1^{st} Topic

Integral Calculus

Double Integrals

(Where limits are given)

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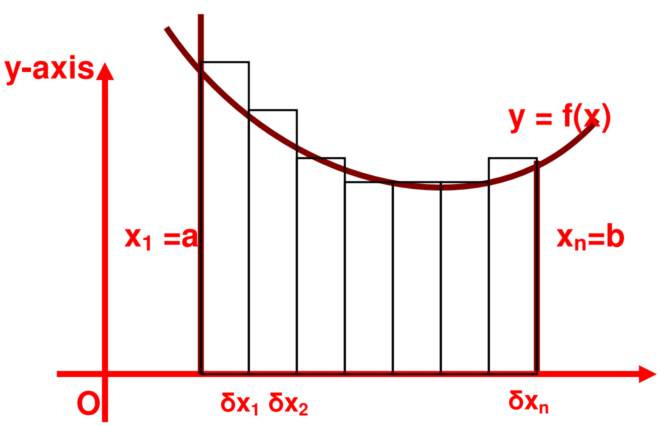
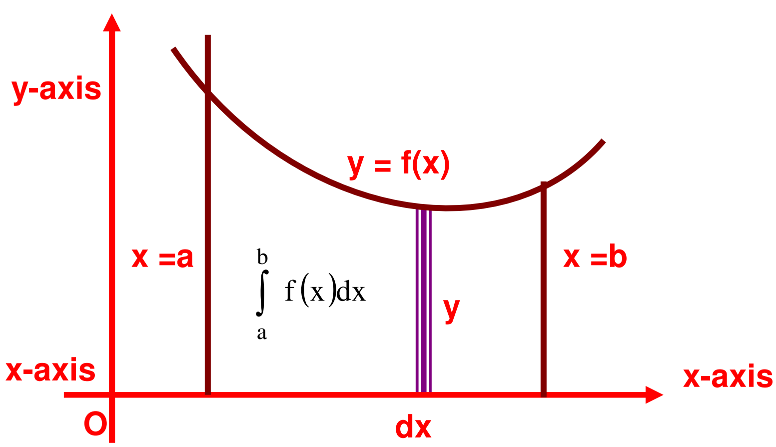
DOUBLE INTEGRALS:

The definite integral $\int_a^b f(x)dx$ is defined as the limit of the sum

$f(x_1)\delta x_1 + f(x_2)\delta x_2 + \ldots\ldots\ldots\ldots + f(x_n)\delta x_n$ i.e. $\sum_{r=1}^{n} f(x_r)\delta x_r$,

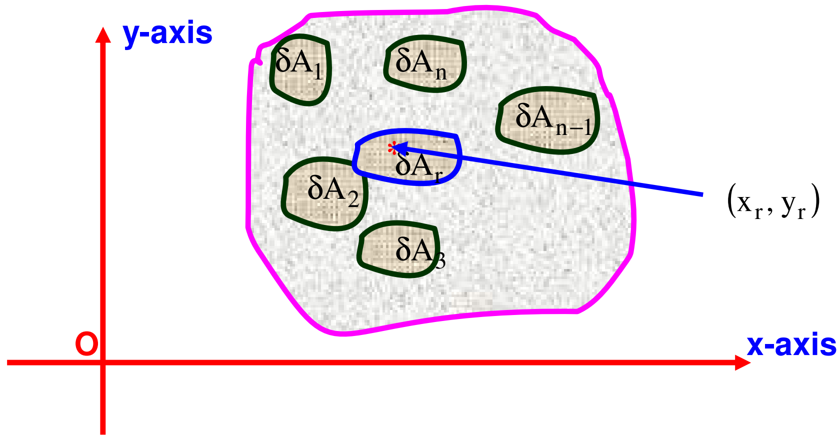
where $n \to \infty$ and each of the lengths $\delta x_1, \delta x_2, \ldots\ldots\ldots\ldots\ldots\ldots$ tends to zero.





A double integral is its counterpart in two dimensions

Consider a function f (x, y) of the independent variables x, y defined at each point in the finite region R of the xy-plane. Divide R into n elementary areas $\delta A_1, \delta A_2, \ldots\ldots\ldots, \delta A_n$. Let (x_r, y_r) be any point with in the rth elementary area δA_r.



Now consider the sum

$$f(x_1, y_1)\delta A_1 + f(x_2, y_2)\delta A_2 + \ldots\ldots\ldots\ldots + f(x_n, y_n)\delta A_n \quad \text{i.e.} \sum_{r=1}^{n} f(x_r, y_r)\delta A_r.$$

Definition:

The limit of this sum, if it exists, as the number of sub-division increases indefinitely (i.e. $n \to \infty$) and consequently (as a result) the area of each sub-division (i.e. δA_r) decreases to zero (i.e. $\delta A_r \to 0$), is defined as the double integral of f (x, y) over the region R and is written as $\int_R f(x, y)dA$.

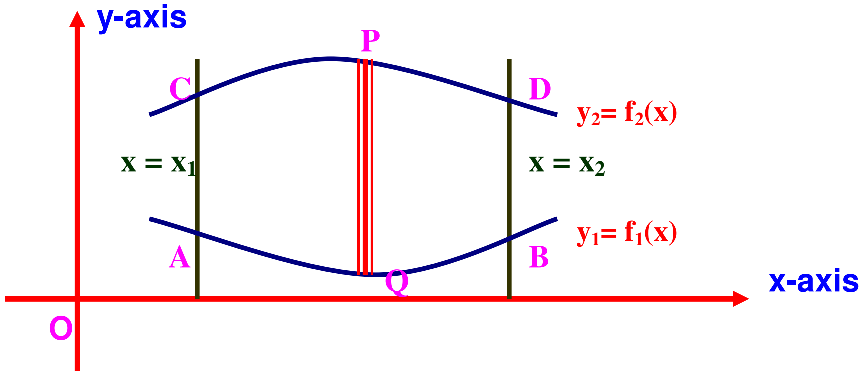
Thus $$\int_R f(x, y)dA = \lim_{\substack{n \to \infty \\ \delta A_r \to 0}} \sum_{r=1}^{n} f(x_r, y_r)\delta A_r. \qquad \text{(i)}$$

For purpose of evaluation, (i) is expressed as the repeated Integrals

$$\int_R f(x, y)dA = \int_{x_1}^{x_2} \int_{y_1}^{y_2} f(x, y)dxdy.$$

Its value is found as follows:

Case 1: When y_1, y_2 are functions of x and x_1, x_2 are constants.



Here f(x, y) is first integrated w. r. t. y keeping x fixed between limits y_1, y_2 and then the resulting expression is integrated w. r. t. x within the limits x_1, x_2 i.e.

$$I_1 = \boxed{\int_{x_1}^{x_2} \boxed{\int_{y_1}^{y_2} f(x,y)\,dy}\, dx},$$

where integration is carried from the inner to the outer rectangle.

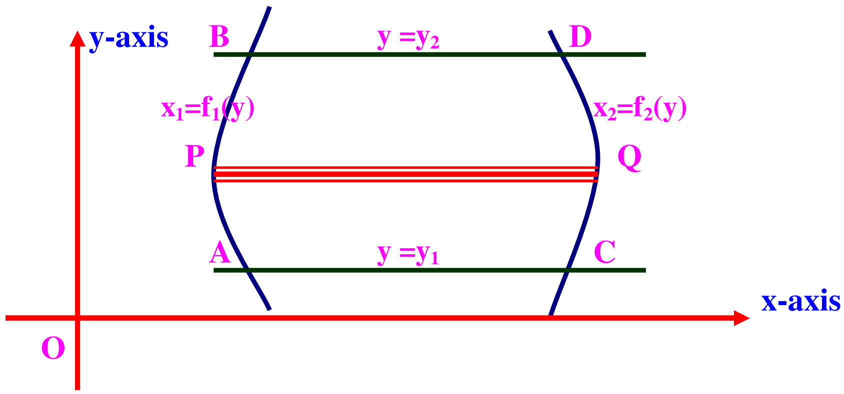
Geometrical illustration with Figure:

Here AB and CD are the two curves whose equations are $y_1 = f_1(x)$ and $y_2 = f_2(x)$. PQ is a vertical strip of width dx.

The inner rectangle integral means that the integration is along one edge of the strip PQ from P to Q (x remaining constant), while the outer rectangle integral corresponds to the sliding of the strip from AC to BD.

Thus, the whole region of integration is the area ABDC.

Case 2: When x_1, x_2 are functions of y and y_1, y_2 are constants.



Here f(x,y) is first integrated w. r. t. x keeping y fixed, with in the limits x_1, x_2 and the resulting expression is integrated w. r. t. y between the limits y_1, y_2 i.e.

$$I_2 = \boxed{\int_{y_1}^{y_2} \boxed{\int_{x_1}^{x_2} f(x,y)dx} dy} ,$$

Geometrical illustration with Figure:

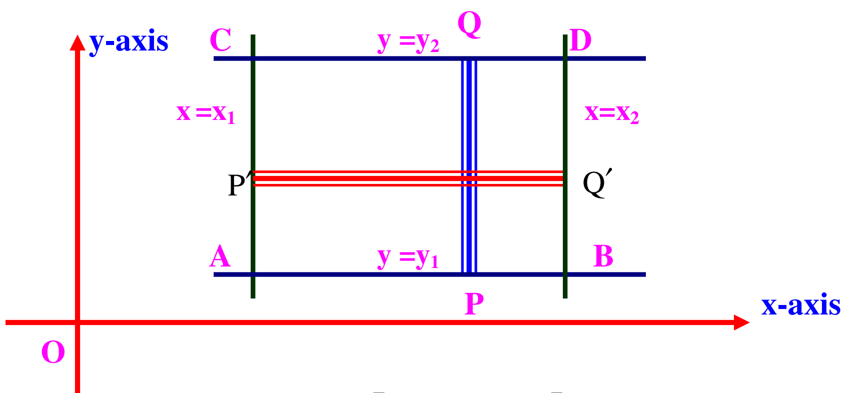
Here AB and CD are the curves $x_1 = f_1(y)$ and $x_2 = f_2(y)$. PQ is horizontal strip of width dy.

Then the inner rectangle indicates that the integration is along one edge of this strip from P to Q while the outer rectangle corresponds to the sliding of this strip from AC to BD.

Thus the whole region of integration is the area ABDC.

Case 3: When both pairs of limits are constant.

The region of integration is the rectangle ABDC as shown in the figure.



$$I_1 = \int_{x_1}^{x_2}\left[\int_{y_1}^{y_2} f(x,y)\,dy\right]dx,$$

In I_1, we integrate along the vertical strip PQ and then slide it from AC to BD.

$$I_2 = \int_{y_1}^{y_2}\left[\int_{x_1}^{x_2} f(x,y)\,dx\right]dy,$$

In I_2, we integrate along horizontal strip $P'Q'$ and then slide it from AB to CD.

Here obviously $I_1 = I_2$.

Thus for constant limits, it hardly matters whether we first integrate w.r.t. x and then w.r.t. y or vice versa.

Here we will discuss those problems in double integrals, where limits are given. By observing the limits, we will decide the order of integration. Since limits are given, so rough sketch of the region of integration is not required.

Q.No.1.: Evaluate the integral $\int_1^2\int_1^3 xy^2\,dx\,dy$.

Sol.: Let $I = \int_1^2\left(\int_1^3 xy^2dx\right)dy = \int_1^2\left[\frac{x^2y^2}{2}\right]_1^3 dy = \int_1^2\left[\frac{(3)^2y^2}{2} - \frac{y^2}{2}\right]dy$

$$= \int_1^2\left(\frac{9y^2 - y}{2}\right)dy = \int_1^2 4y^2dy = \left[\frac{4y^3}{3}\right]_1^2 = \frac{4}{3}(2)^3 - \frac{4}{3}(1)^3 = \frac{4\times 8}{3} - \frac{4}{3} = \frac{28}{3}\text{ .Ans.}$$

Q.No.2.: Evaluate $\int_0^5 \int_0^{x^2} x\left(x^2+y^2\right)dxdy$.

Sol.: $I=\int_0^5\left(\int_0^{x^2}\left(x^3+xy^2\right)dy\right)dx=\int_0^5\left[x^3y+x.\frac{y^3}{3}\right]_0^{x^2}dx=\int_0^5\left[x^3.x^2+x.\frac{x^6}{3}\right]dx$

$$=\int_0^5\left(x^5+\frac{x^7}{3}\right)dx=\left|\frac{x^6}{6}+\frac{x^8}{24}\right|_0^5=5^6\left[\frac{1}{6}+\frac{5^2}{24}\right]=18880.2 \text{ nearly. Ans.}$$

Q.No.3.: Evaluate the integral $\int_0^1 \int_x^{\sqrt{x}} \left(x^2+y^2\right)dxdy$.

Sol.: Let $I=\int_0^1\left(\int_x^{\sqrt{x}}\left(x^2+y^2\right)dy\right)dx=\int_0^1\left[x^2y+\frac{y^3}{3}\right]_x^{\sqrt{x}}dx$

$$=\int_0^1\left[x^2\sqrt{x}+\frac{x^{3/2}}{3}-x^3-\frac{x^3}{3}\right]dx$$

$$=\int_0^1\left(x^{5/2}+\frac{x^{3/2}}{3}-\frac{4x^3}{3}\right)dx=\left[\frac{x^{7/2}}{7/2}+\frac{1}{3}.\frac{x^{5/2}}{5/2}-\frac{x^4}{3}\right]_0^1$$

$$=\left[\frac{2}{7}(1)^{7/2}+\frac{2}{15}(1)^{5/2}-\frac{1}{3}\right]=\frac{2}{7}+\frac{2}{15}-\frac{1}{3}=\frac{30+14-35}{105}=\frac{9}{105}=\frac{3}{35}. \text{ Ans.}$$

Q.No.4.: Evaluate the integral $\int_0^4 \int_0^{x^2} e^{y/x}dxdy$.

Sol.: Let $I=\int_0^4\left(\int_0^{x^2}e^{y/x}dy\right)dx=\int_0^4\left[xe^{y/x}\right]_0^{x^2}dx=\int_0^4\left[xe^{x^2/x}-xe^0\right]dx=\int_0^4\left[xe^x-x\right]dx$

$$=\int_0^4 xe^xdx-\int_0^4 xdx=x\int_0^4 e^xdx-\int_0^4\left[\frac{d}{dx}(x)\int e^xdx\right]-\left[\frac{x^2}{2}\right]_0^4$$

$$=\left[xe^x\right]_0^4-\left[e^x\right]_0^4-\left[\frac{x^2}{2}\right]_0^4=\left(4e^4-0\right)-\left(e^4-1\right)-\left[\frac{(4)^2}{2}-0\right]$$

$$=4e^4-e^4+1-8=3e^4-7. \text{ Ans.}$$

Q.No.5.: Evaluate the integral $\int_0^1 \int_0^{\sqrt{(1+x^2)}} \frac{dxdy}{1+x^2+y^2}$.

Sol.: Let $I=\int_0^1 \left(\int_0^{\sqrt{1+x^2}} \frac{dy}{1+x^2+y^2}\right)dx = \int_0^1 \left[\frac{1}{\sqrt{1+x^2}}\tan^{-1}\left(\frac{y}{\sqrt{1+x^2}}\right)\right]_0^{\sqrt{1+x^2}} dx$

$$=\int_0^1 \left[\frac{1}{\sqrt{1+x^2}}\tan^{-1}\frac{\sqrt{1+x^2}}{\sqrt{1+x^2}}-\frac{1}{\sqrt{1+x^2}}\tan^{-1}0\right]dx$$

$$=\int_0^1 \frac{\pi/4}{\sqrt{1+x^2}}dx=\frac{\pi}{4}\left[\log\left(x+\sqrt{x^2+1}\right)\right]_0^1=\frac{\pi}{4}\left[\log\left(1+\sqrt{2}\right)-\log 1\right].$$ Ans.

Q.No.6.: Evaluate $\int_0^{4a}\int_{\frac{y^2}{4a}}^{y}\left(\frac{x^2-y^2}{x^2+y^2}\right)dxdy$.

Sol.: Let $I=\int_0^{4a}\left[\int_{\frac{y^2}{4a}}^{y}\left(\frac{x^2-y^2}{x^2+y^2}\right)dx\right]dy=\int_0^{4a}\left[\int_{\frac{y^2}{4a}}^{y}\left(\frac{x^2-y^2}{x^2+y^2}-1+1\right)dx\right]dy$

$$=\int_0^{4a}\left[\int_{\frac{y^2}{4a}}^{y}\left(\frac{-2y^2}{x^2+y^2}+1\right)dx\right]dy=\int_0^{4a}\left(\left[-2y^2\times\frac{1}{y}\times\tan^{-1}\left(\frac{x}{y}\right)+x\right]_{\frac{y^2}{4a}}^{y}\right)dy$$

$$=\int_0^{4a}\left(\left(-2y\times\frac{\pi}{4}+y\right)-\left[-2y\times\tan^{-1}\left(\frac{y}{4a}\right)+\frac{y^2}{4a}\right]\right)dy$$

$$=\int_0^{4a}\left[-\frac{\pi y}{2}+y+2y.\tan^{-1}\frac{y}{4a}-\frac{y^2}{4a}\right]dy$$

$$=\left[\frac{-\pi}{2}.\frac{y^2}{2}+\frac{y^2}{2}+\left\{\left(y^2+16a^2\right)\tan^{-1}\frac{y}{4a}-4ay\right\}-\frac{y^3}{12a}\right]_0^{4a}$$

$$=\left[\frac{-\pi}{4}\times16a^2+\frac{16a^2}{2}+\left(16a^2+16a^2\right)\times\frac{\pi}{4}-16a^2-\frac{64a^3}{12a}\right]$$

$$= \left[-4\pi a^2 + 8a^2 + 8\pi a^2 - 16a^2 - \frac{16}{3}a^2 \right] = a^2\left[4\pi - \frac{40}{3} \right] = 8\left[\frac{\pi}{2} - \frac{5}{3} \right]a^2. \text{ Ans.}$$

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Home Assignments

Q.No.1.: Evaluate the integral $\int_0^{2a}\left(\int_0^{x^2/4a} xy\,dy\right)dx$.

Q.No.2.: Evaluate the integral $\int_0^{a}\left(\int_0^{\sqrt{a^2-y^2}} xy\,dx\right)dy$.

Q.No.3.: Evaluate the integral $\int_{-b}^{b}\left[\int_{-a\sqrt{1-\frac{y^2}{b^2}}}^{a\sqrt{1-\frac{y^2}{b^2}}} (x+y)^2\,dx\right]dy$.

Q.No.4.: Evaluate the integral $\int_0^{1}\left[\int_{x^2}^{x} xy(x+y)\,dy\right]dx$.

Q.No.5.: Evaluate the integral $\int_1^{2}\left(\int_0^{4-x^2} (x+y)\,dy\right)dx$.

Q.No.6.: Evaluate the integral $\int_0^{4a}\left(\int_{y^2/4a}^{2\sqrt{ay}} dx\right)dy$.

Q.No.7.: Evaluate the integral $\int_0^{1}\left[\int_{ay^2}^{ay} \left(x^2+y^2\right)dx\right]dy$.

Q.No.8.: Evaluate the integral $\int_0^{a}\left[\int_0^{y^2/a} \frac{dx}{\sqrt{1-\frac{a^2x^2}{y^4}}}\right]dy$.

Q.No.9.: Evaluate the integral $\int_0^a \left[\int_0^{\frac{a}{b}\sqrt{a^2-x^2}} x^3 y dy \right] dx$.

Q.No.10.: Evaluate the integral $I = \int_1^{e^2} \left[\int_{\log y}^{2} dx \right] dy$.

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1^{st} Topic

Integral Calculus

Triple Integrals

[Where limits are given]

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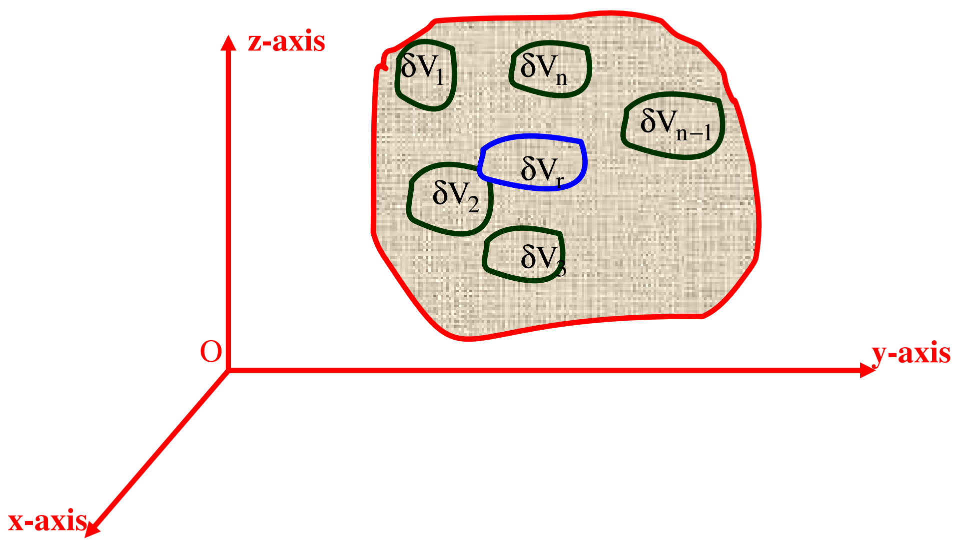
Triple Integrals:

Consider a function f(x, y, z) is defined at every point of the 3-demensional finite region V. Divide V into n elementary volumes $\delta V_1, \delta V_2, \ldots, \delta V_r, \ldots, \delta V_n$. Let (x_r, y_r, z_r) be any point within the r^{th} sub-division δV_r.



Now consider the sum $\sum_{r=1}^{n} f(x_r, y_r, z_r)\,\delta V_r$.

The limit of this sum, if it exist, as $n \to \infty$ and consequently $\delta V_r \to 0$ is called the triple integral of $f(x, y, z)$ over the region V and is denoted by $\int f(x, y, z)dV$.

For purposes of the evaluation, it can be expressed as the repeated integrals

$$\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} f(x, y, z)dxdydz.$$

If x_1, x_2 are constants; y_1, y_2 are either constants or functions of x and z_1, z_2 are either constant or functions of x and y, then this integral is evaluated as follows:

First $f(x, y, z)$ is integrating w. r. t. z between the limits z_1, and z_2 keeping x and y fixed. The resulting expression is integrated w. r. t. y between the limits y_1, and y_2 keeping x constant. The result just obtained is finally integrated w. r. t. x from x_1, and x_2 .

Thus $I = \left[\int_{x_1}^{x_2}\left\{\int_{y_1(x)}^{y_2(x)}\left(\int_{z_1(x,y)}^{z_2(x,y)} f(x, y, z)dz\right)dy\right\}dx\right]$,

where the integration is carried out from the innermost bracket to the outermost bracket. This order of integration may be different for different type of limits.

Here we will discuss those problems in triple integrals, where limits are given. By observing the limits, we will decide the order of integration. Since limits are given, so rough sketch of the region of integration is not required.

Q.No.1.: Evaluate $\int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z}(x + y + z)dxdydz$.

Sol.: We have $I = \int_{-1}^{1}\left\{\int_{0}^{z}\left(\int_{x-z}^{x+z}(x + y + z)dy\right)dx\right\}dz = \int_{-1}^{1}\left(\int_{0}^{z}\left|xy + \frac{y^2}{2} + yz\right|_{x-z}^{x+z}dx\right)dz$

$$= \int_{-1}^{1}\left(\int_{0}^{z}\left\{x[(x+z)-(x-z)] + \frac{1}{2}\left[(x+z)^2-(x-z)^2\right] + [(x+z)-(x-z)]z\right\}dx\right)dz$$

$$= \int_{-1}^{1}\left(\int_{0}^{z}\left[(x+z)(2z)+\frac{1}{2}4xz\right]dx\right)dz$$

$$= 2\int_{-1}^{1}\left|\frac{x^2z}{2}+z^2x+\frac{x^2z}{2}\right|_0^z dz = 2\int_{-1}^{1}\left[\frac{z^3}{2}+z^3+\frac{z^3}{2}\right]dz = 4\left|\frac{z^4}{4}\right|_{-1}^{1} = 0.\ \text{Ans.}$$

Q.No.2.: Evaluate $\int_0^1 \int_0^{\sqrt{(1-x^2)}} \int_0^{\sqrt{(1-x^2-y^2)}} xyz\,dx\,dy\,dz$.

Sol.: We have $I = \int_0^1 x\left[\int_0^{\sqrt{(1-x^2)}} y\left\{\int_0^{\sqrt{(1-x^2-y^2)}} z\,dz\right\}dy\right]dx = \int_0^1 x\left\{\int_0^{\sqrt{(1-x^2)}} y.\left|\frac{z^2}{2}\right|_0^{\sqrt{(1-x^2-y^2)}} dy\right\}dx$

$$= \int_0^1 x\left\{\int_0^{\sqrt{(1-x^2)}} y.\frac{1}{2}\left(1-x^2-y^2\right)dy\right\}dx = \frac{1}{2}\int_0^1 x\left|\left(1-x^2\right)\frac{y^2}{2}-\frac{y^4}{4}\right|_0^{\sqrt{(1-x^2)}} dx$$

$$= \frac{1}{8}\int_0^1\left[\left(1-x^2\right)^2.2x-\left(1-x^2\right)^4.x\right]dx = \frac{1}{8}\int_0^1\left(x-2x^3+x^5\right)dx$$

$$= \frac{1}{8}\left|\frac{x^2}{2}-\frac{2x^4}{4}+\frac{x^6}{6}\right|_0^1 = \frac{1}{8}\left(\frac{1}{2}-\frac{1}{2}+\frac{1}{6}\right) = \frac{1}{48}.\ \text{Ans.}$$

Q.No.3.: Evaluate $\int_0^1 \int_0^2 \int_1^2 x^2yz\,dz\,dy\,dx$.

Sol.: We have $I = \int_0^1 \int_0^2 \int_1^2 x^2yz\,dz\,dy\,dx = \int_0^1\left[\int_0^2\left(\int_1^2 x^2yz\,dz\right)dy\right]dx$

$$= \int_0^1\left[\int_0^2\left(\frac{x^2yz^2}{2}\right)_1^2 dy\right]dx = \int_0^1\left[\int_0^2\left(2x^2y-\frac{x^2y}{2}\right)dy\right]dx$$

$$= \int_0^1\left[\frac{2x^2y^2}{2}-\frac{x^2y^2}{4}\right]_0^2 dx = \int_0^1\left[4x^2-x^2\right]dx = \int_0^1 3x^2dx$$

$$= \left[\frac{3x^3}{3}\right]_0^1 = \left[x^3\right]_0^1 = [1-0] = 1.\ \text{Ans.}$$

Q.No.4.: Evaluate $\int_{-c}^{c}\int_{-b}^{b}\int_{-a}^{a}\left(x^2+y^2+z^2\right)dx\,dy\,dz$.

Sol.: We have $I=\int_{-b}^{c}\left\{\int_{-a}^{b}\left(\int_{-a}^{a}\left(x^2+y^2+z^2\right)dx\right)dy\right\}dz=8\int_{0}^{c}\left\{\int_{0}^{b}\left[\frac{x^3}{3}+y^2x+z^2x\right]_0^a dy\right\}dz$

$$=8\int_0^c\left\{\int_0^b\left[\frac{a^3}{3}+ay^2+az^2\right]dy\right\}dz=8\int_0^c\left[\frac{a^3}{3}y+\frac{ay^3}{3}+az^2y\right]_0^b dz$$

$$=8\int_0^c\left[\frac{a^3b}{3}y+\frac{ab^3}{3}+abz^2\right]dz=8\left[\frac{a^3b}{3}z+\frac{ab^3}{3}z+\frac{abz^3}{3}\right]_0^c dz$$

$$=\frac{8}{3}\left[a^3bc+ab^3c+abc^3\right]=\frac{8}{3}abc\left[a^2+b^2+c^2\right]. \text{ Ans.}$$

Q.No.5.: Evaluate $\int_0^4\int_0^{2\sqrt{z}}\int_0^{\sqrt{\left(4z-x^2\right)}}dy\,dx\,dz$.

