac{n\pi x}{\ell}}{\left(\frac{n\pi}{\ell}\right)^2}\right]_0^a+\frac{2h}{\ell(a-\ell)}\left[\frac{-(x-\ell)\cos\frac{n\pi x}{\ell}}{\frac{n\pi}{\ell}}+\frac{\sin\frac{n\pi x}{\ell}}{\left(\frac{n\pi}{\ell}\right)^2}\right]_a^\ell$$

$$=\frac{2}{\ell}\frac{h}{a}\left[\frac{-a\cos\frac{n\pi a}{\ell}}{\frac{n\pi}{\ell}}+\frac{\sin\frac{n\pi a}{\ell}}{\left(\frac{n\pi}{\ell}\right)^2}\right]+\frac{2h}{\ell(a-\ell)}\left[\frac{(a-\ell)\cos\frac{n\pi a}{\ell}}{\frac{n\pi}{\ell}}-\frac{\sin\frac{n\pi a}{\ell}}{\left(\frac{n\pi}{\ell}\right)^2}\right]$$

$$=\frac{2h}{\ell}\left[-\cos\frac{\frac{n\pi a}{\ell}}{\frac{n\pi}{\ell}}+\frac{1}{a}\frac{\sin\frac{n\pi a}{\ell}}{\left(\frac{n\pi}{\ell}\right)^2}+\cos\frac{\frac{n\pi a}{\ell}}{\frac{n\pi}{\ell}}-\frac{1}{(a-\ell)}\frac{\sin\frac{n\pi a}{\ell}}{\left(\frac{n\pi}{\ell}\right)^2}\right]$$

$$=\frac{2h}{\ell}\times\frac{\ell^2}{4^2\pi^2}\left[\frac{1}{a}-\frac{1}{a-\ell}\right]\sin\frac{n\pi a}{\ell}=\frac{2h\ell^2}{\ell 4^2\pi^2}\left(\frac{a-\ell-a}{a(a-\ell)}\right)\sin\frac{n\pi a}{\ell}$$

$$=\frac{2h\ell^2}{\pi^2}\left(\frac{1}{a(\ell-a)}\right)\sin\frac{n\pi a}{\ell}.$$

Thus $b_1=\frac{2\ell^2 h}{a(\ell-a)\pi^2}\sin\frac{\pi a}{\ell}$, $\quad b_2=\frac{2\ell^2 h}{a(\ell-a)\pi^2}\sin\frac{2\pi x}{\ell}$, ...etc.

Hence, the Fourier sine series for f(x) over the half range $(0,\ \ell)$ is

$$f(x)=\frac{2\ell^2 h}{a(\ell-a)\pi^2}\left[\sin\frac{\pi a}{\ell}\sin\frac{\pi x}{\ell}+\frac{1}{2^2}\sin\frac{2\pi a}{\ell}\sin\frac{2\pi x}{\ell}+\frac{1}{3^2}\sin\frac{3\pi a}{\ell}\sin\frac{3\pi x}{\ell}+\ldots\ldots..\right].\text{ Ans.}$$

Q.No.20.: If $f(x)=1-\frac{x}{L}$ in $0<x<L$, find **(a)** **Fourier cosine series** **(b)** **Fourier sine series** of f(x).

Graph the corresponding periodic continuation of f(x).

Sol.: The given function f(x) is neither periodic nor odd. In order to obtain

(a). Fourier cosine series of nonperiodic, not even function f(c). Define (or construct) a new function g(x) such that (i) g(x) = f(x) in (0, L). (ii) g(x) is even periodic function in $(-L, L)$. Now obtain the Fourier cosine series of g(x) in $(-L, L)$ which is the required Fourier cosine series of f(x) in (0, L) since g(x) coincide in (0, L).

Define $g(x) = f(x) = 1 - \frac{x}{L}$ in $0 < x < L$

$g(x) = 1 - \frac{x}{L}$ in $-L < x < 0$

and $g(x + 2L) = g(x)$.