Sol.: We have $I=\int_0^4\left\{\int_0^{2\sqrt{z}}\left(\int_0^{\sqrt{4z-x^2}}dy\right)dx\right\}dz=\int_0^4\left\{\int_0^{2\sqrt{z}}[y]_0^{\sqrt{4z-x^2}}dx\right\}dz$

$$=\int_0^4\left\{\int_0^{2\sqrt{z}}\sqrt{4z-x^2}\,dx\right\}dz$$

Put $2\sqrt{z}=\rho\Rightarrow 4z=\rho^2$.

$$\therefore I=\int_0^4\left\{\int_0^{\rho}\sqrt{\left(\rho^2-x^2\right)}dx\right\}dz.$$

Put $x=\rho\sin\theta\Rightarrow dx=\rho\cos\theta d\theta$.

When $x=0,\ \theta=0;\quad x=\rho,\ \theta=\frac{\pi}{2}$.

$$\therefore I=\int_0^4\left\{\int_0^{\pi/2}\sqrt{\rho^2-\rho^2\sin^2\theta}.\rho\cos\theta d\theta\right\}dz=\int_0^4\left\{\int_0^{\pi/2}\rho^2\cos^2\theta d\theta\right\}dz=\int_0^4 4z\left(\frac{1}{2}\times\frac{\pi}{2}\right)dz$$

$$=\pi\int_0^4 zdz=\pi\left[\frac{z^2}{2}\right]_0^4=\pi.\frac{16}{2}=8\pi.\text{ Ans.}$$

Q.No.6.: Evaluate $\int_0^a\int_0^x\int_0^{x+y}e^{x+y+z}dz\,dy\,dx$.

Sol.: We have $I=\int_0^a\int_0^x\int_0^{x+y}e^{x+y+z}dz\,dy\,dx=\int_0^a\left[\int_0^x\left(\int_0^{x+y}e^{x+y+z}dz\right)dy\right]dx$

$$=\int_0^a\left[\int_0^x\left(e^{2(x+y)}-e^{x+y}\right)dy\right]dx=\int_0^a\left[\frac{e^{2(x+y)}}{2}-e^{x+y}\right]_0^x dx$$

$$=\int_0^a\left[\frac{e^{4x}}{2}-e^{2x}-\frac{e^{2x}}{2}+e^x\right]dx=\left[\frac{e^{4x}}{8}-\frac{e^{2x}}{2}-\frac{e^{2x}}{4}+e^x\right]_0^a$$

$$=\frac{e^{4a}}{8}-\frac{e^{2a}}{2}-\frac{e^{2a}}{4}+e^a-\frac{1}{8}+1+\frac{1}{2}-1$$

$$=\frac{e^{4a}}{8}-\frac{3e^{2a}}{4}+e^a-\frac{3}{8}.\text{ Ans.}$$

Q.No.7.: Evaluate $\int_1^e\int_1^{\log y}\int_1^{e^x}\log z\,dz\,dx\,dy$.

Sol.: We have $I=\int_1^e\left\{\int_1^{\log y}\left(\int_1^{e^x}\log z dz\right)dx\right\}dy=\int_1^e dy\int_1^{\log y}dx\left(\int_1^{e^x}\log zd\right)z$

$$=\int_1^e dy\int_1^{\log y}dx\left[\log z.z.z\right]_1^{e^x}=\int_1^e dy\int_1^{\log y}dx\left[\log e^x.e^x-1\log z\right]$$

$$=\int_1^e dy\int_1^{\log y}dx\left[x.e^x-e^x+1\right]dx=\int_1^e dy\left[x.e^x-e^x-e^x+x\right]_1^{\log y}$$

$$=\int_1^e dy\left[x.e^x-2e^x+x\right]_1^{\log y}$$

$$=\int_1^e\left(\log ye^{\log y}-2e^{\log y}+\log y-1.e^1+2e^1-1\right)dy$$

$$=\int_1^e (y\log y-2y+\log y-e-1).dy$$

$$=\left[\frac{y^2}{2}\log y-\frac{y^2}{4}+y\log y-y-y^2+y-y\right]_1^e$$

$$=\left(\frac{e^2}{2}-\frac{e^2}{4}+e-e-e^2+e^2-e\right)-\left(-\frac{1}{4}-1-1+e-1\right)$$

$$=\frac{e^2}{4}-2e+\frac{13}{4}=\frac{1}{4}\left(e^2-8e+13\right). \text{ Ans.}$$

Q.No.8.: Evaluate $\int_0^{\pi/2}\int_0^{a\sin\theta}\int_0^{\left(a^2-r^2\right)/a} rdz\,dr\,d\theta$.

Sol.: We have $I=\int_0^{\pi/2}\int_0^{a\sin\theta}\int_0^{\left(a^2-r^2\right)/a} rdz\,dr\,d\theta=\int_0^{\pi/2}\left[\int_0^{a\sin\theta}\left(\int_0^{\left(a^2-r^2\right)/a} r.dz\right)dr\right]d\theta$

$$=\int_0^{\pi/2}\left[\int_0^{a\sin\theta}(rz)_0^{\left(a^2-r^2\right)/a}.dr\right]d\theta=\int_0^{\pi/2}\left[\int_0^{a\sin\theta}\frac{r\left(a^2-r^2\right)}{a}.dr\right]d\theta$$

$$=\int_0^{\pi/2}\left[a\frac{r^2}{2}-\frac{r^4}{4a}\right]_0^{a\sin\theta}d\theta=\int_0^{\pi/2}\left[\frac{a}{2}\left(a^2\sin^2\theta\right)-\frac{a^4\sin^4\theta}{4a}\right]d\theta$$

$$=\int_0^{\pi/2}\left[\frac{a^3}{2}\sin^2\theta-\frac{a^3}{4}\sin^4\theta\right]d\theta=\frac{a^3}{2}\left[\frac{1}{2}\times\frac{\pi}{2}\right]-\frac{a^3}{4}\left[\frac{3.1}{4.2}\times\frac{\pi}{2}\right]$$

$$=\frac{a^3\pi}{8}-\frac{3a^3\pi}{64}=\left(\frac{8-3}{64}\right)a^3\pi=\frac{5a^3\pi}{64}. \text{ Ans.}$$

Q.No.9.: Evaluate $\int_0^1\int_{y^2}^1\int_0^{1-x} xdzdxdy$.

Sol.: $\int_0^1\int_{y^2}^1\int_0^{1-x} xdzdxdy=\int_0^1\left[\int_{y^2}^1\left(\int_0^{1-x} xdz\right)dx\right]dy=\int_0^1\left[\int_{y^2}^1 (xz)_0^{1-x}dx\right]dy$

$$=\int_0^1\left[\int_{y^2}^1 x(1-x)dx\right]dy$$

$$=\int_0^1\left[\frac{x^2}{2}-\frac{x^3}{3}\right]_{y^2}^1 dy=\int_0^1\left[\left(\frac{1}{2}-\frac{1}{3}\right)-\left(\frac{y^4}{2}-\frac{y^6}{3}\right)\right]dy$$

$$=\left[\frac{y}{2}-\frac{y}{3}-\frac{y^5}{10}+\frac{y^7}{21}\right]_0^1=\left[\frac{1}{2}-\frac{1}{3}-\frac{1}{10}+\frac{1}{21}\right]$$

$$=\frac{105-70-21+10}{210}=\frac{24}{210}=\frac{4}{35}\text{. Ans.}$$

Q.No.10.: Evaluate $\int_{-3}^{3}\int_0^1\int_1^2 (x+y+z)dzdydx$.

Sol.: $I=\int_{-3}^{3}\int_0^1\int_1^2 (x+y+z)dzdydx=\int_{-3}^{3}\left[\int_0^1\left\{\int_1^2 (x+y+z)dz\right\}dy\right]dx$

$$=\int_{-3}^{3}\left[\int_0^1\left\{xz+yz+\frac{z^2}{2}\right\}_1^2 dy\right]dx=\int_{-3}^{3}\left[\int_0^1 (2x+2y+2)-\left(x+y+\frac{1}{2}\right)dy\right]dx$$

$$=\int_{-3}^{3}\left[2xy+y^2+2y-xy-\frac{y^2}{2}-\frac{y}{2}\right]_0^1 dx=\int_{-3}^{3}\left(2x+1+2-x-\frac{1}{2}-\frac{1}{2}\right)dx$$

$$=\int_{-3}^{3}(x+3-1)dx=\int_{-3}^{3}(x+2)dx=\left[\frac{x^2}{2}+2x\right]_{-3}^{3}=\left(\frac{9}{2}+6\right)-\left(\frac{9}{2}-6\right)$$

$$=\frac{21}{2}+\frac{3}{2}=\frac{24}{2}=12\text{. Ans.}$$

Q.No.11.: Evaluate $\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} dzdydx$.

Sol.: $$I=\int_0^1\int_0^{\sqrt{1-x^2}}\sqrt{1-x^2-y^2}\,dydx$$

$$=\int_0^1\left[\int_0^{\sqrt{1-x^2}}\frac{y}{2}\sqrt{1-x^2-y^2}+\frac{\left(\sqrt{1-x^2}\right)^2}{2}\sin^{-1}\frac{y}{\sqrt{1-x^2}}\right]dx$$

$$=\int_0^1\left[\frac{\sqrt{1-x^2}}{2}\times 0+\frac{1-x^2}{2}\sin^{-1}\frac{\sqrt{1-x^2}}{\sqrt{1-x^2}}\right]dx=\int_0^1\left[\frac{\left(1-x^2\right)}{2}\sin^{-1}1\right]dx$$

$$=\int_0^1\left[\frac{\left(1-x^2\right)}{2}\times\frac{\pi}{2}\right]dx=\frac{\pi}{4}\int_0^1\left(1-x^2\right)dx=\frac{\pi}{4}\left[x-\frac{x^3}{3}\right]_0^1$$

$$=\frac{\pi}{4}\left[1-\frac{1}{3}\right]=\frac{\pi}{4}\times\frac{2}{3}=\frac{\pi}{6}.\text{ Ans.}$$

Q.No.12.: Evaluate $\int_0^a\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_0^{mx} z^2dzdydx$.

Sol.: $$I=\int_0^a\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\left[\int_0^{mx}z^2dz\right]dydx=\int_0^a\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\left[\frac{z^3}{3}\right]_0^{mx}dydx$$

$$=\int_0^a\left[\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\left(\frac{m^3x^3}{3}\right)dy\right]dx=\int_0^a\frac{m^3x^3}{3}\left[\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}dy\right]dx=\int_0^a\frac{m^3x^3}{3}[y]_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}dx$$

$$=\int_0^a\frac{m^3x^3}{3}\left[\sqrt{a^2-x^2}+\sqrt{a^2-x^2}\right]dx=\int_0^a\left[\frac{2}{3}m^3x^3\sqrt{a^2-x^2}\right]dx$$

$$=\frac{2}{3}m^3\int_0^a\left(x^3\sqrt{a^2-x^2}\right)dx$$

Put $x=a\sin\theta$ $\quad\therefore dx=a\cos\theta d\theta$

$$=\frac{2}{3}m^3\int_0^{\pi/2}a^3\sin^3\theta\sqrt{a^2-a^2\sin^2\theta}.a\cos\theta d\theta=\frac{2}{3}m^3\int_0^{\pi/2}a^5\sin^3\theta\sqrt{1-\sin^2\theta}.\cos\theta d\theta$$

$$= \frac{2}{3}a^5m^3 \int_0^{\pi/2} \sin^3\theta.\cos^2\theta d\theta$$

Now $\cos\theta = t$, $\therefore -\sin\theta d\theta = dt$

$$= \frac{2}{3}a^5m^3 \int_0^1 -\left(1-t^2\right)t^2 dt = \frac{2}{3}a^5m^3 \int_0^1 \left(t^2 - t^4\right)dt = \frac{2}{3}m^3a^5\left(\frac{t^3}{3} - \frac{t^5}{5}\right)_0^1 dt$$

$$= \frac{2}{3}m^3a^5\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{2}{3}m^3a^5\left(\frac{5-3}{15}\right) = \frac{2}{3}m^3a^5\left(\frac{2}{15}\right) = \frac{4m^3a^5}{45}. \text{ Ans.}$$

Q. No. 13.: Evaluate $\int_0^{\log 2}\int_0^x\int_0^{x+\log y} e^{x+y+z} dzdydx$.

Sol.: $I = \int_0^{\log 2}\int_0^x\int_0^{x+\log y} e^{x+y+z} dzdydx = \int_0^{\log 2}\int_0^x \left[e^{x+y+z}\right]_0^{x+\log y} dydx$

$$= \int_0^{\log 2}\int_0^x \left[e^{x+y+x+\log y} - e^{x+y}\right]dydx = \int_0^{\log 2}\int_0^x \left[e^{2x}e^y.y - e^x.e^y\right]dydx$$

$$= \int_0^{\log 2} e^{2x}\int_0^x \left[e^y.ydy - e^x\int_0^x e^y dy\right]dx = \int_0^{\log 2}\left[e^{2x}\left(y.e^y - e^y\right)_0^x - e^x\left(e^y\right)_0^x\right]dx$$

$$= \int_0^{\log 2}\left[e^{2x}\left(x.e^x - e^x + 1\right) - e^x\left(e^x - 1\right)\right]dx = \int_0^{\log 2}\left[xe^{3x} - e^{3x} + e^{2x} - e^{2x} + e^x\right]dx$$

$$= \int_0^{\log 2} xe^{3x}dx - \int_0^{\log 2} e^{3x} + \int_0^{\log 2} e^x = \left[\frac{xe^{3x}}{3} - \frac{e^{3x}}{9}\right]_0^{\log 2} - \left[\frac{e^{3x}}{3}\right]_0^{\log 2} + \left[e^x\right]_0^{\log 2}$$

$$= \left[\frac{8\log 2}{3} - \frac{8}{9} + \frac{1}{9}\right] - \left[\frac{8}{3} - \frac{1}{3}\right] + [2-1] = \left[\frac{8\log 2}{3} - \frac{7}{9}\right] - \left[\frac{7}{3}\right] + 1$$

$$= \frac{8\log 2}{3} - \frac{7}{9} - \frac{7}{3} + 1 = \frac{8\log 2}{3} - \left(\frac{7+21-9}{9}\right) = \frac{8\log 2}{3} - \left(\frac{28-9}{9}\right)$$

$$= \frac{8\log 2}{3} - \frac{19}{9}. \text{ Ans.}$$

Q.No.14.: Evaluate $\int_0^4 \int_0^{\frac{12-3z}{4}} \int_0^{\frac{12-4y-3z}{6}} dxdydz$.

Sol.: $I=\int_0^4\int_0^{\frac{12-3z}{4}}\int_0^{\frac{12-4y-3z}{6}}dxdydz=\int_0^4\left[\int_0^{\frac{12-3z}{4}}\left\{\int_0^{\frac{12-4y-3z}{6}}dx\right\}dy\right]dz$

$$=\int_0^4\left[\int_0^{\frac{12-3z}{4}}\frac{12-4y-3z}{6}dy\right]dz=\frac{1}{6}\int_0^4 12[y]_0^{\frac{12-3z}{4}}-\frac{4}{2}\left[y^2\right]_0^{\frac{12-3z}{4}}-3z[y]_0^{\frac{12-3z}{4}}dz$$

$$=\frac{1}{6}\int_0^4 12\left(\frac{12-3z}{4}\right)-2\left(\frac{12-3z}{4}\right)^2-3z\left(\frac{12-3z}{4}\right)dz$$

$$=\frac{1}{6}\int_0^4\left(\frac{12-3z}{4}\right)\left[12-\frac{2(12-3z)}{4}-3z\right]dz=\frac{1}{6}\int_0^4\left(\frac{12-3z}{4}\right)\left[\frac{(24-12+3z-6z)}{2}\right]dz$$

$$=\frac{1}{6}\int_0^4\left(\frac{12-3z}{4}\right)\left[12-\frac{2(12-3z)}{4}-3z\right]dz=\frac{1}{8}\int_0^4\left[(4-z)\frac{(12-3z)}{2}\right]dz$$

$$=\frac{3}{16}\int_0^4(4-z)^2dz=\frac{3}{16}\left[\frac{-(4-z)^3}{3}\right]_0^4=\frac{3}{16}\times\frac{4^3}{3}=4\text{. Ans.}$$

Q.No.15.: Evaluate $\int_0^1\int_0^{1-x}\int_0^{x+y}e^x dxdydz$.

Sol.: $I=\int_0^1\int_0^{1-x}\int_0^{x+y}e^x dxdydz=\int_0^1\left\{\int_0^{1-x}\left(\int_0^{x+y}e^x dz\right)dy\right\}dx$

$$=\int_0^1\left\{\int_0^{1-x}\left(e^x z\right)_0^{x+y}dy\right\}dx=\int_0^1\left\{\int_0^{1-x}e^x(x+y)dy\right\}dx=\int_0^1\left\{e^x\left(xy+\frac{y^2}{2}\right)_0^{1-x}\right\}dx$$

$$=\int_0^1\left\{e^x\left(x(1-x)+\frac{(1-x)^2}{2}\right)\right\}dx=\int_0^1\left\{e^x\left(x-x^2+\frac{1+x^2-2x}{2}\right)\right\}dx$$

$$=\int_0^1\left\{e^x\left(-\frac{x^2}{2}+\frac{1}{2}\right)\right\}dx=-\frac{1}{2}\left[\left(x^2e^x\right)_0^1-\int_0^1 2xe^x dx\right]+\left(\frac{e^x}{2}\right)_0^1$$

$$= -\frac{1}{2}\left[(e-0) - 2\left\{\left(xe^x\right)_0^1 - \int_0^1 e^x dx\right\}\right] + \left(\frac{e}{2} - \frac{1}{2}\right)$$

$$= -\frac{e}{2} + \{e - (e-1)\} + \left(\frac{e}{2} - \frac{1}{2}\right) = -\frac{e}{2} + 1 + \left(\frac{e}{2} - \frac{1}{2}\right) = \frac{1}{2}\text{. Ans.}$$

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2^{nd} Topic

Integral Calculus

Double Integrals

[Where limits are not given,

but region of integration is given]

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Here we will discuss those problems in double integrals, where limits are not given, but region of integration is given.

Since limits are not given, so rough sketch of the region of integration is required. So in that case, we can integrate first w.r.t. x or y depends upon our desire.

If we suppose, strip is parallel to x-axis (horizontal strip), then integrate w.r.t. x first and then w.r.t. y, whereas if we suppose strip is parallel to y-axis (vertical strip), then integrate w.r.t. y first and then w.r.t. x.

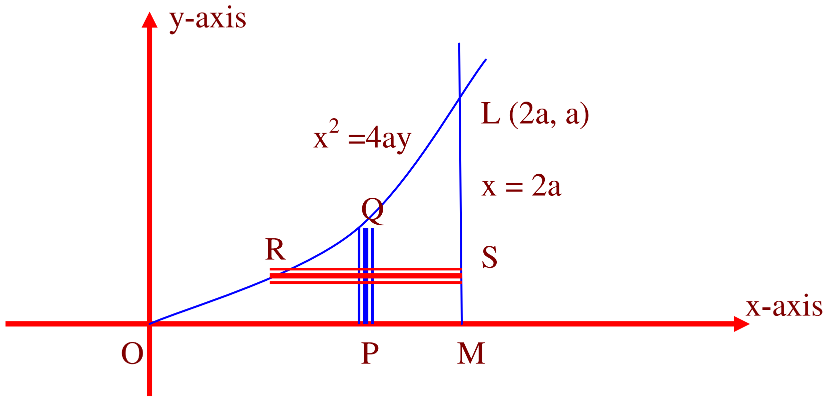
Let us clear this concept with the help of problems given below.

Q.No.1.: Evaluate $\iint\limits_A xy\,dx\,dy$, where A is the domain bounded by x-axis, ordinate $x = 2a$ and the curve $x^2 = 4ay$.

Sol.: **First way to solve this problem:**

Let us consider the strip, parallel to y-axis

The line $x = 2a$ and the parabola $x^2 = 4ay$ intersect at $L(2a, a)$. This figure shows the domain A which is the area OML.



Now let us suppose strip is parallel to y-axis. In that case integrating first over a vertical strip PQ, w.r.t. y from $P(y = 0)$ to $Q\left(y = \frac{x^2}{4a}\right)$ on the parabola and then w.r.t. x from x = 0 to x = 2a, we have

$$\iint_A xydxdy = \int_0^{2a}\left(\int_0^{x^2/4a} xydy\right)dx = \int_0^{2a} x\left[\frac{y^2}{2}\right]_0^{x^2/4a} dx$$

$$= \frac{1}{32a^2}\int_0^{2a} x^5dx = \frac{1}{32a^2}\left|\frac{x^6}{6}\right|_0^{2a} = \frac{a^4}{3}. \text{ Ans.}$$

Second way to solve this problem:

Let us consider the strip, parallel to x-axis

Now let us suppose strip is parallel to x-axis. In that case integrating first over a horizontal strip RS, w. r. t. x from $R\left(x = 2\sqrt{ay}\right)$ on the parabola to $S(x = 2a)$ and then w. r. t. y from y = 0 to y = a, we get

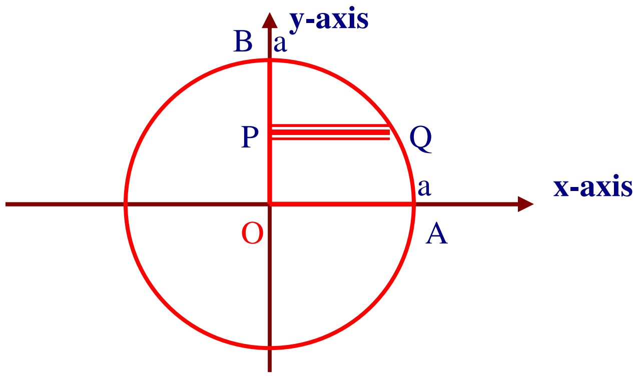
$$\iint_A xydxdy = \int_0^a\left(\int_{2\sqrt{(ay)}}^{2a} xydx\right)dy = \int_0^a y\left[\frac{x^2}{2}\right]_{2\sqrt{(ay)}}^{2a} dy$$

$$= 2a\int_0^a \left(ay - y^2\right)dy = 2a\left|\frac{ay^2}{2} - \frac{y^3}{3}\right|_0^a = \frac{a^4}{3}. \text{ Ans.}$$

Q.No.2.: Evaluate the integral $\iint xydxdy$ over the positive quadrant of the circle

$$x^2 + y^2 = a^2.$$

Sol.: The region OAB, represents the positive quadrant of the circle $x^2 + y^2 = a^2$.



Let us suppose that the strip is parallel to x-axis, then in this region, x varies from 0 to $\sqrt{a^2 - y^2}$ and y varies from 0 to a,. Hence

$$I = \int_0^a \left(\int_0^{\sqrt{a^2-y^2}} xydx\right)dy = \int_0^a \left[\frac{x^2y}{2}\right]_0^{\sqrt{a^2-y^2}} dy = \int_0^a y\frac{\sqrt{\left(a^2-y^2\right)^2}}{2}dy = \frac{1}{2}\int_0^a y\left(a^2-y^2\right)dy$$

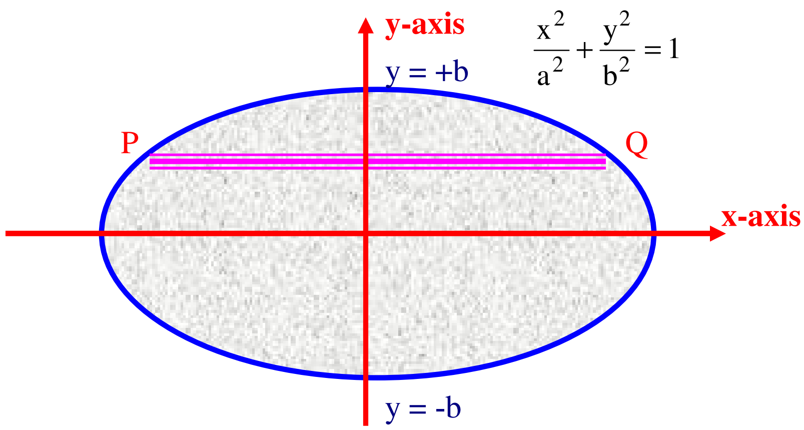
$$= \frac{1}{2}\int_0^a \left(a^2y - y^3\right)dy = \frac{1}{2}\left[\frac{a^2y^2}{2} - \frac{y^4}{4}\right]_0^a = \frac{1}{2}\left[\frac{a^2(a)^2}{2} - \frac{\left(a^4\right)}{4}\right] - (0-0)$$

$$= \frac{1}{2}\left[\frac{a^4}{2} - \frac{a^4}{4}\right] = \frac{a^4}{8}. \text{Ans.}$$

Q.No.3.: Evaluate the integral $\iint (x+y)^2 dxdy$ over the area bounded by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Sol.:



Since $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow x^2 = a^2\left(1-\frac{y^2}{b^2}\right) \Rightarrow x = \pm a\sqrt{1-\frac{y^2}{b^2}}$.

$$\therefore I = \int_{-b}^{b}\left[\int_{-a\sqrt{1-\frac{y^2}{b^2}}}^{a\sqrt{1-\frac{y^2}{b^2}}}(x+y)^2 dx\right]dy = \int_{-b}^{b}\left[\int_{-a\sqrt{1-\frac{y^2}{b^2}}}^{a\sqrt{1-\frac{y^2}{b^2}}}\left(x^2+y^2+2xy\right)dx\right]dy$$

$$= \int_{-b}^{b}\left[2\int_{0}^{a\sqrt{1-\frac{y^2}{b^2}}}\left(x^2+y^2\right)dx\right]dy = 2\int_{-b}^{b}\left[\left(\frac{x^3}{3}+y^2x\right)_0^{a\sqrt{1-\frac{y^2}{b^2}}}\right]dy$$

$$= 2\int_{-b}^{b}\left[\frac{a^3}{3}\left(1-\frac{y^2}{b^2}\right)^{3/2} + y^2a\sqrt{1-\frac{y^2}{b^2}}\right]dy$$

Since function is even, then we get

$$\Rightarrow I = 4\int_{0}^{b}\left[\frac{a^3}{3}\left(1-\frac{y^2}{b^2}\right)^{3/2} + ay^2\sqrt{1-\frac{y^2}{b^2}}\right]dy$$

Put $y = b\sin\theta \Rightarrow dy = b\cos\theta d\theta$. Also when $y = 0,\ \theta = 0$ and when $y = b,\ \theta = \frac{\pi}{2}$.

Then $I = 4\int_{0}^{\pi/2}\left[\frac{a^3}{3}\cos^3\theta + ab^2\sin^2\theta\cos\theta\right] b\cos\theta d\theta$

$$= 4ab\left[\frac{a^2}{3}\int_0^{\pi/2}\cos^4\theta d\theta + b^2\int_0^{\pi/2}\sin^2\theta\cos^2\theta d\theta\right] \qquad \text{(i)}$$

Now first evaluate $\int_0^{\pi/2}\cos^4\theta d\theta = \frac{3.1}{4.2}.\frac{\pi}{2} = \frac{3\pi}{16}$,

and $\int_0^{\pi/2}\sin^2\theta\cos^2\theta d\theta = \frac{1.1}{4.2}.\frac{\pi}{2} = \frac{\pi}{16}$

Hence

$$I = 4ab\left[\frac{a^2}{3}.\frac{3\pi}{16} + b^2\frac{\pi}{16}\right] = \frac{ab\pi}{4}\left(a^2 + b^2\right). \text{ Ans.}$$

Q.No.4.: Evaluate the integral $\iint xy(x+y)dxdy$ over the area between $y = x^2$ and $y = x$.

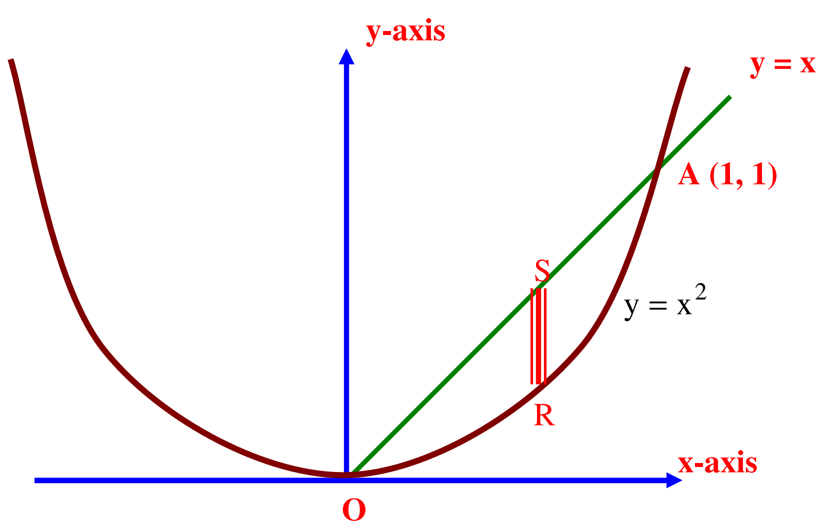
Sol.: Solving $y = x^2$ and $y = x$, we get $x = x^2 \Rightarrow x - x^2 = 0 \Rightarrow x(1-x) = 0$.

$\therefore x = 0$ and $x = 1$

When x = 0, y = 0 and x = 1, y = 1.

$\therefore$ The points of intersection are O(0, 0) and A (1, 1).

Let us suppose, the strip is parallel to y-axis. In that case integrating first over a horizontal strip RS, w. r. t. y from $y = x^2$ to $y = x$ and then w. r. t. x from x= 0 to x = 1, we get



$$I = \iint xy(x+y)\,dx\,dy = \int_0^1 \left[\int_{x^2}^{x} xy(x+y)\,dy\right]dx = \int_0^1 \left[\int_{x^2}^{x} \left(x^2y + xy^2\right)dy\right]dx$$

$$= \int_0^1 \left[\frac{x^2y^2}{2} + \frac{xy^3}{3}\right]_{x^2}^{x} dx = \int_0^1 \left[\frac{x^4}{2} + \frac{x^4}{3} - \frac{x^6}{2} - \frac{x^7}{3}\right]dx = \left[\frac{x^5}{10} + \frac{x^5}{15} - \frac{x^7}{14} - \frac{x^8}{24}\right]_0^1$$

$$= \left[\frac{1}{10} + \frac{1}{15} - \frac{1}{14} - \frac{1}{24}\right] = [0.1 + 0.06667 - 0.07143 - 0.0417] = 0.5357 \text{ . Ans.}$$

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2^{nd} Topic

Integral Calculus

Triple Integrals

[Where limits are not given, but the region of integration is given]

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Here we will discuss those problems in triple integrals, where limits are not given. Now when limits are not given then how we can evaluate the limits without rough sketch of the region of integration. With the basic understanding of the problems given below, we can easily evaluate the limits in future.

Q.No.1.: Evaluate $\iiint (x+y+z)dxdydz$ over the tetrahedron bounded by the planes

$$x=0,\ y=0,\ z=0 \text{ and } x+y+z=1 .$$

Sol.: The region here is a tetrahedron bounded by the planes

$$x=0, y=0, z=0,\ x+y+z=1 .$$

$$\because x+y+z \le 1$$

$$\Rightarrow x \le 1,\ (x+y) \le 1,\ (x+y+z) \le 1$$

$$\Rightarrow x \le 1,\ y \le (1-x),\ z \le (1-x-y)$$

$$\therefore R = \{(x,y,z),\ 0 \le x \le 1,\ 0 \le y \le (1-x),\ 0 \le z \le (1-x-y)\}.$$

$$\therefore I = \iiint_V (x+y+z)dxdydz = \int_0^1 \left[\int_0^{1-x} \left\{ \int_0^{1-x-y} (x+y+z)dz \right\} dy \right] dx$$

$$= \int_0^1 \left\{ \int_0^{1-x} \left[\frac{(x+y+z)^2}{2} \right]_0^{1-x-y} dy \right\} dx \qquad \left[\because \int (az+b)^n dz = \frac{(az+b)^{n+1}}{(n+1)a} (n \neq -1) \right]$$

$$= \frac{1}{2}\int_0^1 \left(\int_0^{1-x} \left[1-(x+y)^2\right] dy \right) dx = \frac{1}{2}\int_0^1 \left(y - \frac{(x+y)^3}{3} \right)_0^{1-x} dx$$

$$= \frac{1}{2}\int_0^1 \left[\left(1-x-\frac{1}{3}\right) - \left(0-\frac{x^3}{3}\right) \right] dx = \frac{1}{2}\int_0^1 \left(\frac{2}{3} - x + \frac{x^3}{3} \right) dx$$

$$= \frac{1}{2}\left[\frac{2}{3}x - \frac{x^2}{2} + \frac{x^4}{12} \right]_0^1 = \frac{1}{2}\left[\frac{2}{3} - \frac{1}{2} + \frac{1}{12} \right] = \frac{1}{2}\left(\frac{8-6+1}{12} \right) = \frac{3}{24} = \frac{1}{8}. \text{ Ans.}$$

Q.No.2.: Compute triple integral $\iiint \frac{dxdydz}{(1+x+y+z)^3}$, if the region of integration is bounded by the co-ordinate planes and the plane $x+y+z=1$.

Sol.: The region here is a bounded by the co-ordinate planes and the plane $x+y+z=1$.

i.e. $x=0, y=0, z=0, \ x+y+z=1$.

$\because x+y+z \le 1$

$\Rightarrow x \le 1, \ (x+y) \le 1, \ (x+y+z) \le 1$

$\Rightarrow x \le 1, \ y \le (1-x), \ z \le (1-x-y)$

$\therefore R = \{(x,y,z), \ 0 \le x \le 1, \ 0 \le y \le (1-x), \ 0 \le z \le (1-x-y)\}$.

$$\text{Thus } I = \int_0^1 \left\{ \int_0^{1-x} \left(\int_0^{1-x-y} \frac{1}{(1+x+y+z)^3} dz \right) dy \right\} dx = \int_0^1 \left\{ \int_0^{1-x} \left| -\frac{1}{2(1+x+y+z)^2} \right|_0^{1-x-y} dy \right\} dx$$

$$= -\frac{1}{2}\int_0^1 \left\{ \int_0^{1-x} \left(\frac{1}{(1+x+y+1-x-y)^2} - \frac{1}{(1+x+y)^2} \right) dy \right\} dx$$

$$= -\frac{1}{2}\int_0^1 \left\{ \int_0^{1-x} \left(\frac{1}{4} - \frac{1}{(1+x+y)^2} \right) dy \right\} dx = -\frac{1}{2}\int_0^1 \left\{ \left(\frac{1}{4}|y|_0^{1-x} + \left| \frac{1}{(1+x+y)} \right|_0^{1-x} \right) \right\} dx$$

$$= -\frac{1}{2}\int_0^1 \left(\frac{(1-x)}{4} + \frac{1}{1+x+1-x} - \frac{1}{1+x}\right)dx = -\frac{1}{2}\int_0^1 \left(\frac{1}{4} - \frac{x}{4} + \frac{1}{2} - \frac{1}{1+x}\right)dx$$

$$= -\frac{1}{2}\int_0^1 \left(\frac{3}{4} - \frac{x}{4} - \frac{1}{1+x}\right)dx = -\frac{1}{2}\left[\frac{3}{4}(x)_0^1 - \frac{1}{4}\left(\frac{x^2}{2}\right)_0^1 - \left|\log(1+x)_0^1\right|\right]$$

$$= -\frac{1}{2}\left[\frac{3}{4} - \frac{1}{8} - \log 2\right] = \frac{1}{2}\log 2 - \frac{5}{16}. \text{ Ans.}$$

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3rd Topic

Integral Calculus

Double Integrals

(Change of order of Integration)

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Change of order of Integration:

In a double integral, if the limits of integration are constant, then the order of integration is immaterial, provided the limits of integration are changed accordingly.

Thus $\int_c^d \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx$.

In a double integral with variable limits, the change of order of integrationchanges the limits of integration. While doing so, sometimes it is required to split up the region of integration and the given integral is expressed as the sum of a number of double integrals with changed limits. To fix up the new limits, it is always advisable to draw a rough sketch of the region of integration.

The change of order of integration, to a certain extent makes easy, the evaluation of a double integral. The following examples will make these ideas clear.

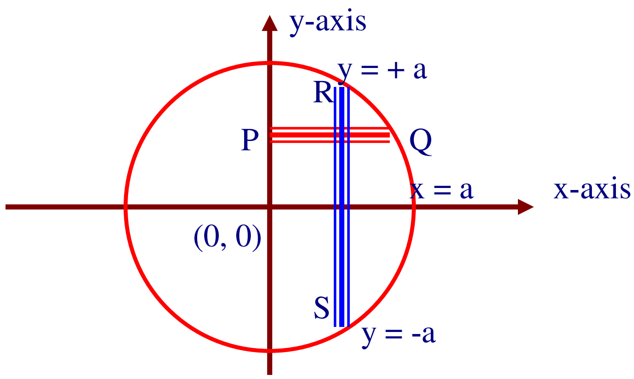
Here we will discuss those problems in double integrals, where limits are given, but we have to change the order of integration. So in this type of problems rough sketch of the region of integration is required. Let us clear this concept with the help of problems given below.

Q.No.1.: Change the order of integration in the integral $I = \int_{-a}^{a} \int_{0}^{\sqrt{(a^2-y^2)}} f(x, y)dxdy$.

Sol.: Here the elementary strip is parallel to x-axis (such as PQ) and extends from x = 0 to $x = \sqrt{(a^2 - y^2)}$ (i. e. to the circle $x^2 + y^2 = a^2$) and strip slides from $y = -a$ to $y = a$.

This shaded semi-circle area is therefore, the region of integration, as shown in the figure.

On changing the order of integration, we first integrate w. r. t. y along a vertical strip RS which extends from $R\left[y = -\sqrt{(a^2 - x^2)}\right]$ to $S\left[y = \sqrt{(a^2 - x^2)}\right]$. To cover the given region, we then integrate w. r. t. x from x = 0 to x = a.



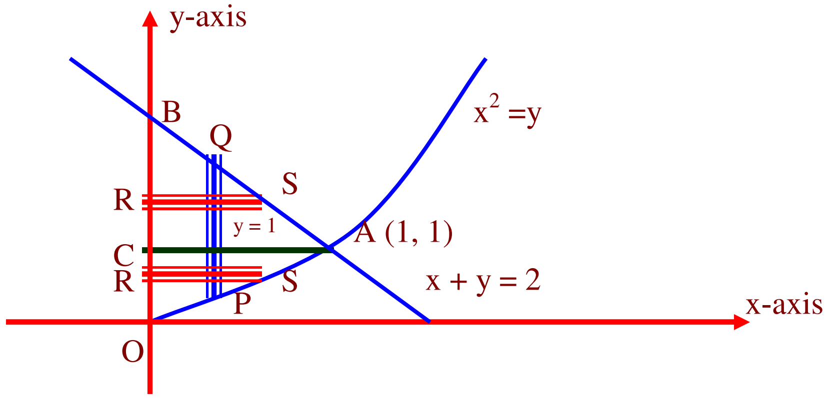
Thus, on reversing the order of integration, we get

$$I = \int_{0}^{a} \left(\int_{-\sqrt{(a^2-x^2)}}^{\sqrt{(a^2-x^2)}} f(x, y)dy \right) dx \text{. Ans.}$$

Q.No.2.: Change the order of integration in $I = \int_{0}^{1} \int_{x^2}^{2-x} xy\, dx\, dy$, and hence evaluate the same.

Sol.: Here the integration is first w. r. t. y along a vertical strip PQ which extends from P on the parabola $y = x^2$ to Q on the line $y = 2 - x$. Such a strip slides from x = 0 to x = 1, giving the region of integration as the curvilinear triangle OAB(as shown in figure).

When we change the order of integration, we first integrate w. r. t. x along a horizontal strip RS and that requires the splitting up of the region OAB into two parts by the line AC(y = 1), the curvilinear triangle OAC and the triangle ABC.



For the region OAC, the limits of integration for x are from x = 0 to $x = \sqrt{y}$ and those for y are from y = 0 to y = 1. So the contribution to I from the region OAC is

$$I_1 = \int_0^1 \left(\int_0^{\sqrt{y}} xydx \right) dy.$$

For the region ABC, the limits of integration for x are from x = 0 to $x = 2 - y$ and for those for y are from y = 1 to y = 2. So the contribution to I from the region ABC is

$$I_2 = \int_1^2 \left(\int_0^{2-y} xydx \right) dy.$$

Hence, on reversing the order of integration, we get

$$I = \int_0^1 \left(\int_0^{\sqrt{y}} xydx \right) dy + \int_1^2 \left(\int_0^{2-y} xydx \right) dy = \int_0^1 \left| \frac{x^2}{2} .y \right|_0^{\sqrt{y}} dy + \int_1^2 \left| \frac{x^2}{2} .y \right|_0^{2-y} dy$$

$$=\frac{1}{2}\int_0^1 y^2dy+\frac{1}{2}\int_1^2 y(2-y)^2dy=\frac{1}{2}\left[\frac{y^3}{3}\right]_0^1+\frac{1}{2}\left[4.\frac{y^2}{2}+\frac{y^4}{4}-4.\frac{y^3}{3}\right]_1^2$$

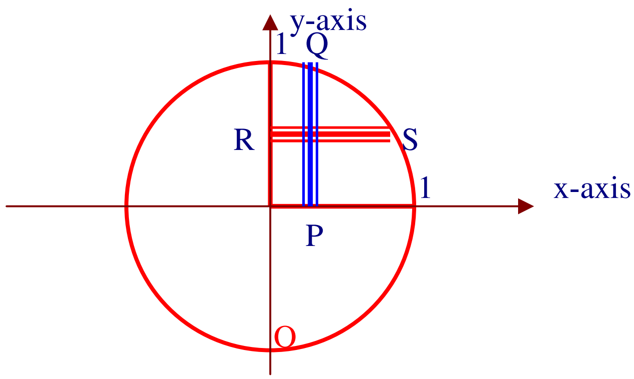
$$=\frac{1}{2}\left[\frac{1}{3}\right]+\frac{1}{2}\left[\left(8+4-\frac{32}{3}\right)-\left(2+\frac{1}{4}-\frac{4}{3}\right)\right]=\frac{1}{6}+\frac{1}{2}\left(\frac{4}{3}-\frac{11}{12}\right)=\frac{1}{6}+\frac{1}{2}\left(\frac{5}{12}\right)$$

$$=\frac{1}{6}+\frac{5}{24}=\frac{3}{8}.\ \text{Ans.}$$

Q.No.3.: Evaluate the integral by changing the order of integration $\int_0^1\int_0^{\sqrt{(1-x^2)}} y^2dxdy$.

Sol.: Given integral is $I=\int_0^1\int_0^{\sqrt{(1-x^2)}} y^2dxdy$.

The strip PQ parallel to y-axis is made from y = 0 to $y=\sqrt{1-x^2}$ and this strip PQ slides from x = 0 to x = 1 (radius of the circle)



Now when we change the order of integration, we make strip RS parallel to x-axis, we see that strip is made from x = 0 to $x=\sqrt{1-y^2}$, and it slides from y = 0 to y = 1

Hence, on reversing the order of integration, we get

$$I=\int_0^1\left(\int_0^{\sqrt{1-y^2}} y^2dx\right)dy=\int_0^1\left(y^2|x|_0^{\sqrt{1-y^2}}\right)dy=\int_0^1 y^2\left(\sqrt{1-y^2}\right)dy$$

Putting $y=\sin\theta$ to solve $\int y^2\sqrt{1-y^2}dy$

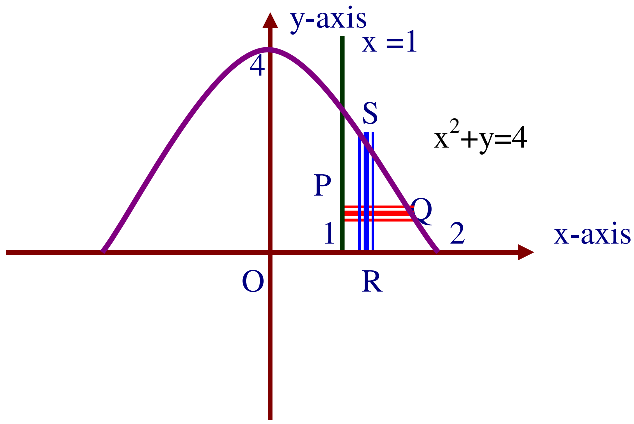
Also when $y = 0,\ \theta = 0;\quad y = 1,\ \theta = \frac{\pi}{2}$ and $\frac{dy}{d\theta} = \cos\theta$.

$$\therefore I = \int_0^{\pi/2} \sin^2\theta\cos^2\theta d\theta = \frac{1.1}{4.2}.\frac{\pi}{2} = \frac{\pi}{16}.\ \text{Ans.}$$

Q.No.4.: Evaluate the following integral by changing the order of integration

$$\int_0^3 \left(\int_1^{\sqrt{(4-y)}} (x+y)dx \right) dy.$$

Sol.: Here the strip PQ parallel to x-axis is made from x = 1 to $x = \sqrt{4-y}$ and this strip slides from y = 0 to y = 3.



Now when we change the order of integration, we make strip parallel to y-axis, which is made from y = 0 to $y = 4 - x^2$ and slides from x = 1 to x = 2.

Hence, on reversing the order of integration, we get

$$I = \int_1^2 \left(\int_0^{4-x^2} (x+y)dy \right) dx = \int_1^2 \left| xy + \frac{y^2}{2} \right|_0^{4-x^2} dx$$

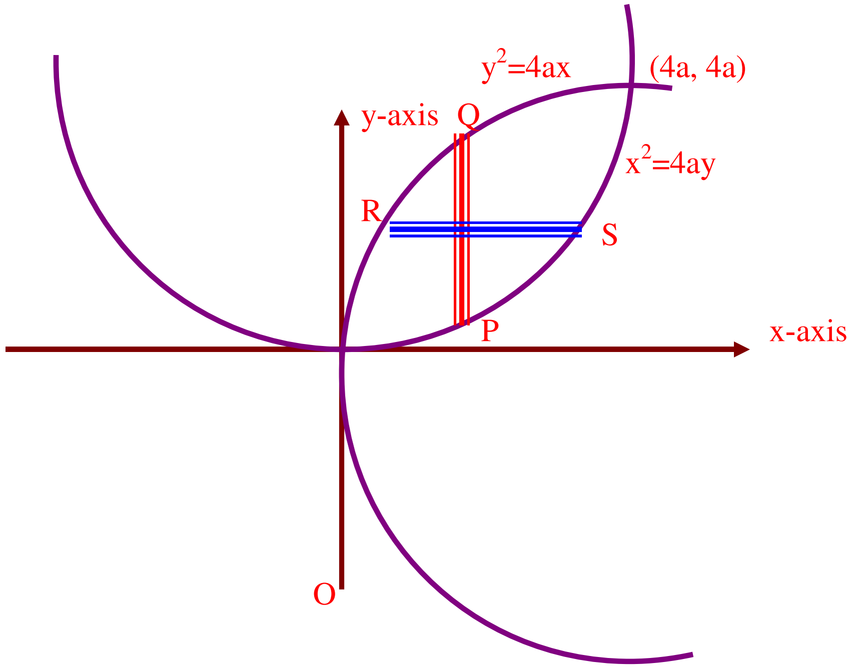
$$= \int_1^2 \left[x\left(4-x^2\right) + \frac{\left(16+x^4-8x^2\right)}{2} \right] dx = \left[4.\frac{x^2}{2} - \frac{x^4}{4} + 8x + \frac{x^5}{5} - 4.\frac{x^3}{3} \right]_1^2$$

$$= 6 - \frac{15}{4} + 8 + \frac{31}{10} - \frac{28}{3} = \frac{360-225+480+186-560}{60} = \frac{241}{60} = 4.02.\ \text{Ans.}$$

Q.No.5.: Evaluate the integral by changing the order of integration $\int_0^{4a} \left(\int_{x^2/4a}^{2\sqrt{ax}} dy \right) dx$.

Sol.: Here the strip PQ parallel to y-axis is made from $y = \frac{x^2}{4a}$ to $y = 2\sqrt{ax}$ and this strip PQ found above slides fromx = 0 to x = 4a.

When we changing the order of integration, we make strip RS parallel to x-axis and this strip is made from $x = \frac{y^2}{4a}$ to $x = 2\sqrt{ay}$ and slides from y = 0 to y = 4a.



Hence, on reversing the order of integration, we get

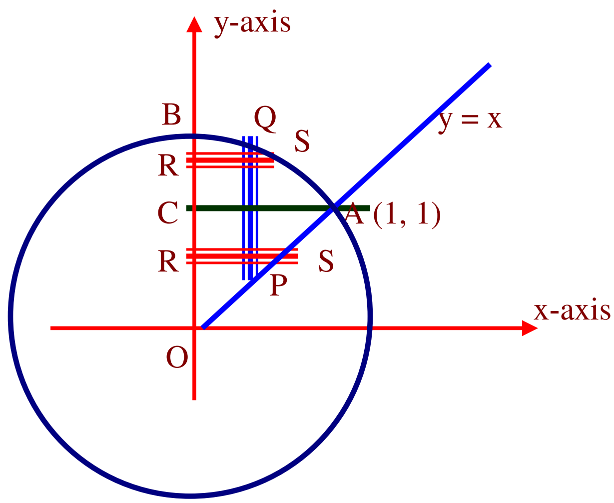
$$I = \int_0^{4a}\left(\int_{y^2/4a}^{2\sqrt{ay}} dx\right)dy = \int_0^{4a}\left[2\sqrt{a}\ \ y^{1/2} - \frac{1}{4a}y^2\right]dy = 2\sqrt{a}\left(\frac{y^{3/2}}{\frac{3}{2}}\right)_0^{4a} - \frac{1}{4a}\left(\frac{y^3}{3}\right)_0^{4a}$$

$$= \frac{32}{3}a^2 - \frac{1}{12a}.64a^3 = \frac{32}{3}a^2 - \frac{16}{3}a^2 = \frac{16a^2}{3}.\text{ Ans.}$$

Q.No.6.: Evaluate the integral by changing the order of integration

$$\int_0^1\left(\int_x^{\sqrt{(2-x^2)}} \frac{x}{\sqrt{x^2+y^2}}dy\right)dx.$$

Sol.: Here the strip PQ parallel to y-axis is made from y = x to $y = \sqrt{2 - x^2}$, and this strip PQ slides from x = 0 to x = 1.



When we change the order of integration, strip RS has nonviable character.

When we change the order of integration, we first integrate w. r. t. x along a horizontal strip RS and that requires the splitting up of the region OAB into two parts by the line AC(y = 1), the triangle OAC and thecurvilinear triangle ABC.

For the region OAC, the limits of integration for x are from x = 0 to $x = y$ and those for y are from y = 0 to y = 1. So the contribution to I from the region OAC is

$$I_1 = \int_0^1 \left(\int_0^y \frac{x}{\sqrt{x^2 + y^2}} dx \right) dy .$$

For the region ABC, the limits of integration for x are from x = 0 to $x = \sqrt{2 - y^2}$ and for those for y are from y = 1 to y $= \sqrt{2}$. So the contribution to I from the region ABC is

$$I_2 = \int_1^{\sqrt{2}} \left(\int_0^{\sqrt{2-y^2}} \frac{x}{\sqrt{x^2 + y^2}} dx \right) dy .$$

Hence, on reversing the order of integration, we get

$$I = \int_0^1 \left(\int_0^y \frac{x}{\sqrt{x^2 + y^2}} dx \right) dy + \int_1^{\sqrt{2}} \left(\int_0^{\sqrt{2-y^2}} \frac{x}{\sqrt{x^2 + y^2}} dx \right) dy \quad .$$

Now $I_1 = \frac{1}{2}\int_0^1 \left(\int_0^y \frac{2x}{\sqrt{x^2+y^2}}dx\right)dy = \frac{1}{2}\int_0^1 2\left|\sqrt{x^2+y^2}\right|_0^y dy = \int_0^1 \left(\sqrt{2}\;y - y\right)dy$

$= \left[\sqrt{2}\frac{y^2}{2} - \frac{y^2}{2}\right]_0^1 = \frac{1}{\sqrt{2}} - \frac{1}{2}$,

and $I_2 = \frac{1}{2}\int_1^{\sqrt{2}} \left(\int_0^{\sqrt{2-y^2}} \frac{2x}{\sqrt{x^2+y^2}}dx\right)dy = \frac{1}{2}\int_1^{\sqrt{2}} 2\left|\sqrt{x^2+y^2}\right|_0^{\sqrt{2-y^2}} dy = \int_1^{\sqrt{2}} \left|\sqrt{2} - y\right|dy$

$= \left[\sqrt{2}y - \frac{y^2}{2}\right]_1^{\sqrt{2}} = 2 - \sqrt{2} - \left(1 - \frac{1}{2}\right) = \frac{3}{2} - \sqrt{2}$.

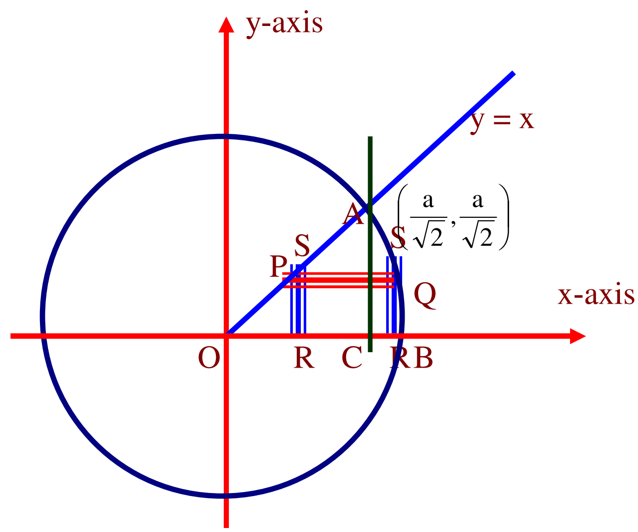
$\therefore$ The required integrals is

$I = I_1 + I_2 = \left(\frac{1}{\sqrt{2}} - \frac{1}{2}\right) + \left(\frac{3}{2} - \sqrt{2}\right) = \frac{1}{\sqrt{2}} + 1 - \sqrt{2} = \frac{1+\sqrt{2}-2}{\sqrt{2}} = \frac{\sqrt{2}-1}{\sqrt{2}} = 1 - \frac{1}{\sqrt{2}}$. Ans.

Q.No.7.: Evaluate the integral by changing the order of integration

$$\int_0^{a/\sqrt{2}} \left(\int_y^{\sqrt{(a^2-y^2)}} \log\left(x^2+y^2\right)dx\right)dy \qquad (a > 0).$$

Sol.:



Here the strip PQ parallel to x-axis is made from x = y to $x=\sqrt{a^2-y^2}$ and this strip slides from y = 0 to $y=\frac{a}{\sqrt{2}}$.

When we change the order of integration, strip RS has nonviable character.