Now g(x) is even in $(-L, L))$ and is periodic with period 2L.

The graph is the even periodic continuation (or extension) of f(x) .

The Fourier cosine series of g(x) is

$$g(x) = \frac{a_0}{2} + \sum a_n \cos\frac{n\pi x}{L}.$$

Here $a_0 = \frac{2}{L}\int_0^L f(x)dx = \frac{2}{L}\int_0^L \left(1 - \frac{x}{L}\right)dx = \left[\left(x - \frac{x^2}{2L}\right)\Big|_0^L\right] = \frac{2}{L}\left[L - \frac{L^2}{2L}\right] = 1.$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos n\frac{\pi x}{L}dx \quad = \frac{2}{L}\int_0^L \left(1 - \frac{x}{L}\right)\cos n\frac{\pi x}{L}dx$$

$$= \frac{2}{L}\left[\left(1 - \frac{x}{L}\right).\frac{L}{\pi x}\sin\frac{n\pi x}{L} - \left(-\frac{1}{L}\right).\frac{L^2}{n^2\pi^2}\left(-\cos\frac{n\pi x}{L}\right)\right]_0^L = \frac{2}{n^2\pi^2}\left[1 - (-1)^n\right]$$

So $g(x) = \frac{1}{2} + \frac{2}{\pi^2}\sum \frac{\left[1 - (-1)^n\right]}{n^2}\cos\left(\frac{n\pi x}{L}\right).$

Thus, the required Fourier cosine series of f(x) in the interval (0,L) is

$$f(x) = g(x) = \frac{1}{2} + \frac{2}{\pi^2}\sum_{n=1}^{\infty} \frac{\left[1 - (-1)^n\right]}{n^2}\cos\left(\frac{n\pi x}{L}\right).$$

Since f(x) is equal to g(x) in the interval (0, L).

(b). Fourier sine series of f(x) in (0, L): On similar lines define a new function (i) h(x) = f(x) in (0, L) and (ii) h(x) is odd periodic function.

Define $h(x) = f(x) = 1 - \frac{x}{L}$ in 0, L

$h(x) = 1 - \frac{x}{L}$ in $-L,0$

and $h(x + 2L) = h(x)$.

Thus g(x) is odd in $(-L, L))$ and is periodic with period 2L. The graph in Fig….. is the odd periodic continuation (or extension) of f(x) shown in Fig …. The Fourier cosine series of g(x) in the interval is $(-L, L)$

$$g(x) = \frac{a_0}{2} + \sum a_n \cos\frac{n\pi x}{L}.$$

Here $b_n = \frac{2}{L}\int_0^L f(x)\sin n\frac{\pi x}{L}dx \quad = \frac{2}{L}\int_0^L \left(1-\frac{x}{L}\right)\sin n\frac{\pi x}{L}dx$

$$= \frac{2}{L}\left[\left(1-\frac{x}{L}\right)(-1)\left(-\cos\frac{n\pi x}{L}\right).\frac{L}{\pi x} - \left(-\frac{1}{L}\right).\frac{L^2}{n^2\pi^2}\sin\frac{n\pi x}{L} - \right]_0^L$$

$$b_n = \frac{2}{n\pi}.$$

So $h(x) = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{n}\sin\left(\frac{n\pi x}{L}\right).$

Thus, the required Fourier sine series of f(x) in the interval (0,L) is

$$f(x) = h(x) = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{n}\sin\left(\frac{n\pi x}{L}\right).$$

Q.No.21.: Represent $f(x) = \sin\frac{\pi x}{L}$ in $0 < x < L$ by a Fourier cosine series, Graph the corresponding periodic continuation of f(x).

Sol.: In 0 <x < L, f(x) is neither periodic nor odd nor even.

Construct $g(x) = f(x) = \sin\frac{\pi x}{L}$ in $0 < x < L$

$$g(x) = -\sin\frac{\pi x}{L} \text{ in } -L < x < 0$$

g(x) is even, periodic with 2L.