When we change the order of integration, we first integrate w. r. t. y along a vertical strip RS and that requires the splitting up of the region OAB into two partsby the line AC($y=\frac{a}{\sqrt{2}}$),, the triangle OAC and thecurvilinear triangle ABC.

For the region OAC, the limits of integration for y are from y = 0 to $y = x$ and those for y are from x = 0 to $x=\frac{a}{\sqrt{2}}$. So the contribution to I from the region OAC is

$$I_1=\int_0^{a/\sqrt{2}}\left[\int_0^x \log\left(x^2+y^2\right)dy\right]dx .$$

For the region ABC, the limits of integration for y are from y = 0 to $y=\sqrt{a^2-x^2}$ and for those for x are from $x=\frac{a}{\sqrt{2}}$ to x =a. So the contribution to I from the region ABC is

$$I_2=\int_{a/\sqrt{2}}^{a}\left[\int_0^{\sqrt{a^2-x^2}} \log\left(x^2+y^2\right)dy\right]dx .$$

Hence, on reversing the order of integration, we get

$$I=\int_0^{a/\sqrt{2}}\left(\int_y^{\sqrt{a^2-y^2}} \log\left(x^2+y^2\right)dx\right)dy$$

$$=\int_0^{a/\sqrt{2}}\left[\int_0^x \log\left(x^2+y^2\right)dy\right]dx+\int_{a/\sqrt{2}}^{a}\left[\int_0^{\sqrt{a^2-x^2}} \log\left(x^2+y^2\right)dy\right]dx=I_1+I_2$$

$$\Rightarrow I=I_1+I_2 \quad \text{[say]}$$

where $I_1 = \int_0^{a/\sqrt{2}} \left[\int_0^x \log(x^2+y^2)dy \right] dx$ and $I_2 = \int_{a/\sqrt{2}}^{a} \left[\int_0^{\sqrt{a^2-x^2}} \log(x^2+y^2)dy \right] dx$

Now first evaluate $\int \log(x^2+y^2)dy$: Integrating by parts, we get

$$\int \log(x^2+y^2)dy = \{\log(x^2+y^2)\}y - \int \frac{2y}{x^2+y^2}.ydy$$

$$= y\log(x^2+y^2) - 2\int \frac{y^2+x^2-x^2}{x^2+y^2}dy = y\log(x^2+y^2) - 2\int \left\{1 - \frac{x^2}{x^2+y^2}\right\}dy$$

$$= y\log(x^2+y^2) - 2\left\{y - x^2.\left(\frac{1}{x}\tan^{-1}\frac{y}{x}\right)\right\} = y\log(x^2+y^2) - 2y + 2x\tan^{-1}\frac{y}{x}$$

Now $\quad I_1 = \int_0^{a/\sqrt{2}} \left[y\log(x^2+y^2) - 2y + 2x\tan^{-1}\frac{y}{x} \right]_0^x dx$

$$= \int_0^{a/\sqrt{2}} \left[x\log 2x^2 - 2x + 2x\tan^{-1}(1) - 0 \right] dx$$

$$= \int_0^{a/\sqrt{2}} \left[x\log 2 - 2x\log x - 2x + \frac{2x\pi}{4} \right] dx$$

$$= \left[\log 2.\frac{x^2}{2} - \left\{ 2\log x.\frac{x^2}{2} - \int_0^{a/\sqrt{2}} \frac{2}{x}.\frac{x^2}{2}dx \right\} - 2\frac{x^2}{2} + \frac{\pi}{2}.\frac{x^2}{2} \right]_0^{a/\sqrt{2}}$$

$$= \left[\left(\log 2 - 2 + \frac{\pi}{2} + 2\log x \right) \frac{x^2}{2} \right]_0^{a/\sqrt{2}} - \int_0^{a/\sqrt{2}} \frac{2}{x}.\frac{x^2}{2}dx$$

$$= \left(\log 2 - 2 + \frac{\pi}{2} + 2\log\frac{a}{\sqrt{2}} \right)\frac{a^2}{4} - \left[\frac{x^2}{2} \right]_0^{a/\sqrt{2}} = \left(\log 2 - 2 + \frac{\pi}{2} + 2\log\frac{a}{\sqrt{2}} \right)\frac{a^2}{4} - \frac{a^2}{4}$$

$$= \frac{a^2}{8}\left(2\log 2 - 4 + \pi + 4\log\left(\frac{a}{\sqrt{2}}\right) - 2 \right) = \frac{a^2}{8}(2\log 2 + 4\log a - 2\log 2 - 6 + \pi)$$

$$I_1 = \frac{a^2}{8}\{\pi + 4\log a - 6\} \qquad \text{(i)}$$

Similarly, $I_2 = \int\limits_{a/\sqrt{2}}^{a} \left[y\log\left(x^2+y^2\right) - 2y + 2x\tan^{-1}\frac{y}{x} \right]_0^{\sqrt{a^2-x^2}} dx$

$$= \int\limits_{a/\sqrt{2}}^{a} \left\{ \sqrt{a^2-x^2}\log a^2 - 2\sqrt{a^2-x^2} + 2x\tan^{-1}\frac{\sqrt{a^2+x^2}}{x} \right\} dx$$

$$= \left(\log a^2 - 2\right) \int\limits_{a/\sqrt{2}}^{a} \sqrt{a^2-x^2}\,dx + 2\int\limits_{a/\sqrt{2}}^{a} x\tan^{-1}\frac{\sqrt{a^2+x^2}}{x}\,dx$$

$$= \left(\log a^2 - 2\right) \int\limits_{a/\sqrt{2}}^{a} \sqrt{a^2-x^2}\,dx + I_3, \quad \text{where } I_3 = 2\int\limits_{a\sqrt{2}}^{a} x\tan^{-1}\frac{\sqrt{a^2-x^2}}{x}\,dx$$

$$I_2 = \left[\left(\log a^2 - 2\right) \times \left(\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} \right) \right]_{a/\sqrt{2}}^{a} + I_3.$$

Now evaluate $I_3 = 2\int\limits_{a\sqrt{2}}^{a} x\tan^{-1}\frac{\sqrt{a^2-x^2}}{x}\,dx$.

Let $x = a\cos\theta \therefore dx = -a\sin\theta d\theta$

Now when $x = \frac{a}{\sqrt{2}} \Rightarrow a\cos\theta = \frac{a}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4}$

and $x = a \Rightarrow a\cos\theta = a \Rightarrow \theta = 0$.

$$\therefore I_3 = 2\int\limits_{\pi/4}^{0} a\cos\theta\tan^{-1}\frac{\sqrt{a^2\left(1-\cos^2\theta\right)}}{a\cos\theta} \times -a\sin\theta d\theta = -a^2\int\limits_{\pi/4}^{0} 2\sin\theta\cos\theta\tan^{-1}\left(\tan\theta\right)d\theta$$

$$= -a^2\int\limits_{\pi/4}^{0} \theta\sin 2\theta d\theta = a^2\int\limits_{0}^{\pi/4} \theta\sin 2\theta d\theta = a^2\left[\frac{-\theta\cos 2\theta}{2}\right]_0^{\pi/4} + a^2\int\limits_{0}^{\pi/4}\frac{1.\cos 2\theta}{2}d\theta$$

$$= a^2\left[0-0\right]_0^{\pi/4} + a^2\left[\frac{\sin 2\theta}{4}\right]_0^{\pi/4} = \frac{a^2}{4}\left[1-0\right] = \frac{a^2}{4}.$$

Thus $I_2 = \left[\left(\log a^2 - 2\right) \times \left(\left(\frac{a}{2}\times 0\right) + \left(\frac{a^2}{2}.\frac{\pi}{2}\right) - \frac{a}{2\sqrt{2}}.\frac{a}{\sqrt{2}} - \frac{a^2}{2}.\frac{\pi}{4} \right) \right] + \frac{a^2}{4}$

$$= \left(2\log a - 2\right)\left(\frac{\pi a^2}{8} - \frac{a^2}{4}\right) + \frac{a^2}{4} \qquad \text{(ii)}$$

By (i) and (ii), we get

$$I = \left\{\frac{a^2}{8}\{\pi + 4\log a - 6\}\right\} + \left\{(2\log a - 2)\left(\frac{\pi a^2}{8} - \frac{a^2}{4}\right) + \frac{a^2}{4}\right\}$$

$$I = \frac{a^2}{8}\{\pi + 4\log a - 6\} + \frac{a^2}{8}\{2\pi\log a - 2\pi - 4\log a + 4 + 2\}$$

$$= \frac{a^2}{8}\{\pi + 4\log a - 6 + 2\pi\log a - 2\pi - 4\log a + 6\} = \frac{a^2}{8}\{2\pi\log a - \pi\}$$

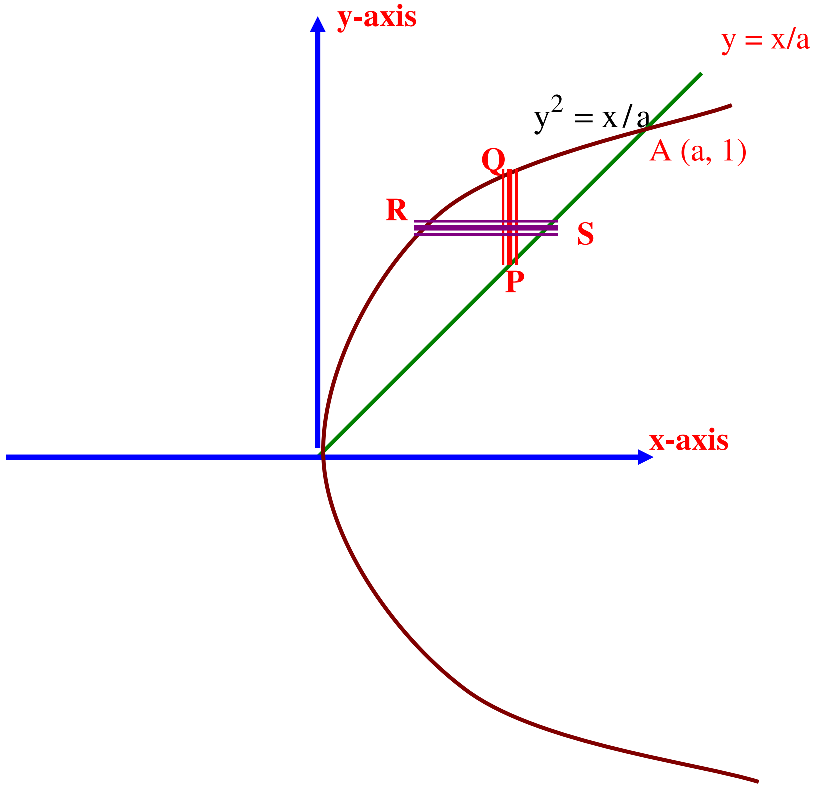
$$\therefore I = \frac{\pi a^2}{4}\left(\log a - \frac{1}{2}\right).$$ Ans.

Q.No.8.: Evaluate the integral by changing the order of integration

$$\int_0^a \left(\int_{x/a}^{\sqrt{(x/a)}} (x^2 + y^2)dy\right)dx \quad .$$

Sol.: The strip PQ parallel to y-axis is made from $y = \frac{x}{a}$ to $y = \sqrt{\frac{x}{a}}$ and this strip PQ slides from x = 0 to x = a.

When we change the order of integration, the strip is changed from PQ to RS which is now parallel to x-axis, is made from $x = ay^2$ to x = ay and slides from y = 0 and y = 1.



Hence, on reversing the order of integration, we get

$$I = \int_0^1 \left[\int_{ay^2}^{ay} \left(x^2 + y^2\right) dx \right] dy = \int_0^1 \left[\frac{x^3}{3} + y^2 x \right]_{ay^2}^{ay} dy$$

$$= \int_0^1 \left[\frac{a^3y^3 - a^3y^6}{3} \right] dy + \int_0^1 \left[ay^3 - ay^4\right] dy = \frac{a^3}{3}\left[\frac{y^4}{4} - \frac{y^7}{7} \right]_0^1 + a\left[\frac{y^4}{4} - \frac{y^5}{5} \right]_0^1$$

$$= \frac{a^3}{28} + \frac{a}{20}. \text{ Ans.}$$

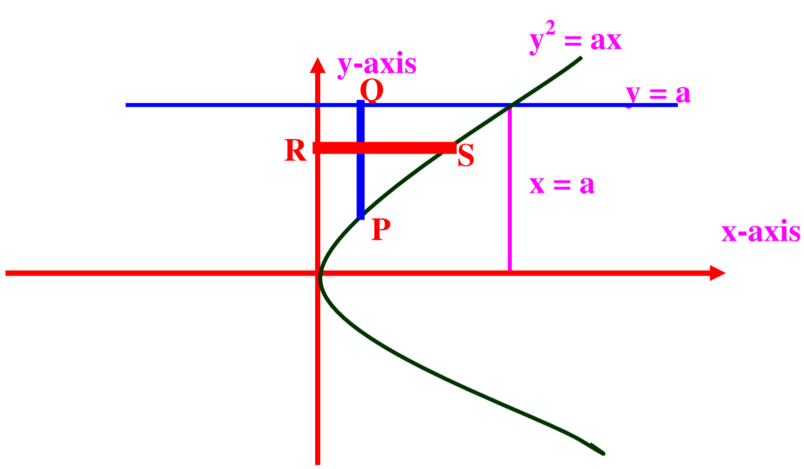
Q.No.9.: Evaluate the integral by changing the order of integration

$$\int_0^a \int_{\sqrt{ax}}^a \frac{y^2 dy dx}{\sqrt{y^4 - a^2x^2}} \quad .$$

Sol.: Given integral is $\int_0^a \int_{\sqrt{ax}}^a \frac{y^2 dy dx}{\sqrt{y^4 - a^2x^2}} = \int_0^a \left(\int_{\sqrt{ax}}^a \frac{y^2}{\sqrt{y^4 - a^2x^2}} dy \right) dx$

The strip PQ parallel to y-axis is made from $y = \sqrt{ax}$ to $y = a$, and this strip PQ slides from x = 0 to x = a.

When we change the order of integration, the strip is changed from PQ to RS which is now parallel to x-axis, is made from x = 0 to $x = \frac{y^2}{a}$ and slides from y = 0 and y = a.



Hence, on reversing the order of integration, we get

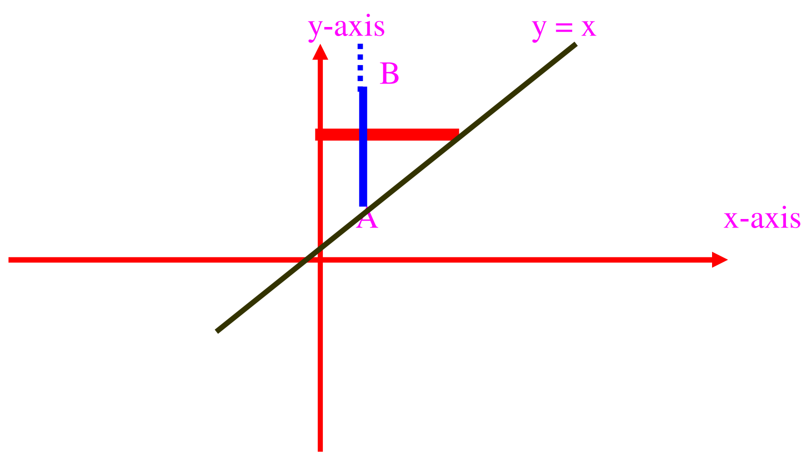
$$I = \int_0^a \left[\int_0^{y^2/a} \frac{dx}{\sqrt{1 - \frac{a^2x^2}{y^4}}} \right] dy = \int_0^a \left[\frac{y^2}{a} \sin^{-1}\left(\frac{ax}{y^2} \right) \right]_0^{y^2/a} dy$$

$$= \int_0^a \left[\frac{y^2}{a} \left\{ \sin^{-1}\left(\frac{a}{y^2} \cdot \frac{y^2}{a} \right) - \sin^{-1} 0 \right\} \right] dy = \int_0^a \frac{y^2}{a} \sin^{-1} 1 dy = \int_0^a \frac{y^2}{a} \cdot \frac{\pi}{2} dy$$

$$= \frac{\pi}{2a} \left[\frac{y^3}{3} \right]_0^a = \frac{\pi a^2}{6}. \text{ Ans.}$$

Q.No.10.: Evaluate the integral by changing the order of integration $\int_0^\infty \left(\int_x^\infty \frac{e^{-y}}{y} dy \right) dx$.

Sol.: The elementary strip given in the problem is a strip AB parallel to the y-axis whose one end lies on the line y = x and the other end extends to infinity (i.e. $y \to \infty$), and this strip slides from x = 0 to $x \to \infty$.



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On changing the order of the above problem, we first integrate it along a horizontal strip which extends from x = 0 to x = y, and this strip slides from y = 0 to $y \to \infty$.

(The region of integration is the area above the line x = y which extends up to infinity to cover this area)

Hence, on reversing the order of integration, we get

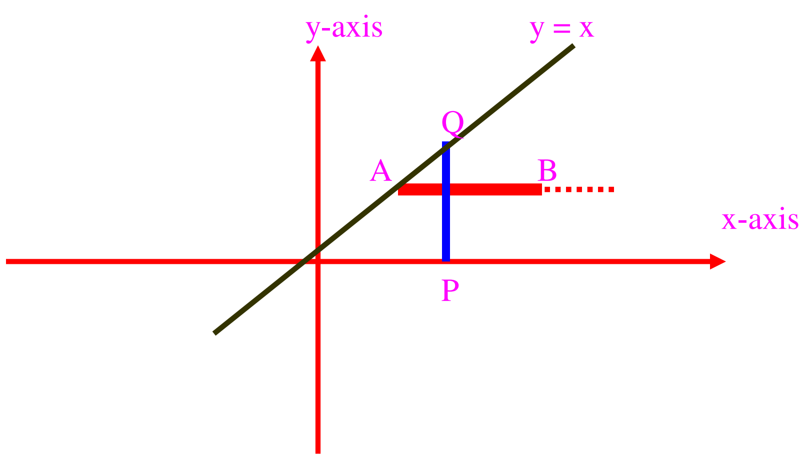
$$I=\int_0^{\infty}\left[\int_0^{y}\frac{e^{-y}}{y}dx\right]dy=\int_0^{\infty}\left[\frac{e^{-y}}{y}.x\right]_0^{y}dy=\int_0^{\infty}\frac{e^{-y}}{y}(y-0)dy$$

$$=\int_0^{\infty}e^{-y}dy=\left|e^{-y}\right|_{\infty}^{0}=e^{-0}-e^{-\infty}=1-\frac{1}{e^{\infty}}=1-0=1\text{ . Ans.}$$

Q.No.11.: Evaluate the integral by changing the order of integration

$$\int_0^{\infty}\left(\int_0^{x}xe^{-x^2/y}dy\right)dx\ .$$

Sol.: The region of integration or the given integral is the area bounded by $y=0,\ y=x$ and $x=0,\ x=\infty$.



The elementary strip given in the problem is a strip PQ parallel to the y-axis whose one end lies on the line y= 0 and other end lies on the line y = x and this strip slides from x = 0 to $x\to\infty$.

On changing the order of the above problem, we first integrate it along a horizontal strip which extends from x = 0 to $x\to\infty$, and this strip slides from y = 0 to $y\to\infty$.

(The region of integration is the area below the line x = y which extends up to infinity to cover this area)

Hence, on reversing the order of integration, we get

$$I=\int_0^{\infty}\left(\int_y^{\infty}xe^{-x^2/y}dx\right)dy$$

Let $e^{-x^2/y} = t \Rightarrow \dfrac{-2x}{y}e^{-x^2/y}dx = dt \Rightarrow xe^{-x^2/y}dx = -\dfrac{y}{2}dt$

Also when x = y, $t = e^{-y}$ and when $x = \infty$, t = 0.

$$\therefore I = \int_0^\infty \left(\int_y^\infty xe^{-x^2/y}dx \right) dy = \int_0^\infty \left(\int_{e^{-y}}^0 \left(-\frac{y}{2}\right)dt \right) dy = \int_0^\infty \left[-\frac{y}{2}.t \right]_{e^{-y}}^0 dy = \int_0^\infty \frac{1}{2}ye^{-y}dy$$

$$= \frac{1}{2}\left[-y.e^{-y} + \int e^{-y}dy \right]_0^\infty = \frac{1}{2}\left[-ye^{-y} - e^{-y} \right]_0^\infty = \frac{1}{2}. \text{ Ans.}$$

Q.No.12.: Change the order of integration in the following

$$\textbf{(i)} \int_0^1 \int_{-\sqrt{1-y^2}}^{1-y} f(x,y)dydx\,, \quad \textbf{(ii).} \int_0^1 \int_{x^2}^{\sqrt{2-x^2}} f(x,y)dydx\,.$$

Sol.: **(i)** The region of integration is one region bounded by the curves $x^2 + y^2 = 1$ and $x + y = 1$.

Initially the strip was parallel to x-axis having its two ends on $x = -\sqrt{1-y^2}$ and $x = 1 - y$, sliding from 0 to 1.

Changing the order of integration:

Taking one strip parallel to y-axis. Now, the strip will slide parallel to y-axis having its two ends on y = 0 and $x^2 + y^2 = 1$ from $x = -1$ to x = 0. This will give us the integral

$\int_{-1}^0 \int_0^{\sqrt{1-x^2}} f(x,y)dydx$. After that when it will coincide with y-axis it will start sliding with its two ends on y = 0 and $y = 1 - x$. In the positive direction of x., which will give us the integral $\int_0^1 \int_0^{1-x} f(x,y)dydx$ as it moves from x = 0 to x = 1.

Thus the integral becomes

$$\int_{-1}^0 \int_0^{\sqrt{1-x^2}} f(x,y)dydx + \int_0^1 \int_0^{1-x} f(x,y)dydx\,.$$

(ii) The region of given interval is bounded by $y = x^2$,

$y=\sqrt{2-x^2} \Rightarrow y^2=2-x^2 \Rightarrow x^2+y^2=\left(\sqrt{2}\right)^2$, and $x=0,\ x=1$.

The graph of the region is the area bounded by the curve is OAB. When we change the order of integration in the given interval, we have to first integrate w. r. t. x y as constant and then w. r. t. y. this is done by covering the area OAB by drawing an elementary strip PQ parallel to x-axis and then moving it parallel to x-axis as to cover whole of the area. On the strip PQ y is constant and x varies first from x = 0 to $y=\sqrt{y}$ and then from x = 0 to $x=\sqrt{2-y^2}$. And when the strip PQ moves parallel to x-axis, so as to cover the whole area, y vanishes first from y = 0 to y = 1 and then from y = 1 to $y=\sqrt{y}$.

Thus the integral becomes

$$\int_0^1\left[\int_{x^2}^{\sqrt{y}}(x,y)dx\right]dy+\int_0^{\sqrt{2}}\left[\int_{x^2}^{\sqrt{2-y^2}}(x,y)dx\right]dy\text{. Ans.}$$

Q.No.13.: Evaluate the following integral by changing the order of integration

$$\int_0^b\int_0^{\frac{a}{b}\sqrt{b^2-y^2}}x^3ydxdy.$$

Sol.: The region of the given integral is bounded by x = 0 and $x=\frac{a}{b}\sqrt{b^2-y^2}$

$$\Rightarrow x^2b^2=a^2\left(b^2-y^2\right)=\frac{x^2b^2}{a^2}=b^2-y^2 \Rightarrow \frac{x^2b^2}{a^2}+y^2=b^2$$

$$\Rightarrow \frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

This equation represent an ellipse.

Now to change the order

$$b^2x^2+a^2y^2=a^2b^2 \Rightarrow a^2y^2=a^2b^2-b^2x^2 \Rightarrow a^2y^2=b^2\left(a^2-x^2\right)$$

$$\Rightarrow y=\frac{b}{a}\sqrt{a^2-x^2}.$$

$\therefore$ Integral becomes $\int_0^a \left[\int_0^{\frac{a}{b}\sqrt{a^2-x^2}} x^3 y dy \right] dx$. Ans.

$$I = \int_0^a \left[\int_0^{\frac{a}{b}\sqrt{a^2-x^2}} x^3 y dy \right] dx = \int_0^a \left[x^3 \left[\frac{y^2}{2} \right]_0^{\frac{b}{a}\sqrt{a^2-x^2}} \right] dx = \int_0^a \left\{ \frac{x^3}{2} \left[\frac{b^2}{a^2} \left(a^2 - x^2\right) \right] \right\} dx$$

$$= \int_0^a \left\{ \frac{x^3}{2} \left[\frac{b^2\left(a^2 - x^2\right)}{a^2} \right] \right\} dx = \int_0^a \left\{ \frac{x^3}{2} \left[\frac{\left(a^2 - b^2 x^2\right)}{a^2} \right] \right\} dx = \frac{1}{2} \int_0^a \left\{ x^3 b^2 - \frac{b^2 x^5}{a^2} \right\} dx$$

$$= \frac{1}{2} \left[b^2 \left[\frac{x^4}{4} \right]_0^a - \frac{b^2}{a^2} \left[\frac{x^6}{6} \right]_0^a \right] = \frac{1}{2} \left[\frac{b^2 a^4}{4} - \frac{b^2}{a^2} \frac{a^2 a^4}{6} \right] = \frac{1}{2} b^2 a^4 \left[\frac{1}{4} - \frac{1}{6} \right]$$

$$= \frac{1}{2} b^2 a^4 \times \frac{2}{24} = \frac{a^4 b^2}{24} \text{ . Ans.}$$

Q.No.14.: Determine the limit of integration for $\iint_R f(x, y) dydx$, where the region is bounded by $y = 0, \ y = 1 - x^2$ and hence change the order.

Sol.: The region of the given integral is bounded by $y = 0$ and $y = 1 - x^2 \Rightarrow x^2 = 1 - y$.

Thus the shaded region is the region of integration. To find the limit of integral $\iint_R f(x, y) dydx$.

Let us take an element strip parallel to y-axis. Now moving the strip parallel to y-axis so as to cover the whole of the area. On the strip parallel to y-axis, the x is constant and y-varies from y = 0 to $y = 1 - x^2$ and when the strip is parallel to x-axis and is moved parallel to x-axis, so as to cover whole of the area.

Thus the integral becomes.

$$\int_{-1}^{1} \left[\int_0^{1-x^2} f(x, y) dy \right] dx .$$

Now we have to change the limit of integration (change of order).

Now taking a strip parallel to x-axis and moving it parallel to x-axis so as to cover the whole area. Now when we change the order of integration in the given integral, we have to first integrate w. r. t. x keeping y as constant and then w. r. t. y.

As y is constant. Therefore x varies from $x=-\sqrt{1-y}$ to $x=\sqrt{1-y}$, and when the strip is parallel to y-axis, the strip moved parallel to y-axis. So as to cover the whole area. Therefore y varies from $y=0$ to $y=1$.

Thus the equation becomes

$$\int_0^1\left[\int_{-\sqrt{1-y}}^{\sqrt{1-y}} f(x,y)dx\right]dy.$$

Q.No.15.: Evaluate the following integral by changing the order of integration

$$\int_{-a}^{a}\int_0^{\sqrt{a^2-y^2}} f(x,y)dxdy.$$

Sol.: $I=\int_{-a}^{a}\int_0^{\sqrt{a^2-y^2}} f(x,y)dxdy$ [given]

In the given equation the elementary strip is parallel to x-axis (say AB) and is from $x=0$ to $x=\sqrt{a^2-y^2}$

And from $x=\sqrt{a^2-y^2}\Rightarrow x^2=a^2-y^2$ which is a circle with radius O.

The same strip is from $y=-a$ to $y=a$.

So area of integration is the area covered by the semicircle (shaded portion)

Now on changing the order of integration we will first integrate w. r. t. y along the vertical strip CD. This vertical is from $y=-\sqrt{a^2-x^2}$ to $y=-\sqrt{a^2-x^2}$ along the strip parallel to x-axis, we will integrate from $x=0$ to $x=a$.