Fourier cosine series of g(x) is $g(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}.$

Here $a_0 = \frac{2}{L}\int_0^L f(x)dx = \frac{2}{L}\sin\left(\frac{\pi x}{L}\right)dx = \frac{2}{L}\frac{L}{\pi}\left(-\cos\frac{\pi x}{L}\right)\Big|_0^L = \frac{-2}{\pi}[-1-1] = \frac{4}{\pi}.$

For $n \neq 1$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx = \frac{2}{L}\int_0^L \sin\left(\frac{\pi x}{L}\right).\cos\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{2}{L}\frac{1}{2}\int_0^L \left[\sin(1n)\frac{\pi x}{L} + (1-n)\frac{\pi x}{L}\right]dx$$

$$= \frac{1}{L}\left[\frac{-L}{\pi(n+1)}.\cos(1+n)\frac{\pi x}{L} - \frac{L}{\pi(1-n)}.\cos(1-n)\frac{\pi x}{L}\right]_0^L = \frac{1}{\pi}\left[\frac{1}{n+1} - \frac{1}{n-1}\right]\left[(-1)^n + 1\right]$$

$$= -\frac{4}{\pi}\frac{1}{n^2-1} \text{ if n is even}$$

$$= 0 \qquad \text{if n is odd.}$$

i.e., $a_{2n} = -\frac{4}{\pi}\frac{1}{4n^2-1} = \frac{-4}{\pi(2n-1)(2n+1)}$.

For n = 1,

$$a_1 = \frac{2}{L}\int_0^L \sin\frac{\pi x}{L}.\cos\frac{\pi x}{L}dx = 0.$$

Thus

$$g(x) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)(2n+1)}\cos\frac{2n\pi x}{L}$$

$$= \frac{2}{\pi} - \frac{4}{\pi}\left(\frac{1}{1.3}\cos\frac{2\pi x}{L} + \frac{1}{3.5}\cos\frac{4\pi x}{L} + \frac{1}{5.7}\cos\frac{6\pi x}{L} + ...\right)$$

Hence, the Fourier cosine series representation of f(x) in (0, L) is

$$f(x) = g(x) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)(2n+1)}\cos\left(\frac{2n\pi x}{L}\right)$$

Q.No.22.: Show that in the interval (0, 1), $\cos\pi x = \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{n}{4n^2-1}\sin 2n\pi x$.

Sol.: This is a Fourier sine series representation of $\cos\pi x$ in the interval $0 < x < 1$. Put $\pi x = z$, for $0 < x < 1$. then $0 < z < \pi$. Rewriting

$$\cos z = \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{n}{4n^2-1}\sin 2nz.$$

To expand $\cos z$ in Fourier sine series in $0, \pi$

Define

$g(z) = f(z) = \cos z \quad \text{in } 0 < z < \pi$

$g(z) = -\cos z \quad \text{in } -\pi < z < 0$

Now g(z) is an odd function in $(-\pi, \pi)$, and is periodic of period 2π, then

$$g(z) = \sum_{n=1}^{\infty} b_n \sin nz$$

For $n \neq 1$

$$b_n = \frac{2}{\pi}\int_0^{\pi} g(x)\sin nzdz = \frac{2}{\pi}\int_0^{\pi} \cos z.\sin nzdz = \frac{2}{\pi}\int_0^{\pi} \frac{1}{2}[\sin(n+1)z + \sin(n-1)z]dz$$

$$= \frac{1}{\pi}\left[\frac{-\cos(n+1)z}{(n+1)} - \frac{\cos(n-1)z}{(n-1)}\right]_0^{\pi} = -\frac{1}{\pi}\left[\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} - \frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$= \frac{2n}{\pi(n^2-1)}\left[(-1)^n + 1\right],$$

$= 0$ if n is odd.