So the changed order of integration is

$$\int_0^a\left[\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} f(x,y)dy\right]dx.$$

Q.No.16.: Evaluate the following integral by changing the order of integration

$$\int_0^4 \int_0^{x(4-x)} dydx.$$

Sol.: Given $I = \int_0^4 \int_0^{x(4-x)} dydx$.

Here strip is parallel to y-axis. Such PQ and extended from $y = 0$ to $y = x(4-x)$

{i. e. parabola $(y-4) = -(x-2)^2$ }., and this strip moves from $x = 0$ to $x = 4$.

Now this shaded region is area of integration. On changing the order of integration we first integrate w. r. t. x along horizontal strip RS. Now, this strip will move

$(x-2)^2 = (y-4) \Rightarrow (x-2) = \pm\sqrt{4-y}$ $x = 2 \pm \sqrt{4-y}$,

from $x = 2-\sqrt{4-y}$ to $x = 2+\sqrt{4-y}$. So to cover the given region we then integrate w. r. t. y from $y = 0$ to $y = 4$.

So, we get

$$I = \int_0^4 \left[\int_{2-\sqrt{4-y}}^{2+\sqrt{4-y}} dx\right] dy = \int_0^4 [x]_{2-\sqrt{4-y}}^{2+\sqrt{4-y}} dy = \int_0^4 \left[2+\sqrt{4-y}-2+\sqrt{4-y}\right] dy$$

$$= \int_0^4 2\sqrt{4-y}dy = 2.\int_0^4 \sqrt{4-y}dy.$$

Put $4-y = t$, $\quad \frac{dt}{dy} = -1 \Rightarrow -dt = dy$

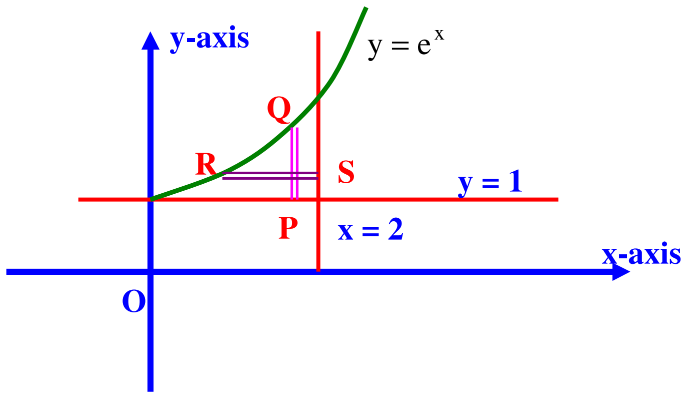
$$\Rightarrow I = -2\int_0^4 \sqrt{t}dt = -2\left[\frac{t}{\frac{1}{2}+1}\right]_0^4 = \frac{-2\times 2}{3}\left[t^{\frac{3}{2}}\right]_0^4 = \frac{-4}{3}\left(4^{3/2}\right) = \frac{-4\times 8}{3} = \frac{-32}{3}.$$

Now neglect negative sign, we get $\frac{32}{3}$ sq. unit is the required result.

Q.No.17.: Evaluate the following integral by changing the order of integration

$$\int_0^2 \int_1^{e^x} dydx.$$

Sol.: Here the elementary strip is parallel to y-axis (such as PQ) and extends from $y_1 = 1$ to $y_2 = e^x$ and this strip slides from $x_1 = 0$ to $x_2 = 2$. This shaded semicircle, area is therefore, the region of integration.



On changing the order of integration, we first integrate w. r. t. x along a strip RS which extends from. $R[x_1 = \log y]$. To cover the given region, we then integrate w. r. t. y from $y_1 = 1$ to $y_2 = e^2$.

So the changed order of integration is $\int_1^{e^2}\int_{\log y}^{2} dxdy = \int_1^{e^2}\left(\int_{\log y}^{2} dx\right)dy$.

$$I = \int_1^{e^2}\left[\int_{\log y}^{2} dx\right]dy = \int_1^{e^2}[x]_{\log y}^{2}dy = \int_1^{e^2}(2 - \log y)dy = \int_1^{e^2} 2dy - \int_1^{e^2}\log ydy$$

$$= (2e^2 - 2) - \int_1^{e^2}\log y.1dy = (2e^2 - 2) - \left[\log y\int dy - \int \frac{d}{dy}(\log y)\int 1.dy\right]$$

$$= (2e^2 - 2) - [y\log y - y]_1^{e^2} = (2e^2 - 2) - [e^2\log e^2 - e^2 - 1\log 1 + 1]$$

$$= 2e^2 - 2 - 2e^2 + e^2 - 1 = e^2 - 3 \begin{bmatrix} \because \log 1 = 0 \\ \log e^2 = 2 \end{bmatrix}$$

$$\int_1^{e^2}\int_{\log y}^{2} dxdy = e^2 - 3. \text{ Ans.}$$

Q.No.18.: Evaluate the following integral by changing the order of integration

$$\int_0^1 \int_{e^x}^{e} \frac{1}{\log y} dydx .$$

Sol.: $I = \int_0^1 \int_{e^x}^{e} \frac{dy}{\log y} dx$.

First we will have to change the order of integral. Taking strip parallel to x-axis, we get the limits from 0 to log y and for dy limits to be taken from 1 to e.

$$I = \int_1^e \int_0^{\log y} \frac{dx}{\log y} dy = \int_1^e \frac{x}{\log y} dy \quad [\text{Where } x = \log y]$$

$$= \int_1^e \frac{dy}{\log y} [x]_0^{\log y} = \int_1^e \frac{\log y}{\log y} = \int_1^e dy = [y]_1^e = e - 1 . \text{ Ans.}$$

Q.No.19.: Evaluate $\int_0^a \int_y^a \frac{xdxdy}{x^2 + y^2}$ by changing the orderof integration.

Sol.: $I = \int_0^a \int_y^a \frac{xdxdy}{x^2 + y^2} = \int_0^a \left[\int_y^a \frac{x}{x^2 + y^2} dx \right] dy$.

Here x = y, x = a and y = a, y = 0.

As strip AB moves from 0 to x.

And ‘x’ changes from 0 to a

$$\therefore I = \int_0^a \left[x \int_0^x \frac{dy}{x^2 + y^2} \right] dx = \int_0^a \left[x \int_0^x \frac{1}{x} \tan^{1} \frac{y}{x} \right] dx = \int_0^a \left[x \times \frac{1}{x} \tan^{1} \frac{y}{x} \right]_0^x dx$$

$$= \int_0^a \left(\tan^{1} \frac{x}{x} - \tan^{-1} 0 \right) dx = \int_0^a \frac{\pi}{4} dx = \frac{\pi}{4} \int_0^a dx = \frac{\pi}{4} [x]_0^a = \frac{\pi}{4} a . \text{ Ans.}$$

Q.No.20.: Evaluate $\int_0^a \int_{y^2/a}^{y} \frac{ydxdy}{(a - x)\sqrt{ax - y^2}} = \frac{1}{2} \pi a$ by changing the orderof integration.

Sol.: The required area of integration is OABC. In the problem we have to first integrate first w. r. t. a horizontal strip PQ w. r. t. x on the parabola $y^2 - ax$ and line y = x. The required region g integration is OABC.

To solve this problem, we have to change the order g integration i. e. we first integrate w. r. t. y along vertical strip P'Q' and then split the area OABPO from that area. Thus for region OABCO, the limit of integration w. r. t. y = 0, $y=\sqrt{ax}$ and y = 0 and y = x and then for x it is for x = 0 to x = a.

Thus the required area.

$$A=\int_0^a\int_{y^2/a}^{y}\frac{ydxdy}{(a-x)\sqrt{ax-y^2}}=\int_0^a\int_0^{\sqrt{ax}}\frac{ydydx}{(a-x)\sqrt{ax-y^2}}-\int_0^a\int_0^x\frac{ydydx}{(a-x)\sqrt{ax-y^2}}$$

$$=\int_0^a\left[\frac{-\sqrt{ax-y^2}}{a-x}\right]_0^{\sqrt{ax}}dx-\int_0^a\left[\frac{-\sqrt{ax-y^2}}{a-x}\right]_0^x dx$$

$$=\int_0^a\left[\frac{-\sqrt{ax-y^2}}{a-x}+\frac{\sqrt{ax}}{a-x}\right]dx-\int_0^a\left[\frac{-\sqrt{ax-y^2}}{a-x}+\frac{\sqrt{ax}}{a-x}\right]dx$$

$$=\int_0^a\frac{\sqrt{ax}}{a-x}-\frac{\sqrt{ax}}{a-x}+\frac{-\sqrt{ax-y^2}}{a-x}dx=\int_0^a\sqrt{x}\,\frac{\sqrt{a-x}}{a-x}dx=\int_0^a\frac{\sqrt{x}}{\sqrt{a-x}}dx$$

We have the property of definite integral.

$$\int_0^a f(x)dx=\int_0^a f(a-x)dx$$

$$2A=\int_0^a\left(\frac{\sqrt{a-x}}{\sqrt{x}}+\frac{\sqrt{x}}{\sqrt{a-x}}\right)dx=\int_0^a\left(\frac{a-x+x}{\sqrt{x}\sqrt{a-x}}\right)dx=\int_0^a\left(\frac{a}{\sqrt{x}\sqrt{a-x}}\right)dx\,.$$

Put $x=a\sin^2\theta$, $\therefore dx=2a\sin\theta\cos\theta d\theta$.

$$2A=\int_0^{\pi/2}\frac{a\times 2a\sin\theta\cos\theta}{a\sin\theta\cos\theta}d\theta=\int_0^{\pi/2}2ad\theta$$

$\Rightarrow A=\int_0^{\pi/2}ad\theta=\frac{\pi}{2}a$, which is the required proof.

Q.No.21.: Show by an example that the interchange the order of integration will not always give the same result.

Sol.: Consider the following two integrals

(i) $\int_0^1 \int_0^1 \frac{x-y}{(x+y)^3} dxdy$ (ii) $\int_0^1 \int_0^1 \frac{x-y}{(x+y)^3} dydx$

Let us evaluate (i):

$$\int_0^1 \frac{x-y}{(x+y)^3} dx = \int_0^1 \frac{x+y-2y}{(x+y)^3} dx = \int_0^1 \frac{1}{(x+y)^2} dx - \int_0^1 \frac{2y}{(x+y)^3} dx$$

$$= \frac{y}{(1+y)^2} - \frac{1}{(1+y)} = \frac{1}{(1+y)^2}.$$

Using the above result in (i), we get $\int_0^1 \frac{1}{(1+y)^2} dy = \left[\frac{1}{1+y}\right]_{y=0}^{y=1} = -\frac{1}{2}$

So $\int_0^1 \int_0^1 \frac{x-y}{(x+y)^3} dxdy = -\frac{1}{2}$. (A)

In similar manner, evaluating (ii), we get $\int_0^1 \int_0^1 \frac{x-y}{(x+y)^3} dydx = \frac{1}{2}$ (B)

From (A) and (B) we see that the interchange of the order of integration will not always give the same result.

Q.No.22.: Change the order of the integration and then evaluate: $\int_0^{2a} \int_0^{\sqrt{2ay-y^2}} dxdy$.

Sol.: The given integral is $\int_0^{2a} \left(\int_0^{\sqrt{2ay-y^2}} dx \right) dy$

$$= \int_0^a \left(\int_{a-\sqrt{a^2-x^2}}^{a+\sqrt{a^2-x^2}} dy \right) dx = \int_0^a [y]_{a-\sqrt{a^2-x^2}}^{a+\sqrt{a^2-x^2}} = 2\int_0^a \sqrt{a^2-x^2} dx$$

Put $x = a\sin\theta \Rightarrow dx = a\cos\theta d\theta$.

Also, when x = 0, $\theta = 0$ and when x = a, then $a = a\sin\theta \Rightarrow \theta = \frac{\pi}{2}$.

Thus $2\int_0^a \sqrt{a^2-x^2} dx = 2\int_0^{\pi/2} a^2\cos^2\theta d\theta = 2a^2 \times \frac{\pi}{4} = \frac{\pi a^2}{2}$. Ans.

Q.No.23.: Change the order of the integration and then evaluate:

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\,dydx$$

Sol.: The given integral is $\int_0^a \left(\int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\,dy \right) dx$

$$= \int_0^a \left(\int_a^{\sqrt{a^2-y^2}} \sqrt{a^2-x^2-y^2}\,dx \right) dy$$

$$= \int_0^a \left(\frac{x}{2}\sqrt{a^2-x^2-y^2} + \left(\frac{\sqrt{a^2-y^2}}{2} \right)^2 \sin^{-1}\left(\frac{x}{\sqrt{a^2-y^2}} \right) \right)_0^{\sqrt{a^2-y^2}} dy$$

$$= \int_0^a \frac{\pi}{2}\left(\frac{a^2-y^2}{2} \right) dy = \frac{\pi}{4}\int_0^a \left(a^2y - \frac{y^3}{3} \right) dy$$

$$= \frac{\pi}{4}\left(a^3 - \frac{a^3}{3} \right) = \frac{\pi}{4}\left(\frac{2a^3}{3} \right) = \frac{\pi a^3}{6} \text{ .Ans.}$$

Q.No.24.: Change the order of the integration and then evaluate: $\int_0^{\sqrt{2}} \int_{-\sqrt{4-2y^2}}^{\sqrt{4-2y^2}} ydxdy$.

Sol.: The given integral is $\int_0^{\sqrt{2}} \left(\int_{-\sqrt{4-2y^2}}^{\sqrt{4-2y^2}} ydx \right) dy = \int_{-2}^{+2} \left(\int_0^{\sqrt{2-\frac{x^2}{2}}} ydy \right) dx = \int_{-2}^{+2} \left(\frac{y^2}{2} \right)_0^{\sqrt{2-\frac{x2}{2}}} dx$

$$= \int_{-2}^{+2} \frac{1}{2}\times\left(2 - \frac{x^2}{2} \right) dx = \int_{-2}^{+2} \left(1 - \frac{x^2}{4} \right) dx = \left| x - \frac{x^3}{12} \right|_{-2}^{+2}$$

$$= 2-(-2)-\left(\frac{8}{12} - \left(-\frac{8}{12} \right) \right) = 4 - \left(2\times\frac{8}{12} \right) = 4 - \frac{4}{3} = \frac{8}{3}. \text{ Ans.}$$

Q.No.25.: Change the order of the integration and then evaluate: $\int_0^a \int_{y^2/a}^{2a-y} xydxdy$

Sol.: The given integral is $\int_0^a \int_{y^2/a}^{2a-y} xydxdy$

Here $x = \frac{y^2}{a} \Rightarrow y^2 = ax$ and $x = 2a - y \Rightarrow x + y = 2a$.

Initially strip is parallel to x-axis

For change of order consider a take strip which is parallel to y-axis.

So we have the following regions for integration

For I_1 y varies from 0 to $\sqrt{ax}$

x varies from 0 to a

For I_2 y varies from 0 to $2a - x$

x varies from a to 2a

$$\therefore \int_0^a \int_{y^2/a}^{2a-y} xydxdy = \int_a^a \left(\int_0^{\sqrt{ax}} xydy \right) dx + \int_a^{2a} \left(\int_0^{2a-x} xydy \right) dx$$

$$= \int_0^a \left| \frac{xy^2}{2} \right|_0^{\sqrt{ax}} dx + \int_0^{2a} \left| \frac{xy^2}{2} \right|_0^{2a-x} dx = \int_0^a \frac{ax^2}{2} dx + \int_a^{2a} \frac{x(2a-x)^2}{2} dx$$

$$= \left| \frac{ax^3}{6} \right|_0^6 + \int_a^{2a} \left(\frac{4a^2x + x^3 - 4ax^2}{2} \right) dx = \frac{a^4}{6} + \left| \frac{4a^2x^2}{4} + \frac{x^4}{8} + \frac{4ax^3}{6} \right|_a^{2a}$$

$$= \frac{a^4}{6} + \left[3a^4 + \frac{15}{8}a^4 - \frac{14}{3}a^4 \right] = \frac{19}{6}a^4 - \frac{28a^4}{6} + \frac{15}{8}a^4 = -\frac{12a^4}{8} + \frac{15a^4}{8} = \frac{3a^4}{8}. \text{ Ans.}$$

Q.No.26.: Change the order of the integration and then evaluate: $\int_0^2 \int_{y^3}^{4\sqrt{2y}} y^2 dxdy$

Sol.: The given integral is $\int_0^2 \int_{y^3}^{4\sqrt{2y}} y^2 dxdy$

In the given problem, firstly, integration is done w.r.t. x and the w.r.t. y

Now let us take strip parallel to y-axis i.e. PQ

Now, PQ slides from $\frac{x^2}{32} \to x^{1/3}$

$$\text{Now} \int_0^2 \left(\int_{y^3}^{4\sqrt{2y}} y^2 dx \right) dy = \int_0^8 \left(\int_{x^2/32}^{x^{1/3}} y^2 dy \right) dx = \int_0^8 \left(\frac{y^3}{3} \right)_{x^2/32}^{x^{1/3}} dx$$

$$= \frac{1}{3} \int_0^8 \left(x - \frac{x^6}{(32)^3} \right) dx = \frac{1}{3} \left(\frac{x^2}{2} - \frac{x^7}{7.(32)^3} \right)_0^8 = \frac{1}{3} \left(32 - \frac{8^7}{7(32)^3} \right)$$

$$=\frac{32}{3}\left(1-\frac{2}{7}\right)=\frac{32}{3}\times\frac{5}{7}=\frac{160}{21}\text{. Ans.}$$

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3^{rd} Topic

Integral Calculus

Triple Integrals

[Change of variables]

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Change of variables:

The evaluation of the triple integrals is greatly simplified by a suitable change of variables. By changing the variables, a given integral can be transformed into a simpler integral involving the new variables.

For triple integrals:

Let the variables x, y, z in the triple integral

$$\iiint_R f(x, y, z)\,dxdydz \qquad \text{(i)}$$

be changed to the new variables u, v, w by the transformation

$$x = \phi(u, v, w),\ \ y = \psi(u, v, w), z = \varphi(u, v, w),$$

where $\phi(u, v, w)$, $\psi(u, v, w)$, $\varphi(u, v, w)$ are continuous and have continuous first order derivatives in some region R'_{uvw} which corresponds to the region R_{xyz}.

The formula corresponds to (i) is

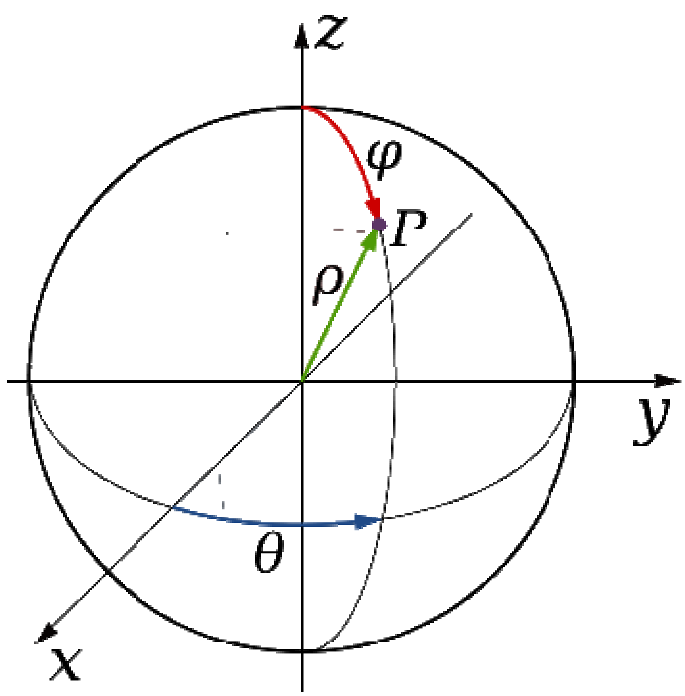
$$\iiint_{R_{xyz}} f(x, y, z)dxdydz = \iiint_{R'_{uvw}} f[x(u, v, w), y(u, v, w), z(u, v, w)]|J|dudvdw,$$

$$\text{where } J = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} \ (\neq 0)$$

is the **Jacobian** of transformation from (x, y, z) to (u, v, w) co-ordinates.

Particular cases:

(I) CONVERSION OF RECTANGULAR TO SPHERICAL SYSTEM



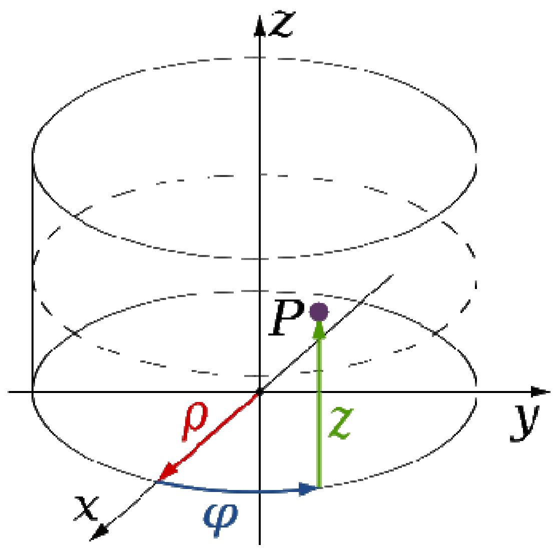
Spherical Coordinates

To change rectangular co-ordinates (x, y, z) to spherical polar co-ordinates (r, θ, ϕ), we have put $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$ and

$$J = \frac{\partial(x, y, z)}{\partial(r, \theta, \phi,)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} = r^2\sin\theta.$$

Then $\iiint_{R_{x,y,z}} f(x, y, z)dxdydz = \iiint_{R'_{r\theta\phi}} f(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta).r^2\sin\theta drd\theta d\phi$.

(II) CONVERSION OF RECTANGULAR TO CYLINDRICALSYSTEM



Cylindrical coordinates.

To change rectangular co-ordinates (x, y, z) to cylindrical co-ordinates (ρ, ϕ, z), we have put $x = \rho\cos\phi$, $y = \rho\sin\phi$, $z = z$ and

$$J = \frac{\partial(x, y, z)}{\partial(\rho, \phi, z)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\phi & -\rho\sin\phi & 0 \\ \sin\phi & \rho\cos\phi & 0 \\ 0 & 0 & 1 \end{vmatrix} = \rho\left(\cos^2\phi + \sin^2\phi\right) = \rho .$$

Then $\iiint_{R_{xyz}} f(x, y, z)dxdydz = \iiint_{R'_{\rho\theta z}} f(\rho\cos\phi, \rho\sin\phi, z).\rho d\rho d\phi dz$.

Now let us solve some problems:

Q.No.1.: Evaluate the following integral by changing to spherical polar co-ordinates:

$$\int_0^1 \int_0^{\sqrt{(1-x^2)}} \int_0^{\sqrt{(1-x^2-y^2)}} \frac{dx\,dy\,dz}{\sqrt{(1-x^2-y^2-z^2)}} .$$

or

Evaluate $\displaystyle\int_0^1 \int_0^{\sqrt{(1-x^2)}} \int_0^{\sqrt{(1-x^2-y^2)}} \frac{dx\,dy\,dz}{\sqrt{(1-x^2-y^2-z^2)}}$,

the integral being extended to the positive octant of the sphere

$x^2+y^2+z^2=1$.

Sol.: We have to solve this problem by changing rectangular co-ordinates (x, y, z) to spherical polar co-ordinates (r, θ, ϕ).

As we know, when we change rectangular co-ordinates (x, y, z) to spherical polar co-ordinates (r, θ, ϕ), we have put $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$ and

$$J = \frac{\partial(x, y, z)}{\partial(r, \theta, \phi,)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} = r^2\sin\theta.$$

Then $\displaystyle\iiint_{R_{xyz}} f(x, y, z)dxdydz = \iiint_{R'_{r\theta\phi}} f(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta).r^2\sin\theta drd\theta d\phi$.

Also $\sqrt{1-x^2-y^2-z^2} = \sqrt{1-r^2}$.

$$\therefore I = \int_0^1 \int_0^{\sqrt{(1-x^2)}} \int_0^{\sqrt{(1-x^2-y^2)}} \frac{dx\,dy\,dz}{\sqrt{(1-x^2-y^2-z^2)}} = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \frac{r^2\sin\theta}{\sqrt{1-r^2}} drd\theta d\phi$$

$$= \int_0^{\pi/2}\left[\int_0^{\pi/2}\left\{\int_0^1 \frac{r^2}{\sqrt{1-r^2}}dr\right\}\sin\theta d\theta\right]d\phi$$

Now evaluate $\displaystyle\int_0^1 \frac{r^2dr}{\sqrt{1-r^2}} = \int_0^{\pi/2}\frac{\sin^2 t}{\sqrt{1-\sin^2 t}}\cos tdt = \int_0^{\pi/2}\sin^2 tdt = \frac{1}{2}\times\frac{\pi}{2} = \frac{\pi}{4}$.

Here we put $r = \sin t$, $\therefore dr = \cos tdt$. And as $r \to 0, t \to 0$ and $r \to 1, t \to \dfrac{\pi}{2}$

$$\therefore I = \int_0^{\pi/2}\left[\int_0^{\pi/2}\frac{\pi}{4}\sin\theta d\theta\right]d\phi = \frac{\pi}{4}\int_0^{\pi/2}\left[\left.-\cos\phi\right|_0^{\pi/2}\right]d\phi = -\frac{\pi}{4}\int_0^{\pi/2}\left[\cos\frac{\pi}{2}-\cos 0\right]d\phi$$

$$= \frac{\pi}{4}\int_0^{\pi/2} d\phi = \frac{\pi}{4}\times\frac{\pi}{2} = \frac{\pi^2}{8}. \text{ Ans.}$$

Q.No.2.: Evaluate the following integral by changing to spherical polar co-ordinates:

$$\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} \frac{dxdydz}{\left(1+x^2+y^2+z^2\right)^2}.$$

Sol.: We have to solve this problem by changing rectangular co-ordinates (x, y, z) to spherical polar co-ordinates (r, θ, ϕ).

As we know, when we change rectangular co-ordinates (x, y, z) to spherical polar co-ordinates (r, θ, ϕ), we have put $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$ and

$$J = \frac{\partial(x, y, z)}{\partial(r, \theta, \phi,)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} = r^2\sin\theta.$$

Then $\iiint_{R_{xyz}} f(x, y, z)dxdydz = \iiint_{R'_{r\theta\phi}} f(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta).r^2\sin\theta drd\theta d\phi$.

To cover the whole region, r varies from 0 to ∞, θ varies from 0 to $\frac{\pi}{2}$, ϕ varies from 0 to $\frac{\pi}{2}$.