$$b_1 = \frac{2}{\pi}\int_0^{\pi} \cos z \sin zdz = 0.$$

The required Fourier sine series in $(0, \pi)$ is

$$f(z) = g(z) = \sum_{n=1}^{\infty} \frac{2n\left[(-1)^n + 1\right]}{\pi(n^2-1)}\sin nz = \sum_{n=1}^{\infty} \frac{2(2n).2}{\pi\left[(2n)^2 - 1\right]}\sin 2nz = \sum_{n=1}^{\infty} \frac{8n}{\pi\left[4n^2 - 1\right]}\sin 2nz$$

Replacing z by πx, we get

$$\cos z = \cos \pi x = \sum_{n=1}^{\infty} \frac{8n}{\pi\left[4n^2 - 1\right]}\sin 2n\pi x \text{ in } 0 < x < 1.$$

Q.No.23.: Expand $\pi x - x^2$ in a **half-range sine series** in the interval $(0, \pi)$ upto the first three terms.

Sol.: Let $f(x) = \pi x^2 - x^2 = \sum_{n=1}^{\infty} b_n \sin nx$.

Here $b_0 = \frac{2}{\pi}\int_0^{\pi} \left(\pi x - x^2\right)\sin nxdx$

$$= \frac{2}{\pi}\left[\left(\pi x - x^2\right)\left(-\frac{\cos \pi x}{n}\right) - \pi - 2x\left(-\frac{\sin nx}{n^2}\right) + (-2)\left(-\frac{\cos nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{2\cos n\pi}{n^3} + \frac{2}{n^3}\right] = \frac{4}{\pi n^3}\left[1-(-1)^n\right]$$

$$= 0 \text{ or } \frac{8}{\pi n^3} \text{ according as n is even or odd.}$$

$$\therefore \pi x - x^2 = \frac{8}{\pi}\left(\sin x + \frac{\sin 3x}{3^3} + \frac{\sin 5x}{5^3} + \ldots\ldots\right)$$

Q.No.24.: Develop $\sin\frac{\pi x}{\ell}$ in **half-range cosine series** in the range $0 < x < \ell$.

Sol.: Let $\sin\frac{\pi x}{\ell} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{\ell}$.

Here $a_0 = \frac{2}{\ell}\int_0^{\ell} \sin\frac{\pi x}{\ell} dx = \frac{2}{\ell}\left[-\frac{\cos\frac{\pi x}{\ell}}{\frac{\pi}{\ell}}\right]_0^{\ell} = -\frac{2}{\pi}[\cos\pi - 1] = \frac{4}{\pi}$.

$$a_n = \frac{2}{\ell}\int_0^{\ell} \sin\frac{\pi x}{\ell}\cos\frac{n\pi x}{\ell} dx = \frac{1}{\ell}\int_0^{\ell}\left[\sin(n+1)\frac{\pi x}{\ell} - \sin(n-1)\frac{\pi x}{\ell}\right]dx$$

$$= \frac{1}{\ell}\left[-\frac{\cos(n+1)\frac{\pi x}{\ell}}{(n+1)\frac{\pi}{\ell}} + \frac{\cos(n-1)\frac{\pi x}{\ell}}{(n-1)\frac{\pi}{\ell}}\right]_0^{\ell}$$

$$= \frac{1}{\pi}\left[\left\{-\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1}\right\} - \left\{-\frac{1}{n+1} + \frac{1}{n-1}\right\}\right]$$

$$= \frac{1}{\pi}\left[-\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right].$$

When n is odd:

$$a_n = \frac{1}{\pi}\left[-\frac{1}{n+1} + \frac{1}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right] = 0.$$

When n is even:

$$a_n = \frac{1}{\pi}\left[\frac{1}{n+1} - \frac{1}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right] = \frac{2}{\pi}\left[\frac{1}{n+1} - \frac{1}{n-1}\right] = \frac{4}{\pi(n+1)(n-1)}.$$

$$\therefore \sin\left(\frac{\pi x}{\ell}\right) = \frac{2}{\pi} - \frac{4}{\pi}\left[\frac{\cos\frac{2\pi x}{\ell}}{1.3} + \frac{\cos\frac{4\pi x}{\ell}}{3.5} + \frac{\cos\frac{6\pi x}{\ell}}{5.7} + \ldots\ldots\ldots\right].$$

Q.No.25.: If f(x) has the Fourier series expansion

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{\pi n x}{\ell} + b_n \sin\frac{\pi n x}{\ell} \text{ in } a \le x \le a + 2\ell,$$

$$\text{show that } \int_a^{a+2\ell} [f(x)]^2\, dx = 2\ell\left[\frac{a_0{}^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty} \left(a_n{}^2 + b_n{}^2\right)\right].$$

Sol.: Given $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{\pi} + b_n \sin\frac{n\pi x}{\pi}\right).$ (i)

Multiplying (i) by (x) and integrating from $-\ell$ to ℓ, we get

$$\int_{-\ell}^{\ell} [f(x)]^2\, dx = \frac{a_0}{2}\left(\int_{-\ell}^{\ell} f(x)dx\right) + \sum_{n=1}^{\infty} a_n\left(\int_{-\ell}^{\ell} f(x)\cos\frac{n\pi x}{\ell}dx\right) + \sum_{n=1}^{\infty} b_n\left(\int_{-\ell}^{\ell} f(x)\sin\frac{n\pi x}{\ell}dx\right)$$

Since $a_0 = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)dx, a_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos\frac{n\pi x}{\ell}dx, b_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin\frac{n\pi x}{\ell}dx$.

$$\int_{-\ell}^{\ell} [f(x)]^2\, dx = \frac{a_0}{2}.(\ell a_0) + \sum_{n=1}^{\infty} a_n.(\ell a_n) + \sum_{n=1}^{\infty} b_n.(\ell b_n) = \ell\left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty} \left(a_n^2 + b_n^2\right)\right]$$

$$= 2\ell\left[\frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty} \left(a_n^2 + b_n^2\right)\right].$$

Q.No.26.: If $f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{\pi n x}{2}$ in $(0, \ell)$. Then show that $\int_0^{\ell} [f(x)]^2 dx = \frac{\ell}{2}\sum_{n=1}^{\infty} b_n^2$.

Sol.: By the definition of half-range sine series, we have

$$b_n = \frac{2}{\ell}\int_0^{\ell} f(x)\sin\frac{n\pi x}{\ell}. \qquad (*)$$

Now given $f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{\pi n x}{2}.$ (i)

Multiply by f(x) and integrating from 0 to ℓ, we get

By using (*), we get

$$\int_0^{\ell} [f(x)]^2 dx = \sum_{n=1}^{\infty} b_n \int_0^{\ell} f(x)\sin\frac{n\pi x}{\ell} dx = \sum_{n=1}^{\infty} b_1.\frac{\ell b_n}{2} = \frac{\ell}{2}\sum_{n=1}^{\infty} b_n^2 .$$

Q.No.27.: If an alternating current wave is represented by the series

$$i = \sum_{n=1}^{\infty} I_{2m-1} \sin[(2m-1)wt + \alpha_{2m+1}].$$

Show that the effective value of the current is $\sqrt{\frac{1}{2}\left(I_1^2 + I_3^2 + I_5^2 + \ldots\ldots..\right)}$.

Sol.: We know that, the effective value of a function y = f(x), $0 < x < \ell$, is given by

$$\bar{y} = \left[\frac{1}{\ell}\int_0^{\ell} y^2 dx\right]^{1/2}. \tag{i}$$

given $i = \sum_{n=1}^{\infty} I_{2m-1} \sin[(2m-1)wt + \alpha_{2m+1}]$, defined in $(0, \ell)$, then by using Parsaval's identity ,

$$\frac{2}{\ell}\int_0^{\ell} [f(t)]^2 dt = \sum_{m=1}^{\infty} b_m^2. \tag{ii}$$

Here $f(t) = i(t), \quad b_m = I_{2m-1}$.