Also $r^2 = x^2 + y^2 + z^2$.

$$\text{Hence } I = \int_0^{\infty}\int_0^{\infty}\int_0^{\infty} \frac{dxdydz}{\left(1+x^2+y^2+z^2\right)^2} = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^{\infty} \frac{1}{\left(1+r^2\right)^2} r^2\sin\theta drd\theta d\phi$$

$$= \int_0^{\pi/2}\left\{\int_0^{\pi/2}\left(\int_0^{\infty}\frac{r^2}{\left(1+r^2\right)^2}dr\right)\sin\theta d\theta\right\}d\phi$$

$$= \int_0^{\pi/2}\left[\int_0^{\pi/2}\left\{\int_0^{\infty}\left(\frac{\left(1+r^2\right)}{\left(1+r^2\right)^2} - \frac{1}{\left(1+r^2\right)^2}\right)dr\right\}\sin\theta d\theta\right]d\phi \qquad \text{(i)}$$

Now first evaluate $\int\limits_0^{\infty}\left(\frac{\left(1+r^2\right)}{\left(1+r^2\right)^2}-\frac{1}{\left(1+r^2\right)^2}\right)dr=\int\limits_0^{\infty}\frac{1}{1+r^2}dr-\int\limits_0^{\infty}\frac{1}{\left(1+r^2\right)^2}dr$

Let $r=\tan\theta \Rightarrow dr=\sec^2\theta d\theta$. Now here θ varies from 0 to $\frac{\pi}{2}$.

$$=\left[\tan^{-1}r\right]_0^{\infty}-\int\limits_0^{\pi/2}\frac{\sec^2\theta}{\sec^4\theta}d\theta=\frac{\pi}{2}-\int\limits_0^{\pi/2}\cos^2\theta d\theta=\frac{\pi}{2}-\frac{1}{2}.\frac{\pi}{2}=\frac{\pi}{4}.$$

Now putting the value in (i), we get

$$=\frac{\pi}{4}\int\limits_0^{\pi/2}\left(\int\limits_0^{\pi/2}\sin\theta d\theta\right)d\phi=\frac{\pi}{4}\int\limits_0^{\pi/2}\left[-\cos\theta\right]_0^{\pi/2}d\phi=\frac{\pi}{4}\int\limits_0^{\pi/2}1.(d\phi)=\frac{\pi}{4}\times\frac{\pi}{2}=\frac{\pi^2}{8}.$$

Q.No.3.: Evaluate $\iiint(ax+by+cz)^2dxdydz$, throughout the sphere $x^2+y^2+z^2=1$, using spherical polar co-ordinates.

Sol.: We have to solve this problem by changing rectangular co-ordinates (x, y, z) to spherical polar co-ordinates (r,θ,ϕ).

As we know, when we change rectangular co-ordinates (x, y, z) to spherical polar co-ordinates (r,θ,ϕ), we have put $x=r\sin\theta\cos\phi$, $y=r\sin\theta\sin\phi$, $z=r\cos\theta$ and

$$J=\frac{\partial(x,y,z)}{\partial(r,\theta,\phi,)}=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi}\\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial\theta} & \frac{\partial z}{\partial\phi}\end{vmatrix}=\begin{vmatrix}\sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi\\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi\\ \cos\theta & -r\sin\theta & 0\end{vmatrix}=r^2\sin\theta.$$

Then $\iiint\limits_{R_{xyz}}f(x,y,z)dxdydz=\iiint\limits_{R'_{r\theta\phi}}f(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta).r^2\sin\theta drd\theta d\phi$.

$$I=\int\limits_0^{2\pi}\int\limits_0^{\pi}\int\limits_0^{1}\left(a^2x^2+b^2y^2+2abxy+c^2z^2+2czax+2czby\right)dxdydz$$

$$=\int\limits_0^{2\pi}\int\limits_0^{\pi}\int\limits_0^{1}(a^2r^2\sin^2\theta\cos^2\phi+b^2r^2\sin^2\theta\sin^2\phi+2abr^2\sin^2\theta\cos\phi\sin\theta+c^2r^2\cos^2\theta$$

$$+2car^2\sin\theta\cos\theta\cos\phi+2bcr^2\sin\theta\cos\theta\cos\phi)r^2\sin\theta drd\theta d\phi$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{1} (a^2r^2\sin^2\theta\cos^2\phi + b^2r^2\sin^2\theta\sin^2\phi + abr^2\sin^2\theta\sin 2\theta + c^2r^2\cos^2\theta$$

$$+ car^2\sin\phi\sin\theta + bcr^2\sin 2\theta\sin\phi)r^2\sin\theta drd\theta d\phi$$

$$= \frac{1}{5}\int_0^{2\pi}\int_0^{\pi} (a^2\sin^3\theta\cos^2\phi + b^2\sin^3\theta\sin^2\phi + ab\sin^2\theta\sin 2\theta + c^2\cos^2\theta\sin$$

$$+ ac\cos\phi\sin 2\theta + bc\sin 2\theta\sin\phi\sin\theta)d\theta d\phi$$

$$= \frac{1}{5}\int_0^{2\pi}\int_0^{\pi} \sin^3\theta 9(a^2\cos^2\phi + b^2\sin^3\theta\sin^2\phi + 2ab\sin\phi\cos\phi + \int_0^{\pi} c\sin^2\theta\sin 2\theta)$$

$$+ (a\cos\phi + b\sin\phi) + \int_0^{\pi} c^2\cos^2\theta\sin\theta)d\theta d\phi$$

$$= \frac{1}{5}\int_0^{2\pi}\left[\int_0^{\pi} \frac{3\sin\theta - \sin 3\theta}{4}(a\cos\phi + b\sin\phi)^2 d\phi + \int_0^{\pi} 2c\sin^2\theta\cos\theta(a\cos\phi + b\sin\phi)d\theta\right.$$

$$\left.+ \int_0^{\pi} c^2(\sin\theta - \sin^3\theta)d\theta\right]d\phi$$

$$= \frac{1}{5}\int_0^{2\pi}\left[\frac{1}{4}(a\cos\phi + b\sin\phi)^2\left\{(-3\cos\phi)_0^{\pi} + \left(\frac{\cos 3\theta}{3}\right)_0^{\pi}\right\} + \int_0^{\pi} 2c(\cos\theta - \cos^3\theta)\right.$$

$$\left.(a\cos\phi + b\sin\phi)d\theta + \frac{c^2}{4}\int_0^{\pi} 4\sin\theta - 3\sin\theta + \sin 3\theta d\theta\right]d\phi$$

$$= \frac{1}{5}\int_0^{2\pi} \frac{4}{3}(a\cos\phi + b\sin\phi)^2 - \frac{c}{2}(a\cos\phi + b\sin\phi)\left[\left(\frac{\sin 3\theta}{3}\right)_9^{\pi} + 3(\sin\theta)_0^{\pi}\right] + \frac{c^2}{4}\left(2 + \frac{2}{3}\right)$$

$$= \frac{1}{5}\int_0^{2\pi} (2a^2\cos\phi + b\sin\phi) + \frac{2}{3}c^2$$

$$= \frac{2}{15}\int_0^{2\pi} (2a^2\cos^2\phi + 2b^2\sin^2\phi + 4ab\cos\phi\sin\phi + c^2)d\phi$$

$$= \frac{2}{15}\int_0^{2\pi}\left[2a^2\left(\frac{1+\cos 2\phi}{2}\right) + 2b^2\left(\frac{1-\sin 2\phi}{2}\right) + 2ab\sin 2\phi + c^2\right]d\phi$$

$$=\frac{2}{15}\left[\begin{array}{c}2.\frac{a^2}{2}\left\{(\phi)_0^{2\pi}+\left(\frac{\sin 2\phi}{2}\right)_0^{2\pi}\right\}+2.\frac{b^2}{2}\left\{(\phi)_0^{2\pi}-\left(\frac{\sin 2\phi}{2}\right)_0^{2\pi}\right\}\\ -2ab\left(\frac{\cos 2\phi}{2}\right)_0^{2\pi}+c^2(\phi)_0^{2\pi}\end{array}\right]$$

$$=\frac{2}{15}\left[a^2 2\pi+b^2 2\pi-ab(1-1)+c^2 2\pi\right]=\frac{4}{15}\pi\left(a^2+b^2+c^2\right). \text{ Ans.}$$

Q.No. 4.: Find the value of $\iiint x^2 dxdydz$, taking throughout the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \text{ using spherical polar co-ordinates..}$$

Sol.: Let $A=\iiint\limits_{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1} x^2 dxdydz$

Putting $\frac{x}{a}=u$, $\frac{y}{b}=v$ and $\frac{z}{c}=w$.

$\therefore dx=adu$, $dy=bdv$ and $dz=cdw$

$$A=\iiint\limits_{u^2+v^2+w^2\le 1} a^2u^2 adu.bdv.cdw=a^3bc\iiint\limits_{u^2+v^2+w^2\le 1} u^2 du.dv.dw=\iiint\limits_{R_{uvw}} f(u,v,w)dudvdw.$$

Now we have to solve this problem by changing rectangular co-ordinates (u, v, w) to spherical polar co-ordinates (r,θ,ϕ).

As we know, when we change rectangular co-ordinates (u, v, w) to spherical polar co-ordinates (r,θ,ϕ), we have put $u=r\sin\theta\cos\phi$, $v=r\sin\theta\sin\phi$, $w=r\cos\theta$ and

$$J=\frac{\partial(u,v,w)}{\partial(r,\theta,\phi,)}=\begin{vmatrix}\frac{\partial u}{\partial r} & \frac{\partial u}{\partial\theta} & \frac{\partial u}{\partial\phi}\\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial\theta} & \frac{\partial v}{\partial\phi}\\ \frac{\partial w}{\partial r} & \frac{\partial w}{\partial\theta} & \frac{\partial w}{\partial\phi}\end{vmatrix}=\begin{vmatrix}\sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi\\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi\\ \cos\theta & -r\sin\theta & 0\end{vmatrix}=r^2\sin\theta.$$

Then $\iiint\limits_{R_{uvw}} f(u,v,w)dudvdw=\iiint\limits_{R'_{r\theta\phi}} f(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta).r^2\sin\theta drd\theta d\phi$.

$$\therefore A=a^3bc\int_0^{2\pi}\left[\int_0^{\pi}\left(\int_0^1 r^2\sin^2\theta\cos^2\phi r^2\sin\theta dr\right)d\theta\right]d\phi$$

$$= a^3bc\int_0^{2\pi}\left[\int_0^{\pi}\left(\int_0^1 r^4 dr\right).\sin^3\theta d\theta\right]\cos^2\phi d\phi = a^3bc\int_0^{2\pi}\left[\int_0^{\pi}\left(\left|\frac{r^5}{5}\right|_0^1\right).\sin^3\theta d\theta\right]\cos^2\phi d\phi$$

$$= \frac{a^3bc}{5}\int_0^{2\pi}\left[\int_0^{\pi}\sin^3\theta d\theta\right]\cos^2\phi d\phi = \frac{a^3bc}{5}\int_0^{2\pi}\left[\int_0^{\pi}\frac{3\sin\theta - \sin 3\theta}{4}d\theta\right]\cos^2\phi.d\phi$$

$$= \frac{a^3bc}{20}\int_0^{2\pi}\left[3(-\cos\theta)_0^{\pi} - \left(\frac{-\cos 3\theta}{3}\right)_0^{\pi}\right]\cos^2\phi.d\phi = \frac{a^3bc}{20}\int_0^{2\pi}\left[3(1+1) - \left(\frac{1+1}{3}\right)\right]\cos^2\phi.d\phi$$

$$= \frac{a^3bc}{20}\int_0^{2\pi}\left(6-\frac{2}{3}\right)\cos^2\phi.d\phi = \frac{a^3bc}{20}\int_0^{2\pi}\frac{18-2}{3}\cos^2\phi.d\phi = \frac{a^3bc}{20}\times\frac{16}{3}\int_0^{2\pi}\frac{1+\cos 2\phi}{2}d\phi$$

$$= \frac{a^3bc}{20}\times\frac{16}{3}\times\frac{1}{2}\left[(\phi)_0^{2\pi} + \left(\frac{\sin 2\phi}{2}\right)_0^{2\pi}\right] = \frac{a^3bc}{20}\times\frac{16}{3}\times\frac{1}{2}\times 2\pi = \frac{4\pi a^3bc}{15}\ \text{Ans.}$$

Q.No.5.: Evaluate $\iiint \frac{1}{x^2+y^2+z^2}dxdydz$ throughout the volume of the sphere

$x^2+y^2+z^2 = a^2$ using spherical polar co-ordinates..

Sol.: We have to solve this problem by changing rectangular co-ordinates (x, y, z) to spherical polar co-ordinates (r, θ, ϕ).

As we know, when we change rectangular co-ordinates (x, y, z) to spherical polar co-ordinates (r, θ, ϕ), we have put $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$ and

$$J = \frac{\partial(x, y, z)}{\partial(r, \theta, \phi,)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} = r^2\sin\theta.$$

Then $\iiint_{R_{xyz}} f(x, y, z)dxdydz = \iiint_{R'_{r\theta\phi}} f(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta).r^2\sin\theta drd\theta d\phi$.

$$\therefore I = 8\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{a}\frac{r^2\sin\theta}{r^2}drd\theta d\phi = 8\int_0^{\pi/2}\left[\int_0^{\pi/2}\left(\int_0^{a}\frac{r^2\sin\theta}{r^2}dr\right)d\theta\right]d\phi$$

$$= 8\int_0^{\pi/2}\int_0^{\pi/2}[r\sin\theta]_0^a d\theta d\phi = 8\int_0^{\pi/2}\left[\int_0^{\pi/2} r\sin\theta d\theta\right]d\phi = 8\int_0^{\pi/2} d\phi = 8a\times\frac{\pi}{2} = 4\pi a\text{. Ans.}$$

Q.No.6.: Evaluate $\iiint xyzdxdydz$ throughout the positive octant of the sphere

$x^2+y^2+z^2=a^2$ using spherical polar co-ordinates.

Sol.: We have to solve this problem by changing rectangular co-ordinates (x, y, z) to spherical polar co-ordinates (r, θ, ϕ).

As we know, when we change rectangular co-ordinates (x, y, z) to spherical polar co-ordinates (r, θ, ϕ), we have put $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$ and

$$J = \frac{\partial(x, y, z)}{\partial(r, \theta, \phi,)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial\theta} & \frac{\partial z}{\partial\phi} \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} = r^2\sin\theta.$$

Then $\iiint\limits_{R_{xyz}} f(x, y, z)dxdydz = \iiint\limits_{R'_{r\theta\phi}} f(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta).r^2\sin\theta drd\theta d\phi$.

$$\therefore \iiint xyzdxdydz = \int_0^{\pi/2}\left[\int_0^{\pi/2}\left(\int_0^a r^5 dr\right)\sin^3\theta\cos\theta d\theta\right]\cos\phi\sin\phi d\phi$$

$$= \int_0^{\pi/2}\left(\int_0^{\pi/2}\frac{a^6}{6}\sin^3\theta\cos\theta d\theta\right)\cos\phi\sin\phi d\phi = \int_0^{\pi/2}\frac{a^6}{24}\cos\phi\sin\phi d\phi$$

$$= \frac{a^6}{24}\int_0^{\pi/2}\cos\phi\sin\phi d\phi = \frac{a^6}{48}\int_0^{\pi/2}\sin 2\phi d\phi = \frac{a^6}{48}\left|\frac{-\cos 2\phi}{2}\right|_0^{\pi/2}$$

$$= \frac{a^6}{48}+\left(-\frac{1}{2}+\frac{1}{2}\right) = \frac{a^6}{48}\text{. Ans.}$$

Q.No.7.: Evaluate the following integral by changing to spherical polar co-ordinates:

$$\int_0^1\int_0^{\sqrt{(1-x^2)}}\int_{\sqrt{(x^2+y^2)}}^1 \frac{dzdydx}{\sqrt{(x^2+y^2+z^2)}}.$$

Sol.: We have to solve this problem by changing rectangular co-ordinates (x, y, z) to spherical polar co-ordinates (r, θ, ϕ).

As we know, when we change rectangular co-ordinates (x, y, z) to spherical polar co-ordinates (r, θ, ϕ), we have put $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$ and in this case

$$J = \frac{\partial(x, y, z)}{\partial(r, \theta, \phi,)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} = r^2\sin\theta .$$

Then $\iiint\limits_{R_{xyz}} f(x, y, z)dxdydz = \iiint\limits_{R'_{r\theta\phi}} f(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta).r^2\sin\theta drd\theta d\phi$.

Now $x^2 + y^2 + z^2 = r^2$.

The region of integration is common to the cone $z^2 = x^2 + y^2$ and the cylinder $x^2 + y^2 = 1$, bounded by the plane z = 1 in the positive octant.

Since z = 1 in the positive octant $\Rightarrow r\cos\theta = 1 \Rightarrow r = \sec\theta$.

Hence, r varies from 0 to $\sec\theta$, θ varies from 0 to $\frac{\pi}{4}$, and ϕ varies from 0 to $\frac{\pi}{2}$.

$\therefore$ The given integral becomes

$$\int_0^{\pi/2}\int_0^{\pi/4}\int_0^{\sec\theta} \frac{1}{r}.r^2\sin\theta drd\theta d\phi = \int_0^{\pi/2}\left(\int_0^{\pi/4}\left|\frac{r^2}{2}\right|_0^{\sec\theta}\sin\theta d\theta\right)d\phi = \int_0^{\pi/2}\left[\int_0^{\pi/4}\frac{\sec^2\theta}{2}\sin\theta d\theta\right]d\phi$$

$$= \frac{1}{2}\int_0^{\pi/2}\left(\int_0^{\pi/4}\sec\theta.\tan\theta d\theta\right)d\phi = \frac{1}{2}\int_0^{\pi/2}\left(\left[\sec\theta\right]_0^{\pi/4}\right)d\phi$$

$$= \frac{\left(\sqrt{2}-1\right)}{2}\int_0^{\pi/2} d\phi = \frac{\left(\sqrt{2}-1\right)\pi}{4}. \text{ Ans.}$$

(II) CONVERSION OF RECTANGULAR TO CYLINDRICALSYSTEM

Q.No.8.: Evaluate the following integral by changing to cylindrical co-ordinates:

$\iiint z^2 dxdydz$, taken over the volume bounded by the surfaces $x^2 + y^2 = a^2$, $x^2 + y^2 = z$ and $z = 0$.

Sol.: We have to solve this problem by changing rectangular co-ordinates (x, y, z) to cylindrical co-ordinates (ρ, ϕ, z).

As we know, when we change rectangular co-ordinates (x, y, z) to cylindrical co-ordinates (ρ, ϕ, z),

we have put $x = \rho\cos\phi$, $y = \rho\sin\phi$, $z = z$ and

$$J = \frac{\partial(x, y, z)}{\partial(\rho, \phi, z)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\phi & -\rho\sin\phi & 0 \\ \sin\phi & \rho\cos\phi & 0 \\ 0 & 0 & 1 \end{vmatrix} = \rho\left(\cos^2\phi + \sin^2\phi\right) = \rho .$$

Then $\iiint\limits_{R_{xyz}} f(x, y, z)dxdydz = \iiint\limits_{R'_{\rho\theta z}} f(\rho\cos\phi, \rho\sin\phi, z).\rho d\rho d\phi dz$.

$\therefore \rho^2 = a^2$, $\rho^2 = z$ and $z = 0$

So here ρ varies from 0 to a, z varies from 0 to ρ^2 and ϕ varies from 0 to 2π.

$$\therefore I = \iiint z^2 dxdydz = \int_0^{2\pi}\int_0^a\int_0^{\rho^2} z^2 \rho d\rho d\phi dz = \int_0^{2\pi}\left[\int_0^a\left\{\int_0^{\rho^2} z^2 dz\right\}\rho d\rho\right]d\phi$$

$$= \int_0^{2\pi}\left[\int_0^a\left[\frac{z^3}{3}\right]_0^{\rho^2}\rho d\rho\right]d\phi$$

$$= \int_0^{2\pi}\left[\int_0^a \frac{\rho^7}{3}d\rho\right]d\phi = \int_0^{2\pi}\left[\frac{\rho^8}{24}\right]_0^a d\phi = \int_0^{2\pi}\frac{a^8}{24}d\phi = \frac{\pi a^8}{12}. \text{ Ans.}$$

Q.No.9.: By transforming into cylindrical coordinates evaluate the integral

$\iiint \left(x^2 + y^2 + z^2\right)dxdydz$ taken over the region $0 \le z \le x^2 + y^2 \le 1$.

Sol.: We have to solve this problem by changing rectangular co-ordinates (x, y, z) to cylindrical co-ordinates (ρ, ϕ, z).

As we know, when we change rectangular co-ordinates (x, y, z) to cylindrical co-ordinates (ρ, ϕ, z), we have put $x = \rho\cos\phi$, $y = \rho\sin\phi$, $z = z$ and

$$J = \frac{\partial(x, y, z)}{\partial(\rho, \phi, z)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\phi & -\rho\sin\phi & 0 \\ \sin\phi & \rho\cos\phi & 0 \\ 0 & 0 & 1 \end{vmatrix} = \rho\left(\cos^2\phi + \sin^2\phi\right) = \rho .$$

Then $\iiint\limits_{R_{xyz}} f(x, y, z)dxdydz = \iiint\limits_{R'_{\rho\theta z}} f(\rho\cos\phi, \rho\sin\phi, z).\rho d\rho d\phi dz$.

$$\int_0^1 \iint\limits_R \left(x^2 + y^2 + z^2\right)dxdydz = \int_0^1 \int_0^{2\pi} \int_0^1 \left(r^2\cos^2\theta + r^2\sin^2\theta + z^2\right)rdrd\theta dz$$

where R: circular region bounded by the circle of radius one and centre at origin: $x^2 + y^2 = 1$, so that r varies from 0 to 1 and θ varies from 0 to 2π.

Thus $\int_0^1 \int_0^{2\pi} \int_0^1 \left(r^2\cos^2\theta + r^2\sin^2\theta + z^2\right)rdrd\theta dz = \int_0^1 \int_0^{2\pi} \int_0^1 \left(r^3 + rz^2\right)drd\theta dz$

$$= \int_0^1 \int_0^{2\pi} \left(\frac{r^4}{4} + \frac{r^2}{2}z^2\right)_0^1 d\theta dz = \int_0^1 \int_0^{2\pi} \left(\frac{1}{4} + \frac{1}{2}z^2\right)d\theta dz = 2\pi.\int_0^1 \left(\frac{1}{4} + \frac{1}{2}z^2\right)dz$$

$$= 2\pi.\left(\frac{z}{4} + \frac{1}{2}\frac{z^3}{3}\right)_0^1 = 2\pi.\left(\frac{1}{4} + \frac{1}{6}\right) = \frac{5\pi}{6} \text{ Ans.}$$

Q.No.10.: By transforming into cylindrical coordinates evaluate the integral

$\iiint\limits_V \left(x^2 + y^2\right)dxdydz$ taken over the region V bounded by the paraboloid

$z = 9 - x^2 - y^2$ and the plane z = 0.

Sol.: We have to solve this problem by changing rectangular co-ordinates (x, y, z) to cylindrical co-ordinates (ρ, ϕ, z).

As we know, when we change rectangular co-ordinates (x, y, z) to cylindrical co-ordinates (ρ, ϕ, z), we have put $x = \rho\cos\phi$, $y = \rho\sin\phi$, $z = z$ and

$$J = \frac{\partial(x, y, z)}{\partial(\rho, \phi, z)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\phi & -\rho\sin\phi & 0 \\ \sin\phi & \rho\cos\phi & 0 \\ 0 & 0 & 1 \end{vmatrix} = \rho\left(\cos^2\phi + \sin^2\phi\right) = \rho .$$

Then $\iiint\limits_{R_{xyz}} f(x, y, z)dxdydz = \iiint\limits_{R'_{\rho\theta z}} f(\rho\cos\phi, \rho\sin\phi, z).\rho d\rho d\phi dz$.

Now $I = \iiint (\rho^2)\rho dz d\rho d\phi$

Now $z = 9 - x^2 - y^2$, $z = 9 - \rho^2$ and $z = 0$

At $z = 0$, $\rho^2 = 9 \Rightarrow \rho = 3$

$$\therefore I = \int_0^{2\pi}\int_0^{3}\int_0^{9-\rho^2} (\rho^2)\rho dz d\rho d\phi$$

$$= \int_0^{2\pi}\int_0^{3} [z]_0^{9-\rho^2}.\rho^3.d\rho.d\phi = \int_0^{2\pi}\left(\int_0^{3}\left(9-\rho^2\right).\rho^3.d\rho\right)d\phi = \int_0^{2\pi}\left(\int_0^{3}\left(9\rho^3-\rho^5\right)d\rho\right).d\phi$$

$$= \int_0^{2\pi}\left[\frac{9\rho^4}{4} - \frac{\rho^6}{6}\right]_0^3 d\phi = \int_0^{2\pi}\left[\frac{9(81)}{4} - \frac{81\times 9}{6}\right]d\phi = \frac{243}{4}\times 2\pi = \frac{243\pi}{2}. \text{ Ans.}$$

(III) CONVERSION OF RECTANGULAR TO ANY OTHERSYSTEM

Q.No.10.: Using the transformation $u = x + y + z$, $uv = y + z$, $uvw = z$, evaluate the integral $\iiint [xyz(1-x-y-z)]^{1/2} dxdydz$ taken over the tetrahedral volume enclosed by the planes $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$.

Sol.: Here we use the transformation

$u = x + y + z$ (i)

$uv = y + z$ (ii)

$uvw = z$ (iii)

Solving (i), (ii) and (iii), we get

$x = u(1-v)$

$y = uv(1-w)$

$z = uvw$ and Jacobian $= J = u^2v$

According to the problem u, v and w vary from 0 to 1 each.

So triple integral becomes:

$$\iiint [xyz(1-x-y-z)]^{1/2}\,dxdydz = \int_0^1\int_0^1\int_0^1 [u(1-v).uv(1-w).uvw(1-u)]^{1/2}.u^2v\,dudvdw$$

Integrating w.r.t. u, we get

$$I = \int_0^1\int_0^1 [(1-v)v(1-w)(vw)]^{1/2}.v\left(\int_0^1 \left[u^3(1-u)\right]^{1/2} u^2du\right)dvdw$$

$$\Rightarrow \int_0^1\int_0^1 \left[v^4(1-v)w(1-w)\right]^{1/2} dv.dw \times \int_0^1 [u^7(1-u)]^{1/2}du \;(iv)$$

Let $u = \sin^2\theta \Rightarrow du = 2\sin\theta d\theta$

So $\int_0^1 \left[u^7(1-u)\right]^{1/2} du = \int_0^{\pi/2} (\sin^7\theta\cos\theta)2\sin\theta\cos\theta d\theta$

$$= \int_0^{\pi/2} 2.\sin^8\theta\cos^2\theta d\theta = 2\int_0^{\pi/2} \left(\sin^8\theta - \sin^{10}\theta\right)d\theta$$

$$= 2.\left[\frac{7\times5\times3\times1}{8\times6\times4\times2}.\frac{\pi}{2} - \frac{9\times7\times5\times3\times1}{10\times8\times6\times4\times2}.\frac{\pi}{2}\right] = \frac{7\pi}{256}.$$

Putting this in (iv), we get

$$\frac{7\pi}{256}\int_0^1 [w(1-w)]^{1/2}dw \times \int_0^1 \left[v^4(1-v)\right]^{1/2} dv \qquad (v)$$

Let $v = \sin^2\theta \Rightarrow dv = 2\sin\theta\cos\theta d\theta$

$$\therefore \int_0^1 [v^4(1-v)]^{1/2}dv = 2\int_0^{\pi/2} \sin^5\theta\cos^2\theta d\theta$$

$$= 2\int_0^{\pi/2} \left(\sin^5\theta - \sin^7\theta\right)d\theta = \left[\frac{4\times2}{5\times3\times1} - \frac{6\times4\times2}{5\times7\times3\times1}\right] = \frac{16}{105}.$$

Putting this in (v), we get

$$I = \frac{7\pi}{256}\times\frac{16}{105}\int_0^1 [w(1-w)]^{1/2}dw$$

Let $w = \sin^2\theta \Rightarrow dw = 2\sin\theta\cos\theta d\theta$

$$I = \frac{7\pi}{256}\times\frac{16}{105}\int_0^{\pi/2} .2\sin^2\theta\cos^2 d\theta = \frac{7\pi}{256}\times\frac{16}{105}\times 2\int_0^{\pi/2}\left(\sin^2\theta - \sin^4\theta\right)d\theta$$

$$= \frac{7\pi}{256}\times\frac{16}{105}\times 2\left[\frac{1}{2}\frac{\pi}{2} - \frac{3\times 1}{4\times 2}.\frac{\pi}{2}\right] = \frac{7\pi}{256}\times\frac{16}{105}\times\frac{\pi}{8} = \frac{\pi^2}{1920}$$

Hence $\iiint \left[xyz(1-x-y-z)\right]^{1/2} dxdydz = \dfrac{\pi^2}{1920}$. Ans.