The required effective value of the current i is given by $\bar{i}$, where

$$i = \left[\frac{1}{\ell}\int_0^{\ell} [i(t)]^2 dt\right]^{1/2} = \left[\frac{1}{2}\sum_{m=1}^{\infty} b_m^2\right]^{1/2} = \left[\frac{1}{2}\sum_{m=1}^{\infty} I_{2m-1}\right]^{1/2} = \sqrt{\frac{1}{2}\left(I_1^2 + I_3^2 + I_5^2 + \ldots\ldots..\right)}$$

Home Assignments

[Interval $0 < x < \ell$]

Q.No.1.: Show that the series $\frac{1}{\pi}\sum_{n=1}^{\infty} \frac{1}{n}\sin\left(-\frac{2n\pi x}{\ell}\right)$ represents $\frac{1}{2}\ell - x$ when $0 < x < \ell$.

Q.No.2.: Find the **two half-range expansions** of

$$f(x)=\begin{cases} 2kx/L & \text{if } 0<x<(L/2) \\ 2k(L-x)/L & \text{if } L/2<x<L \end{cases}$$

Ans.: **(a).** Even periodic extension: Fourier cosine series

$$f(x)=\frac{k}{2}+\frac{4k}{\pi^2}\sum_{n=1}^{\infty}\left(\frac{2\cos\frac{n\pi}{2}-\cos n\pi-1}{n^2}\right)\times\cos\frac{n\pi x}{L}$$

(b). Odd periodic extension: Fourier sine series.

$$f(x)=\frac{8k}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}.\sin\frac{n\pi x}{L}.$$

Q.No.3.: Represent $f(x)=\begin{cases} 1 & \text{if } 0<x<\frac{1}{2} \\ 0 & \text{if } \frac{1}{2}<x<1 \end{cases}$

in **(a) Fourier sine series**

(b) Fourier cosine series

(c) a Fourier series (with periodic 1)

Ans. (a). $f(x)=\frac{2}{\pi}\sum_{n=1}^{\infty}\left(1-\cos\frac{n\pi}{2}\right)\frac{1}{n}\sin n\pi x$

(b). $f(x)=\frac{1}{2}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi}{2}.\cos n\pi x$

(c). $f(x)=\frac{1}{2}+\frac{1}{\pi}\sum\left[1-(-1)^n\right]\frac{1}{n}.\sin n\pi x$.

Q.No.4.: Find the **Fourier sine and cosine series** of $f(x)=\begin{cases}1 & \text{if } 0<x<\dfrac{1}{2}\\ 0 & \text{if } \dfrac{1}{2}<x<1\end{cases}$.

Ans.: Fourier sine series

$$f(x)=\frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{1}{n^2}\sin\frac{n\pi}{2}-\frac{\pi}{2n}.\cos\frac{n\pi}{2}\right]\sin nx$$

Q.No.5.: Find the two **half-range Fourier cosine series** of $f(x)=x^3$ in (0, L).

Ans.: $f(x)=\dfrac{L^3}{4}+\dfrac{6L^3}{\pi^2}\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}\left[(-1)^n+\frac{2}{n^2\pi^2}\left[1-(-1)^n\right]\right]\frac{\cos n\pi x}{L}$.

Q.No.6.: Represent $f(x)=x^2$ in $0<x<L$ by Fourier sine series.

Ans.: $f(x)=\dfrac{2L^2}{\pi}\displaystyle\sum_{n=1}^{\infty}\left[\frac{1}{n^3\pi^2}\left[(-1)^n-1\right]-\frac{(-1)^n}{n}\right]\sin\frac{n\pi x}{L}$

Q.No.7.: Find the **two half range Fourier series** of f(x) =1 in $0<x<L$

Ans.: Sine series $f(x)=1=\dfrac{2}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}\left[1-(-1)^n\right]\sin\frac{n\pi x}{L}$.

Cosine series $f(x)=1=\dfrac{2}{2}=1$ $(a_0=2,\ \ a_n=0,\ \ \text{for } n\geq 1)$.

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