Q.No.11.: Using the transformation u = x + y + z , uv = y + z, uvw = z, evaluate the integral $\iiint (x+y+z)^2 xyz\, dxdydz$ taken over the tetrahedral volume enclosed by the planes $x = 0,\ y = 0,\ z = 0$ and $x + y + z = 1$.

Sol.: Here we use the transformation

$u = x + y + z$ (i)

$uv = y + z$ (ii)

$uvw = z$ (iii)

Solving (i), (ii) and (iii), we get

$x = u(1-v)$

$y = uv(1-w)$

$z = uvw$ and Jacobian = $J = u^2v$

According to the problem u, v and w vary from 0 to 1 each.

So triple integral becomes:

$$\int_0^1\int_0^1\int_0^1 (u)^2u^3v^2w(1-v)(1-w).u^2vdudvdw = \int_0^1\int_0^1\int_0^1 u^7v^3w(1-v)(1-w)dudvdw$$

$$= \int_0^1\int_0^1\int_0^1 u^7\left(v^3 - v^4\right)(w-w^2)dudvdw = \int_0^1\left(\int_0^1\left(\int_0^1 u^2du\right)(v^2-v^4)dy\right)(w-w^2)dw$$

$$= \int_0^1\left(\int_0^1\left[\frac{u^8}{8}\right]_0^1(v^3-v^4)dv\right)(w-w^2)dw = \int_0^1\left(\int_0^1\frac{1}{8}\left(v^3-v^4\right)dv\right)(w-w^2)dw$$

$$= \int_0^1\frac{1}{8}\left[\frac{v^4}{4}-\frac{v^5}{5}\right]_0^1(w-w^2)dw = \int_0^1\frac{1}{8}\left(\frac{1}{4}-\frac{1}{5}\right)(w-w^2)dw = \int_0^1\frac{1}{8}\left(\frac{5-4}{20}-\right)(w-w^2)dw$$

$$=\int_0^1 \frac{1}{160}\left(w-w^2\right)dw=\frac{1}{160}\left[\frac{w^2}{2}-\frac{w^3}{3}\right]_0^1=\frac{1}{160}\left[\frac{1}{2}-\frac{1}{3}\right]=\frac{1}{168}\times\frac{1}{6}=\frac{1}{960}. \text{ Ans.}$$

Q.No.12.: Using the transformation $u = x + y + z$, $uv = y + z$, $uvw = z$, evaluate the integral $\iiint e^{(x+y+z)^3}dxdydz$ taken over the tetrahedral volume enclosed by the planes $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$.

Sol.: Here we use the transformation

$u = x + y + z$ (i)

$uv = y + z$ (ii)

$uvw = z$ (iii)

Solving (i), (ii) and (iii), we get

$x = u(1-v)$

$y = uv(1-w)$

$z = uvw$ and Jacobian $= J = u^2v$

According to the problem u, v and w vary from 0 to 1 each.

So triple integral becomes:

$$\iiint e^{(x+y+z)^3}dxdydz=\iiint e^{[u(1-v)+4v(1-w)+4vw]^3}u^2vdudvdw$$

$$=\int_0^1\int_0^1\int_0^1 e^{u^3}u^2vdudvdw=\int_0^1\left\{\int_0^1\left(\int_0^1 e^{u^3}u^2vdv\right)dw\right\}dv=\frac{1}{2}\int_0^1 e^{u^3}u^2du$$

Put $u^3=t \Rightarrow 3u^2du=dt \Rightarrow u^3du=\frac{dt}{3}$.

When $u = 0$, $t = 0$, $u = 1$, $t = 1$

Then integral becomes

$$\iiint e^{(x+y+z)^3}dxdydz=\frac{1}{6}\int_0^1 e^t dt=\frac{1}{6}[e^t]_0^1=\frac{1}{6}\left[e^1-e^0\right]=\frac{e-1}{6}. \text{ Ans.}$$

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4^{th} Topic

Integral Calculus

Double Integrals

[Double Integrals in Polar co-ordinates]

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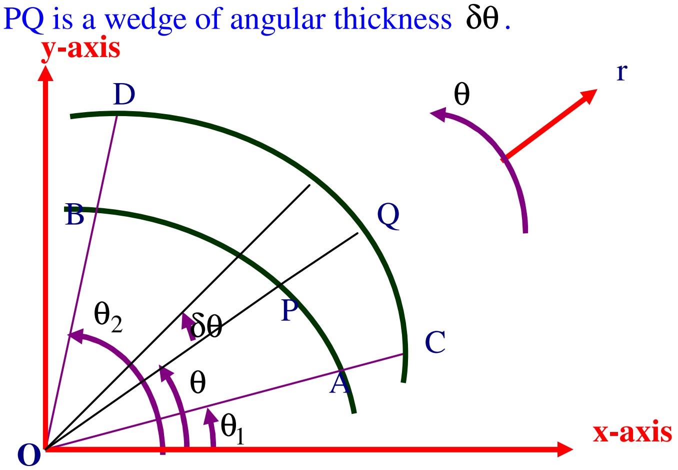
NIT Hamirpur (HP)

Evaluation of Double Integrals in Polar co-ordinates:

To evaluate $\int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} f(r,\theta)\,dr\,d\theta$, we first integrate w.r.t. r between limits $r = r_1$ and $r = r_2$ keeping θ fixed and the resulting expression is integrated w.r.t. θ from θ_1 to θ_2. In this integral r_1, r_2 are functions of θ and θ_1, θ_2 are constants. Figure illustrated the process geometrically.

Here AB and CD are the curves $r_1 = f_1(\theta)$ and $r_2 = f_2(\theta)$ bounded by the lines $\theta = \theta_1$ and $\theta = \theta_2$. PQ is a wedge of angular thickness $\delta\theta$.

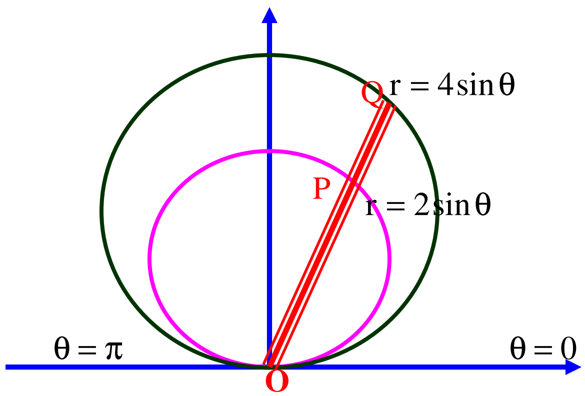


Then $\int_{r_1}^{r_2} f(r,\theta)dr$ indicates that the integration is along PQ from P to Q while the integration w. r. t. θ corresponds to the turning of PQ from AC to BD.

Thus the whole region of integrating is the area ACDB. The order of integration may changed with appropriate changes in the limits.

Q.No.1.: Calculate $\iint r^3 dr d\theta$ over the area included between the circles $r = 2\sin\theta$ and $r = 4\sin\theta$.

Sol.: Given circles are $r = 2\sin\theta$ and $r = 4\sin\theta$, as shown in the figure.



The area between these circles is the region of integration.

If we integrate first w. r. t. r, then its limits are from $P(r = 2\sin\theta)$ to $Q(r = 4\sin\theta)$ and to cover the whole region θ varies from 0 to π. Thus the required integral is

$$I = \int_0^{\pi}\left(\int_{2\sin\theta}^{4\sin\theta} r^3 dr\right)d\theta = \int_0^{\pi}\left[\frac{r^4}{4}\right]_{2\sin\theta}^{4\sin\theta} d\theta = 60\int_0^{\pi} \sin^4\theta d\theta = 60\times 2\int_0^{\pi/2} \sin^4\theta d\theta$$

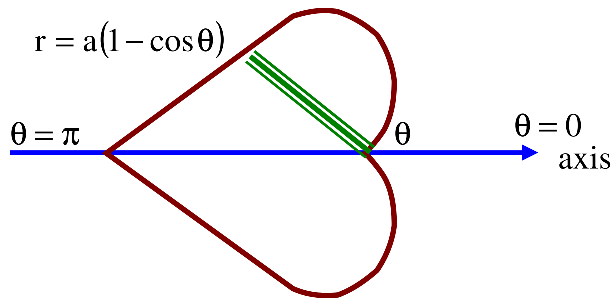
$$= 120\times\frac{3.1}{4.2}.\frac{\pi}{2} = 22.5\pi \text{ .Ans.}$$

Q.No.2.: Evaluate $\iint r\sin\theta dr d\theta$ over the cardioids $r = a(1-\cos\theta)$ above the initial line.

Sol.: The cardioids equation is $r = a(1-\cos\theta)$

The integral $\iint r\sin\theta dr d\theta$ above initial line is

$$I=\int_0^{\pi}\left(\int_0^{a(1-\cos\theta)} r\sin\theta dr\right)d\theta=\int_0^{\pi}\left(\int_0^{a(1-\cos\theta)} rdr\right)\sin\theta d\theta=\int_0^{\pi}\left(\frac{r^2}{2}\right)_0^{a(1-\cos\theta)}\sin\theta d\theta$$



$$=\int_0^{\pi}\frac{a^2(1-\cos\theta)^2}{2}\sin\theta d\theta=\int_0^{\pi}.\frac{a^2\left(2\sin^2\frac{\theta}{2}\right)^2}{2}2\sin\frac{\theta}{2}\cos\frac{\theta}{2}d\theta=\int_0^{\pi}4a^2\sin^5\frac{\theta}{2}\cos\frac{\theta}{2}d\theta$$

$$=4a^2\int_0^{\pi}\sin^5\frac{\theta}{2}\cos\frac{\theta}{2}d\theta=4a^2.2\int_0^{\pi/2}\sin^5\frac{\theta}{2}\cos\frac{\theta}{2}d\theta=4a^2.2\frac{4.2}{6.4.2}\times 1=\frac{4a^2}{3}.\text{ Ans.}$$

Q.No.3.: Sketch the region of integration of $\int_a^{ae^{\pi/4}}\int_{2\log(r/a)}^{\pi/2} f(r,\theta)rdrd\theta$ and change the order of integration.

Sol.: Given Integral is $I=\int_a^{ae^{\pi/4}}\left(\int_{2\log(r/a)}^{\pi/2} f(r,\theta)d\theta\right)rdr$

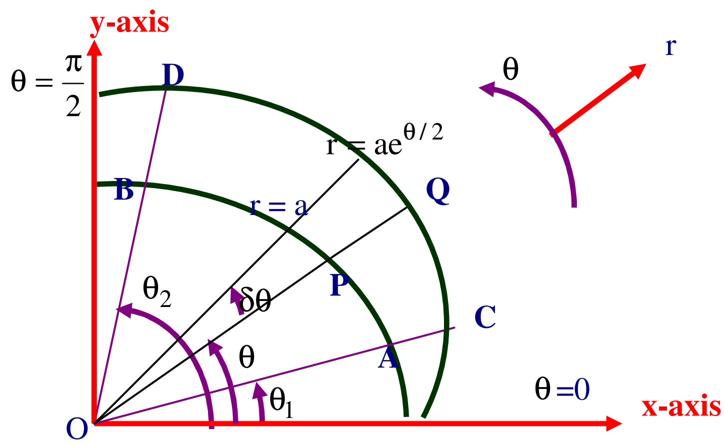
The region of integration is bounded by the curve

$\theta=2\log\frac{r}{a}$, $\theta=\frac{\pi}{2}$, and $r=a$, $r=ae^{\pi/4}$.

Also when $r=a\Rightarrow\frac{r}{a}=1$, then $\theta=2\log\left(\frac{r}{a}\right)=2\log(1)=0$.

and when $r=ae^{\pi/4}\Rightarrow\frac{r}{a}=e^{\pi/4}$ and $\theta=2\log\frac{r}{a}\Rightarrow\frac{r}{a}=e^{\theta/2}$

Then $e^{\theta/2}=e^{\pi/4}\Rightarrow\theta=\frac{\pi}{2}$.



Thus we change the order of integration r varies from $r = a$ to $r = ae^{\theta/2}$ and θ varies from $\theta = 0$ to $\Rightarrow \theta = \frac{\pi}{2}$.

Hence, on reversing the order of integration, we get

$$I = \int_0^{\pi/2}\left[\int_a^{ae^{\theta/2}} f(r,\ \theta) r dr\right] d\theta. \text{ Ans.}$$

Q.No.4.: Show that $\iint_R r^2 \sin\theta dr d\theta = \frac{2a^3}{3}$, where R is the semi-circle $r = 2a\cos\theta$ above the initial line.

Sol.: The given semi-circle is $r = 2a\cos\theta$.

The shaded semicircle shown in the figure is the region of integration. If we integrate first w. r. t. r, then its limits are from r = 0 to $r = 2a\cos\theta$ and to cover the whole area of the semicircle θ varies from 0 to $\frac{\pi}{2}$.

Thus the integral is

$$I = \int_0^{\pi/2}\left[\int_0^{2a\cos\theta} r^2 dr\right]\sin\theta d\theta = \int_0^{\pi/2}\left[\left(\frac{r^3}{3}\right)_0^{2a\cos\theta}\right]\sin\theta d\theta = \int_0^{\pi/2}\left[\frac{8a^3\cos^3\theta}{3}\sin\theta\right]d\theta$$

$$= \frac{-8a^3}{3} \int_0^{\pi/2} \cos^3\theta d(\cos\theta) = \frac{-8a^3}{3}\left[\frac{\cos^4\theta}{4}\right]_0^{\pi/2} = \frac{-8a^3}{3}.\frac{1}{4}[0-1]$$

$$= \frac{-2a^3}{3}(-1) = \frac{2a^3}{3}, \text{ which proves the result.}$$

Q.No.5.: Evaluate $\iint \frac{rdrd\theta}{\sqrt{a^2+r^2}}$ over one loop of the lemniscate $r^2 = a^2\cos 2\theta$.

Sol.: Symmetry: Curve is symmetric about the pole as even power of the r.

Limits: No position of the curve lies between $\theta = \frac{\pi}{4}$ and $\theta = \frac{3\pi}{4}$.

Region of integration is the area bounded by the curve.

$r = 0, \quad r = a\sqrt{\cos 2\theta} \quad$ and $\quad \theta = \frac{-\pi}{4}, \quad \theta = \frac{\pi}{4}$

Thus $I = \int_{-\pi/4}^{\pi/4}\int_0^{a\sqrt{\cos 2\theta}} \frac{r\,dr\,d\theta}{\sqrt{a^2+r^2}} = \int_{-\pi/4}^{\pi/4}\left[\int_0^{a\sqrt{\cos 2\theta}}\left(\frac{1}{2}\frac{2r\,dr}{\sqrt{a^2+r^2}}dr\right)\right]d\theta$

$$= \int_{-\pi/4}^{\pi/4}\left[\frac{1}{2}.\frac{2.(a^2+r^2)}{1}\right]_0^{a\sqrt{\cos 2\theta}} d\theta = \int_{-\pi/4}^{\pi/4}\left[\left(a^2+r^2\right)^{1/2}\right]_0^{a\sqrt{\cos 2\theta}} d\theta$$

$$= \int_{-\pi/4}^{\pi/4}\left[\left(a^2+a^2\cos 2\theta\right)^{1/2} - a\right]d\theta = a\int_{-\pi/4}^{\pi/4}\left[(1+\cos 2\theta)^{1/2} - 1\right]d\theta$$

$$= a\int_{-\pi/4}^{\pi/4}\left[\left(2\cos^2\theta\right)^{1/2} - 1\right]d\theta = a\int_{-\pi/4}^{\pi/4}\left[\left(\sqrt{2}\cos\theta - 1\right)\right]d\theta$$

$$= 2a\int_0^{\pi/4}\left(\sqrt{2}\cos\theta - 1\right)d\theta = 2a\left[\sqrt{2}\sin\theta - \theta\right]_0^{\pi/4}$$

$$= 2a\left[\sqrt{2}.\frac{1}{\sqrt{2}} - \frac{\pi}{4}\right] = 2a\left[1-\frac{\pi}{4}\right] = a\left(2-\frac{\pi}{2}\right). \text{ Ans.}$$

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4^{th} Topic

Integral Calculus

Triple Integrals

[Volume by triple integrals]

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Volume of solids as triple integrals:

Divide the given solid by planes parallel to the co-ordinate planes into rectangular parallelopiped of volume $\delta x\, \delta y\, \delta z$.

$\therefore$ The total volume $= \underset{\substack{\delta x \to 0 \\ \delta y \to 0 \\ \delta z \to 0}}{\text{Lt}} \sum \sum \sum \delta x\, \delta y\, \delta z = \iiint dx\, dy\, dz$,

with appropriate limits of integration.

Q.No.1.: Find the volume of the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$.

Sol.: Let A be the region bounded by the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$.

$$\therefore A = \left\{ (x, y, z) : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1 \right\}$$

$$\because \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1 \Rightarrow \frac{x^2}{a^2} \le 1,\ \frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1 \text{ and } \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1$$

$$\Rightarrow x^2 \le a^2,\quad y^2 \le b^2\left(1 - \frac{x^2}{a^2}\right) \text{ and } z^2 \le c^2\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)$$

$$\Rightarrow -a \le x \le a,\ -b\sqrt{1-\frac{x^2}{a^2}} \le y \le b\sqrt{1-\frac{x^2}{a^2}} \text{ and } -c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}} \le z \le c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}.$$

$$\therefore A = \left\{ \begin{array}{l} (x,\ y,\ z)\ :\ -a \le x \le a,\ -b\sqrt{1-\frac{x^2}{a^2}} \le y \le b\sqrt{1-\frac{x^2}{a^2}}, \\ \qquad -c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}} \le z \le c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}} \end{array} \right\}$$

Hence the volume of the whole ellipsoid $= \iiint dxdydz$

$$= 8\int_0^a \left[\int_0^{b\sqrt{(1-x^2/a^2)}} \left\{ \int_0^{c\sqrt{(1-x^2/a^2-y^2/b^2)}} dz \right\} dy \right] dx$$

$$= 8\int_0^a \left[\int_0^{b\sqrt{(1-x^2/a^2)}} |z|_0^{c\sqrt{(1-x^2/a^2-y^2/b^2)}} dy \right] dx = 8c\int_0^a \left[\int_0^{b\sqrt{(1-x^2/a^2)}} \sqrt{\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)} dy \right] dx$$

$$= \frac{8c}{b}\int_0^a \left[\int_0^{\rho} \sqrt{(\rho^2-y^2)}\,dy \right] dx \quad \text{where, we put } b\sqrt{\left(1-\frac{x^2}{a^2}\right)} = \rho.$$

$$= \frac{8c}{b}\int_0^a \left[\frac{y\sqrt{(\rho^2-y^2)}}{2} + \frac{\rho^2}{2}\sin^{-1}\frac{y}{\rho} \right]_0^{\rho} dx = \frac{8c}{b}\int_0^a \left(\frac{b^2}{2}\left\{1-\frac{x^2}{a^2}\right\}\frac{\pi}{2} \right) dx$$

$$= 2\pi bc\int_0^a \left(1-\frac{x^2}{a^2}\right) dx = 2\pi bc\left[x - \frac{x^2}{3a^2} \right]_0^a = \frac{4\pi abc}{3}. \text{ Cubic units. Ans.}$$

or

Sol.: Volume of the ellipsoid $= \iiint\limits_{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1} 1dxdydz.$

Put $\frac{x}{a} = u,\ \ \frac{y}{b} = v,\ \ \frac{z}{c} = w.$

The given region transforms into the region

$D' = \{(u, v, w) : u^2 + v^2 + w^2 \le 1\}$

$\therefore J = \frac{\partial(x, y, z)}{\partial(u, v, w)} = abc. \qquad \therefore |J| = abc$

Volume of the ellipsoid $= \iiint\limits_{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1} 1dxdydz = \iiint\limits_{u^2+v^2+w^2\le 1} 1.abc.dudvdw$

$$= abc \iiint\limits_{u^2+v^2+w^2\le 1} dudvdw$$

To change rectangular co-ordinates (u, v, w) to spherical polar co-ordinates (r, θ, ϕ), we have put $u = r\sin\theta\cos\phi$, $v = r\sin\theta\sin\phi$, $w = r\cos\theta$ and

$$J = \frac{\partial(u, v, w)}{\partial(r, \theta, \phi,)} = \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial \theta} & \frac{\partial u}{\partial \phi} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial \theta} & \frac{\partial v}{\partial \phi} \\ \frac{\partial w}{\partial r} & \frac{\partial w}{\partial \theta} & \frac{\partial w}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} = r^2\sin\theta$$

Then $\iiint\limits_{R_{uvw}} f(u, v, w)dxdydz = \iiint\limits_{R'_{r\theta\phi}} f(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta).r^2\sin\theta drd\theta d\phi$.

$$\therefore V = abc\int_0^{2\pi}\left[\int_0^{\pi}\left(\int_0^1 r^2dr\right)\sin\theta d\theta\right]d\phi = abc\int_0^{2\pi}\left[\int_0^{\pi}\left(\frac{r^3}{3}\right)_0^1\sin\theta d\theta\right]d\phi$$

$$= abc\int_0^{2\pi}\left(\int_0^{\pi}\frac{1}{3}\sin\theta d\theta\right)d\phi = \frac{abc}{3}\int_0^{2\pi}(-\cos\theta)_0^{\pi}d\phi = -\frac{abc}{3}\int_0^{2\pi}[\cos\pi - \cos 0]d\phi$$

$$= \frac{2abc}{3}\int_0^{2\pi}1d\phi = \frac{2abc}{3}[\phi]_0^{2\pi} = \frac{2abc}{3}(2\pi) = \frac{4\pi}{3}abc.$$ Cubic units. Ans.

Q.No.2.: Find the volume of the tetrahedron bounded by the co-ordinate planes and

plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

Or

Find the volume of the tetrahedron bounded by the planes x = 0, y = 0, z = 0,

$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$, a, b, c are positive.

Sol.: Let A be the region bounded by the four planes of the tetrahedron.

$$\therefore A = \left\{(x, y, z) : x \ge 0, y \ge 0, z \ge 0 \text{ and } \frac{x}{a}+\frac{y}{b}+\frac{z}{c}\le 1\right\}$$

$$\because \frac{x}{a}+\frac{y}{b}+\frac{z}{c}\le 1$$

$$\Rightarrow \frac{x}{a}\le 1,\ \frac{x}{a}+\frac{y}{b}\le 1 \ \text{ and } \ \frac{x}{a}+\frac{y}{b}+\frac{z}{c}\le 1$$

$$\Rightarrow x\le a,\ y\le b\left(1-\frac{x}{a}\right) \ \text{ and } \ x\le c\left(1-\frac{x}{a}-\frac{y}{b}\right)$$

$$\therefore A=\left\{(x,y,z):0\le x\le a, 0\le y\le b\left(1-\frac{x}{a}\right),\quad 0\le z\le c\left(1-\frac{x}{a}-\frac{y}{b}\right)\right\}$$

$$\therefore \text{The required volume } = \int_0^a\left[\int_0^{b(1-x/a)}\left\{\int_0^{c(1-x/a-y/b)}dz\right\}dy\right]dx$$

$$=\int_0^a\left[\int_0^{b(1-x/a)}|z|_0^{c(1-x/a-y/b)}dy\right]dx=\int_0^a\left[\int_0^{b(1-x/a)}c\left(1-\frac{x}{a}-\frac{y}{b}\right)dy\right]dx$$

$$=c\int_0^a\left[\int_0^{b(1-x/a)}\left(1-\frac{x}{a}-\frac{y}{b}\right)dy\right]dx=\int_0^a\frac{c\left[\left(1-\frac{x}{a}-\frac{y}{b}\right)^2\right]_0^{b\left(1-\frac{x}{a}\right)}}{2\left(-\frac{1}{b}\right)}dx$$

$$=-\frac{bc}{2}\int_0^a\left[0-\left(1-\frac{x}{a}\right)^2\right]dx=\frac{bc}{2}\int_0^a\left(1-\frac{x}{a}\right)^2dx$$

$$=\frac{bc}{2}\frac{\left[\left(1-\frac{x}{a}\right)^3\right]_0^a}{3\left(-\frac{1}{a}\right)}=-\frac{abc}{6}[0-1]=\frac{abc}{6}.\ \text{Cubic unit. Ans.}$$

Q.No.3.: Find the volume of the solid surrounded by the surface

$$\left(\frac{x}{a}\right)^{2/3}+\left(\frac{y}{b}\right)^{2/3}+\left(\frac{z}{c}\right)^{2/3}=1.$$

Sol.: The volume of the solid $\left(\frac{x}{a}\right)^{2/3}+\left(\frac{y}{b}\right)^{2/3}+\left(\frac{z}{c}\right)^{2/3}=1$ is

$$V = \iiint\limits_{\left(\frac{x}{a}\right)^{2/3}+\left(\frac{y}{b}\right)^{2/3}+\left(\frac{z}{c}\right)^{2/3}\le 1} 1dxdydz.$$

Put $\left(\frac{x}{a}\right)^{1/3} = u$, $\left(\frac{y}{b}\right)^{1/3} = v$, $\left(\frac{z}{c}\right)^{1/3} = w$.

$\therefore$ The given region transforms into the region $D' = \{(u, v, w) : u^2 + v^2 + w^2 \le 1\}$

$\therefore \frac{x}{a} = u^3,\ \frac{y}{b} = v^3,\ \frac{z}{c} = w^3$

$\Rightarrow x = au^3,\ y = bv^3,\ z = cw^3$ and

$$J = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 3au^2 & 0 & 0 \\ 0 & 3bv^2 & 0 \\ 0 & 0 & 3cw^2 \end{vmatrix} = 27abc\, u^2v^2w^2 .$$

$$\therefore V = \iiint\limits_{u^2+v^2+w^2\le 1} 27abc\, u^2v^2w^2 dudvdw = 27abc \iiint\limits_{u^2+v^2+w^2\le 1} u^2v^2w^2 dudvdw . \qquad \text{(i)}$$

To change rectangular co-ordinates (u, v, w) to spherical polar co-ordinates (r, θ, ϕ), we have put $u = r\sin\theta\cos\phi$, $v = r\sin\theta\sin\phi$, $w = r\cos\theta$ and

$$J = \frac{\partial(u, v, w)}{\partial(r, \theta, \phi,)} = \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial \theta} & \frac{\partial u}{\partial \phi} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial \theta} & \frac{\partial v}{\partial \phi} \\ \frac{\partial w}{\partial r} & \frac{\partial w}{\partial \theta} & \frac{\partial w}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} = r^2\sin\theta$$

Then $\iiint\limits_{R_{uvw}} f(u,v,w)dxdydz = \iiint\limits_{R'_{r\theta\phi}} f(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta).r^2\sin\theta drd\theta d\phi$.

$$\therefore\ V = 27abc \int_0^{2\pi}\int_0^{\pi}\int_0^{1} r^2\sin^2\theta\cos^2\phi.r^2\sin^2\theta\sin^2\phi.r^2\cos^2\theta.r^2\sin\theta.drd\theta d\phi$$

$$= 27abc \int_0^{2\pi}\int_0^{\pi}\int_0^{1} r^8\sin^5\theta\cos^2\theta.\cos^2\phi\sin^2\phi..drd\theta d\phi$$

$$= \frac{27abc}{9}\int_0^{2\pi}\left[\left(\int_0^{\pi} \sin^5\theta\cos^2\theta d\theta\right)\right]\cos^2\phi\sin^2\phi d\phi \qquad \left[\because \int_0^1 r^8 dr = \left(\frac{r^9}{9}\right)_0^1 = \frac{1}{9}\right]$$

$$= \frac{27abc}{9}\int_0^{2\pi}\left[\left(2\int_0^{\pi/2} \sin^5\theta\cos^2\theta d\theta\right)\right]\cos^2\phi\sin^2\phi d\phi$$

$$= \frac{27abc}{9}\int_0^{2\pi}\left[2.\frac{4.2.1}{7.5.3.1}\right]\cos^2\phi\sin^2\phi d\phi = \frac{27abc}{9}\int_0^{2\pi}\left[\frac{16}{105}\right]\cos^2\phi\sin^2\phi d\phi$$

$$= \frac{16}{35}abc\int_0^{2\pi} \cos^2\phi\sin^2\phi d\phi = \frac{64abc}{35}\int_0^{\pi/2} \cos^2\phi\sin^2\phi d\phi$$

$$= \frac{64abc}{35}\frac{1.1}{4.2}\times\frac{\pi}{2} = \frac{4\pi abc}{35}$$. Cubic units. Ans.

Q.No.4.: Find the volume of the portion of the sphere $x^2 + y^2 + z^2 = a^2$ lying inside the cylinder $x^2 + y^2 = ax$.

or

Find the volume cut off from the sphere $x^2 + y^2 + z^2 = a^2$ by the cylinder $x^2 + y^2 = ax$

Sol.: The required volume is easily found by changing to cylindrical co-ordinates (ρ, ϕ, z).

To change rectangular co-ordinates (x, y, z) to cylindrical co-ordinates (ρ, ϕ, z), we have put $x = \rho\cos\phi$, $y = \rho\sin\phi$, $z = z$ and

$$J = \frac{\partial(x, y, z)}{\partial(\rho, \phi, z)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\phi & -\rho\sin\phi & 0 \\ \sin\phi & \rho\cos\phi & 0 \\ 0 & 0 & 1 \end{vmatrix} = \rho\left(\cos^2\phi + \sin^2\phi\right) = \rho .$$

Then $\iiint\limits_{R_{xyz}} f(x, y, z)dxdydz = \iiint\limits_{R'_{\rho\theta z}} f(\rho\cos\phi, \rho\sin\phi, z).\rho d\rho d\phi dz$.

Then the equation of the cylinder becomes $\rho = a\cos\phi$.

The volume inside the cylinder bounded by the sphere is twice the volume shown in the above region for which z varies from 0 to $\sqrt{\left(a^2-\rho^2\right)}$, ρ varies from 0 to $a\cos\phi$ and ϕ varies from 0 to π.

$$\therefore \text{Required volume } = 2\int_0^{\pi}\left\{\int_0^{a\cos\phi}\left(\int_0^{\sqrt{\left(a^2-\rho^2\right)}} dz\right)\rho d\rho\right\}d\phi = 2\int_0^{\pi}\left(\int_0^{a\cos\phi}\rho\sqrt{\left(a^2-\rho^2\right)}d\rho\right)d\phi$$

$$= 2\int_0^{\pi}\left[-\frac{1}{3}\left(a^2-\rho^2\right)^{3/2}\right]_0^{a\cos\phi}d\phi = \frac{2a^3}{3}\int_0^{\pi}\left(1-\sin^3\phi\right)d\phi$$

$$\left[\because \sin 3\phi = 3\sin\phi - 4\sin^3\phi \Rightarrow \sin^3\phi = \frac{3\sin\phi - \sin 3\phi}{4}\right]$$

$$= \frac{2a^3}{3}\int_0^{\pi}\left(1-\frac{3\sin\phi-\sin 3\phi}{4}\right)d\phi = \frac{2a^3}{3}\left(\pi-\frac{1}{4}\left(6-\frac{2}{3}\right)\right) = \frac{2a^3}{3}\left(\pi-\frac{4}{3}\right)$$

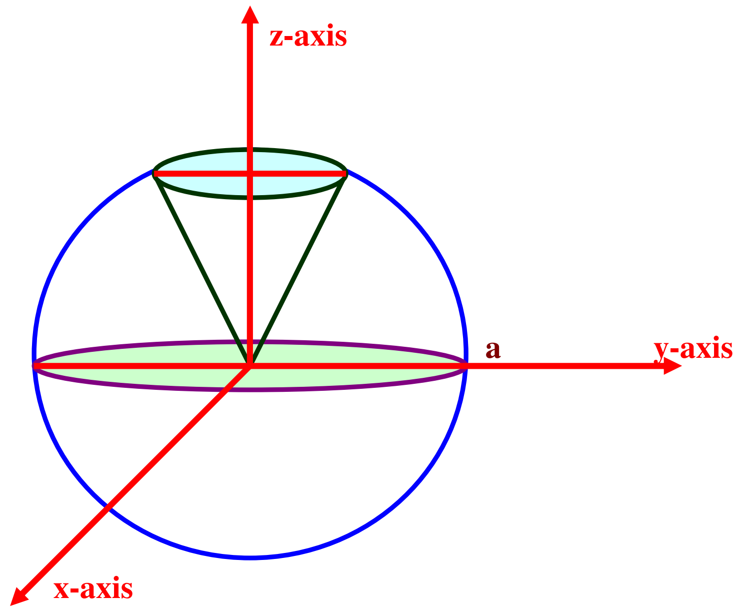
$$= \frac{2a^3}{9}(3\pi-4). \text{ Cubic units. Ans.}$$

Q.No.5.: Find the volume cut from the sphere $x^2+y^2+z^2=a^2$ by the cone $x^2+y^2=z^2$ above xy-plane.

Sol.: The required volume $V = \iiint_R dxdydz$.

To change rectangular co-ordinates (x, y, z) to spherical polar co-ordinates (r, θ, ϕ), we have put $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$ and

$$J = \frac{\partial(x, y, z)}{\partial(r, \theta, \phi,)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} = r^2\sin\theta$$



Then $\iiint\limits_{R_{xyz}} f(x, y, z)dxdydz = \iiint\limits_{R'_{r\theta\phi}} f(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta).r^2\sin\theta drd\theta d\phi$.

$\therefore x^2 + y^2 + z^2 = a^2 \Rightarrow r^2 = a^2$ and $x^2 + y^2 = z^2 \Rightarrow r^2\sin^2\theta = r^2\cos^2\theta$

$\Rightarrow r$ varies from 0 to a, θ varies from 0 to $\frac{\pi}{4}$, ϕ varies from 0 to $\frac{\pi}{2}$.

$\therefore$ Required volume $= 4\int_0^{\pi/2}\int_0^{\pi/4}\int_0^a r^2\sin\theta dr\, d\theta d\phi = 4\int_0^{\pi/2}\int_0^{\pi/4}\left[\frac{r^3}{3}\right]_0^a \sin\theta\, d\theta d\phi$

$$= \frac{4a^3}{3}\int_0^{\pi/2}\int_0^{\pi/4}\sin\theta\, d\theta d\phi = \frac{4a^3}{3}\int_0^{\pi/2}[-\cos\theta]_0^{\pi/4} d\phi$$

$$= \frac{4a^3}{3}\int_0^{\pi/2}\left(1 - \frac{1}{\sqrt{2}}\right)d\phi = 4\frac{a^3}{3}\left(1 - \frac{1}{\sqrt{2}}\right)\frac{\pi}{2}$$

$$= 2\frac{a^3}{3}\left(1 - \frac{1}{\sqrt{2}}\right)\pi = \frac{a^3}{3}\left(2 - \sqrt{2}\right)\pi.$$ Cubic units. Ans.

Q.No.6.: Find the volume common to the cylinders $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$.

Sol.: The required volume $V = \iiint dxdydz$.

Since $x^2 + z^2 = a^2 \Rightarrow z^2 = a^2 - x^2$.

$\Rightarrow z$ varies from $-\sqrt{a^2 - x^2}$ to $\sqrt{a^2 - x^2}$.

Also $x^2 + y^2 = a^2 \Rightarrow y^2 = a^2 - x^2$.

$\Rightarrow$ y varies from $-\sqrt{a^2 - x^2}$ to $\sqrt{a^2 - x^2}$.

Now $x^2 = a^2$, by putting y = 0 and z = 0

$\Rightarrow$ x varies from $-a$ to a.

$$\therefore V = \iiint dxdydz = \int_{-a}^{a}\left[\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\left\{\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dz\right\}dy\right]dx = 8\int_{0}^{a}\left[\int_{0}^{\sqrt{a^2-x^2}}\left\{\int_{0}^{\sqrt{a^2-x^2}} dz\right\}dy\right]dx$$

$$= 8\int_{0}^{a}\left[\int_{0}^{\sqrt{a^2-x^2}} [z]_0^{\sqrt{a^2-x^2}} dy\right]dx = 8\int_{0}^{a}\left(\int_{0}^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2}\,dy\right)dx$$

$$= 8\int_{0}^{a} \sqrt{a^2-x^2}\,[y]_0^{\sqrt{a^2-x^2}} dx = 8\int_{0}^{a} \left(a^2 - x^2\right)dx = 8\left|a^2x - \frac{x^3}{3}\right|_0^a$$

$$= 8\left(a^3 - \frac{a^3}{3}\right) = \frac{16a^3}{3}.$$ Cubic units. Ans.

Q.No.7.: Find the volume bounded by the cylinder $x^2 + y^2 = 4$, and the hyperboloid

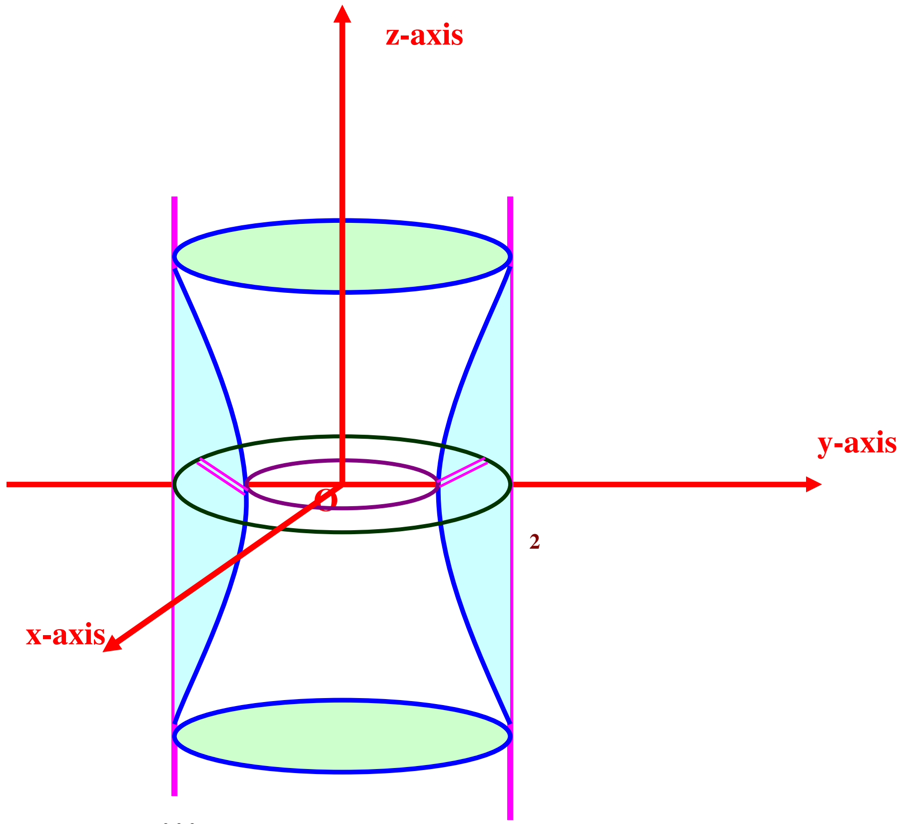
$x^2 + y^2 - z^2 = 1$.

Sol.: The required volume $V = \iiint dxdydz$.

To change rectangular co-ordinates (x, y, z) to cylindrical co-ordinates (ρ, ϕ, z),

we have put $x = \rho\cos\phi$, $y = \rho\sin\phi$, $z = z$ and

$$J = \frac{\partial(x, y, z)}{\partial(\rho, \phi, z)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\phi & -\rho\sin\phi & 0 \\ \sin\phi & \rho\cos\phi & 0 \\ 0 & 0 & 1 \end{vmatrix} = \rho\left(\cos^2\phi + \sin^2\phi\right) = \rho.$$



Then $\iiint\limits_{R_{xyz}} f(x, y, z)dxdydz = \iiint\limits_{R'_{\rho\theta z}} f(\rho\cos\phi, \rho\sin\phi, z).\rho d\rho d\phi dz$.

Then the equation of hyperboloid $x^2 + y^2 - z^2 = 1 \Rightarrow \rho^2 - z^2 = 1$ and that of cylinder $x^2 + y^2 = 4 \Rightarrow \rho^2 = 4$.

The volume inside the cylinder bounded by the hyperboloid is twice the volume above the xy-plane. For which z varies from 0 to $\sqrt{\rho^2 - 1}$, ρ varies from 1 to 2, and ϕ varies from 0 to 2π.

$$\therefore \text{Required volume} = 2\int_0^{2\pi}\left[\int_1^2\left(\int_0^{\sqrt{\rho^2-1}} dz\right)\rho\, d\rho\right]d\phi = 2\int_0^{2\pi}\left[\int_1^2 \rho\sqrt{\rho^2 - 1}d\rho\right]d\phi$$

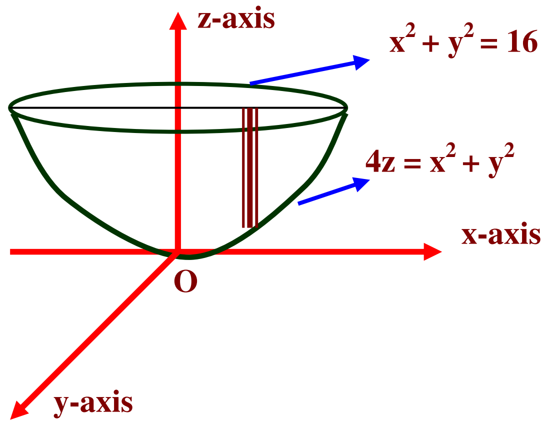
Put $t^2 = \rho^2 - 1$ so that $tdt = \rho d\rho$.

And as ρ varies from 1 to 2; and t varies from 0 to $\sqrt{3}$

$\therefore$ Required volume $= 2\int_0^{2\pi} \left|\frac{t^3}{3}\right|_0^{\sqrt{3}} d\phi = 2\times 2\times \sqrt{3}\,\pi = 4\sqrt{3}\,\pi$. Cubic units. Ans.

Q.No.8.: Find the volume cut from parabolic $4z = x^2 + y^2$ by the plane $z = 4$.

Sol.:



The volume is given by

$$v = 4\int_0^4 \left[\int_0^{\sqrt{16-x^2}} \left\{\int_{\frac{x^2+y^2}{4}}^{4} dz\right\} dy\right] dx = 4\int_0^4 \left[\int_0^{\sqrt{16-x^2}} \left\{4 - \frac{x^2}{4} - \frac{y^2}{4}\right\} dy\right] dx$$

$$= 4\int_0^4 \left[\left(4 - \frac{x^2}{4}\right) y - \frac{y^3}{12}\right]_0^{\sqrt{16-x^2}} dx = 4\int_0^4 \left[\left(4 - \frac{x^2}{4}\right)\sqrt{16-x^2} - \frac{\left(16-x^2\right)^{3/2}}{12}\right] dx$$

$$= 4\int_0^4 \left[\frac{1}{4}\left(16-x^2\right)\sqrt{16-x^2} - \frac{1}{12}\left(16-x^2\right)^{3/2}\right] dx = 4\int_0^4 \left[\frac{1}{4}\left(16-x^2\right)^{3/2} - \frac{1}{12}\left(16-x^2\right)^{3/2}\right] dx$$

$$= 4\int_0^4 \left[\frac{1}{6}\left(16-x^2\right)^{3/2}\right] dx = \frac{2}{3}\int_0^4 \left(16-x^2\right)^{3/2} dx$$

Put $x = 4\sin\theta \Rightarrow dx = 4\cos\theta d\theta$ and $\theta = \frac{\pi}{2}$, when $x = 4$ and $\theta = 0$ when $x = 0$.

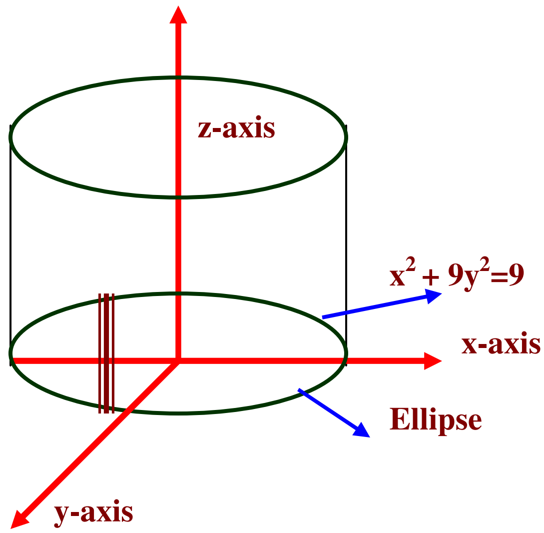
$$V == \frac{2}{3}\int_0^{\pi/2} (16)^{3/2} \cos^3\theta . 4\cos\theta d\theta = \frac{512}{3}\int_0^{\pi/2} \cos^4\theta d\theta = \frac{512}{3} . \frac{3.1}{4.2} . \frac{\pi}{2} = 32\pi$$

$\therefore$ Volume cut from paraboloid $4z = x^2 + y^2$ by plane $z = 4$ is given by 32π. Cubic units.

Q.No.9.: Find the volume bounded by the elliptic Paraboloids $z = x^2 + 9y^2$ and

$$z = 18 - x^2 - 9y^2.$$

Sol.:



The two surfaces intersect on the elliptic cylinder $x^2 + 9y^2 = z = 18 - x^2 - 9y^2$

$$\Rightarrow x^2 + 9y^2 = 9.$$

The projection of this volume onto xy-plane region D enclosed by ellipse having the same equation $\frac{x^2}{3^2} + \frac{y^2}{1^2} = 1^2$.

This volume can be covered as follows:

z: from $z_1(x, y) = x^2 + 9y^2$ to $z_2(x, y) = 18 - x^2 - 9y^2$

y: from $y_1(x, y) = -\sqrt{\frac{9 - x^2}{9}}$ to $y_2(x, y) = \sqrt{\frac{9 - x^2}{9}}$

x: from $x_1(x, y) = -3$ to $x_2(x, y) = 3$.

Thus the volume bounded by the elliptic Paraboloids $z = x^2 + 9y^2$ and $z = 18 - x^2 - 9y^2$ is

$$V = \int_{-3}^{3} \left\{ \int_{-\sqrt{\frac{9-x^2}{9}}}^{\sqrt{\frac{9-x^2}{9}}} \left(\int_{x^2+9y^2}^{18-x^2-9y^2} dz \right) dy \right\} dx$$

$$= \int_{-3}^{3}\left\{\int_{-\sqrt{\frac{9-x^2}{9}}}^{\sqrt{\frac{9-x^2}{9}}}\left\{\left(18 - x^2 - 9y^2\right) - \left(x^2 + 9y^2\right)\right\}dy\right\}dx = 2\int_{-3}^{3}\left\{\int_{-\sqrt{\frac{9-x^2}{9}}}^{\sqrt{\frac{9-x^2}{9}}}\left(9 - x^2 - 9y^2\right)dy\right\}dx$$

$$= 2\int_{-3}^{3}\left\{\left(9y - x^2y - 3y^3\right)_{-\sqrt{\frac{9-x^2}{9}}}^{\sqrt{\frac{9-x^2}{9}}}\right\}dx = \frac{8}{9}\int_{-3}^{3}\left(9 - x^2\right)^{3/2}dx = 72\int_{0}^{\pi}\sin^4\theta d\theta, \text{ where } x = 3\cos\theta$$

$$= 72\times 2\int_{0}^{\pi/2}\sin^4\theta d\theta = 144\times\left(\frac{3\times 1}{4\times 2}\times\frac{\pi}{2}\right) = 27\pi. \text{ Cubic units.}$$

Q.No.10.: Find, by triple integration, the volume in the positive octant bounded by the coordinate planes and the plane $x + 2y + 3z = 4$.

Sol.: Equation of the given plane $x + 2y + 32 = 4 \Rightarrow z = \dfrac{4 - x - 2y}{3}$

i.e. z varies from 0 to $\dfrac{4 - x - 2y}{3}$ and y varies from 0 to $\dfrac{4 - x}{2}$ and similarly x varies from 0 to 4.

Required volume $= \int\int_R\int dzdydx = \int_0^4\int_0^{\frac{4-x}{2}}\int_0^{\frac{4-x-2y}{3}} dzdydx$

$$= \int_0^4\int_0^{\frac{4-x}{2}}\frac{4 - x - 2y}{3}dydx = \int_0^4 \frac{4}{3}\left(\frac{4 - x}{2}\right) - \frac{x}{3}\left(\frac{4 - x}{2}\right) - \frac{2}{3}\times\frac{1}{2}\left(\frac{4 - x}{2}\right)^2 dx$$

$$= \int_0^4 \frac{16 - 4x}{6} - \frac{4x - x^2}{6} - \frac{1}{3}\times\frac{16 + x^2 - 8x}{2\times 2}dx$$

$$= \frac{16}{6}[x]_0^4 - \frac{4}{6}\times\frac{1}{2}\left[x^2\right]_0^4 - \frac{16}{2\times 6}[x]_0^4 - \frac{1}{2\times 6}\times\frac{1}{3}\left[x^3\right]_0^4 + \frac{8}{2\times 6}\times\frac{1}{2}\left[x^2\right]_0^4$$

$$= \frac{16}{6}\times 4 - \frac{4}{12}\times 16 - \frac{16}{12}\times 4 - \frac{1}{18}\times\frac{64}{2} + \frac{8}{12}\times\frac{16}{2}$$

$$= \frac{32}{3} - \frac{16}{3} - \frac{16}{3} + \frac{32}{9} - \frac{16}{3} - \frac{16}{9} + \frac{16}{3} = \frac{32}{9} - \frac{16}{9} = \frac{16}{9}. \text{ Cubic units}$$

Q.No.11.: Find, by triple integration, the volume of the region bounded by the paraboloid $az = x^2 + y^2$ and the cylinder $x^2 + y^2 = R^2$.

Sol.: Given equation of the paraboloid $az = x^2 + y^2 \Rightarrow z = \dfrac{x^2 + y^2}{a}$.

i.e. z varies from 0 to $\dfrac{x^2 + y^2}{a}$, similarly, y varies from 0 to $\sqrt{R^2 - x^2}$ and x varies from 0 to R.

$$\text{Volume required} = \int\int_R\int dzdydx = 4\int_0^R\int_0^{\sqrt{R^2-x^2}}\int_0^{\frac{x^2+y^2}{a}} dzdydx = 4\int_0^R\int_0^{\sqrt{R^2-x^2}} \frac{x^2 + y^2}{a} dydx$$

$$= 4\int_0^R \frac{x^2}{a}[y]_0^{\sqrt{R^2-x^2}} + \frac{1}{3a}\left[y^3\right]_0^{\sqrt{R^2-x^2}} dx = 4\int_0^R \frac{x^2}{a}\sqrt{R^2 - x^2} + \frac{1}{3a}\left(R^2 - x^2\right)^{3/2} dx$$

Putting $x = R\sin\theta \Rightarrow dx = R\cos\theta d\theta$

$$V = 4\int_0^{\pi/2} \frac{R^2}{a}\sin^2\theta\sqrt{R^2\left(1 - \sin^2\theta\right)}R\cos\theta + \frac{1}{3a}\left[R^2\left(1 - \sin^2\theta\right)\right]^{3/2} d\theta\, R\cos\theta$$

$$= \int_0^{\pi/2}\left(\frac{R^4}{a}\sin^2\theta\cos^2\theta + \frac{R^4}{3a}\cos^4\theta\right)d\theta = 4\left(\frac{R^4}{a}\frac{1.1}{4.2}\times\frac{\pi}{2} + \frac{R^4}{3a}\times\frac{3.1}{4.2}\times\frac{\pi}{2}\right)$$

$$= \frac{\pi R^4}{4a} + \frac{\pi R^4}{4a} = \frac{\pi R^4}{2a}.\ \text{Cubic units}$$

Q.No.12.: Find, by triple integration, the volume of the sphere of radius a.

Sol.: Equation of the sphere of radius a

$$x^2 + y^2 + z^2 = a^2 \Rightarrow z = \sqrt{a^2 - x^2 - y^2}$$

$$\text{Required volume} = 8\int\int_R\int dzdydx = 8\int_0^{\pi/2a}\int_0^{\sqrt{a^2-x^2}}\int_0^{\sqrt{a^2-x^2-y^2}} dzdydx$$

$$= 8\int_0^a\int_0^{\sqrt{a^2-x^2}} \sqrt{a^2 - x^2 - y^2}\, dydx$$

Putting $x = r\cos\theta$, $y = r\sin\theta$, $z = z$,

$$x^2 + y^2 = r^2$$

$|J| = r$

$$V = 8\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2 - x^2 - y^2}\,dydx = 8\int_0^{\pi/2}\int_0^a r\sqrt{a^2 - r^2}\,dr\,d\theta$$

$\Rightarrow a^2 - r^2 = t^2 \Rightarrow -2rdr = 2tdt \Rightarrow rdr = -tdt$.

$$V = 8\int_0^{\pi/2}\int_a^0 -t^2dt = 8\int_0^{\pi/2} \frac{1}{3}\left[t^3\right]_0^a d\theta = \frac{8}{3}\int_0^{\pi/2} a^3 d\theta = \frac{8}{3}a^3 \times \frac{\pi}{2} = \frac{4\pi a^3}{3}.$$ Cubic units

Q.No.13.: Find, by triple integration, the volume bounded above by the sphere $x^2 + y^2 + z^2 = 2a^2$ and below the paraboloid $az = x^2 + y^2$.

Sol.: Equation of the given sphere is $x^2 + y^2 + z^2 = 2a^2$ and equation of the given paraboloid is $az = x^2 + y^2$.

i.e. z varies from $z = \dfrac{x^2 + y^2}{a}$ to $z = \sqrt{2a^2 - x^2 - y^2}$.

Now $x^2 + y^2 + z^2 = 2a^2$

$$\Rightarrow az + z^2 = 2a^2 \Rightarrow z^2 + az - 2a^2 = 0 \Rightarrow z = \frac{-a \pm \sqrt{a^2 + 8a^2}}{2} = -2a,\ a$$

Since we have to find volume bounded above by the sphere $x^2 + y^2 + z^2 = 2a^2$ and below the paraboloid $az = x^2 + y^2$. Thus z = -2a (rejected).

Thus equation of circle becomes $x^2 + y^2 + a^2 = 2a^2 \Rightarrow x^2 + y^2 = a^2$

and y varies from $y = -\sqrt{a^2 - x^2}$ to $y = \sqrt{a^2 - x^2}$ and similarly x varies from $x = -a$ to $x = a$.

$$\text{Required volume } = \iiint_R dzdydx = \int_{-a}^{a}\left\{\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\left(\int_{\frac{x^2+y^2}{a}}^{\sqrt{2a^2-x^2-y^2}} dz\right)dy\right\}dx$$

$$= \int_{-a}^{a}\left\{\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\left(\sqrt{2a^2 - x^2 - y^2} - \frac{x^2 + y^2}{a}\right)dy\right\}dx$$

Put $x = r\cos\theta,\ \ y = r\sin\theta,\ \ J = r$, we get

Required volume $=\int_0^{2\pi}\left\{\int_0^a\left(\sqrt{2a^2-r^2}-\frac{r^2}{a}\right)rdr\right\}d\theta=\int_0^{2\pi}\left\{\int_0^a\left(r\sqrt{2a^2-r^2}-\frac{r^3}{a}\right)dr\right\}d\theta$

$$=\int_0^{2\pi}\left\{\int_0^a\left(-\frac{1}{2}\left(\sqrt{2a^2-r^2}\right)(-2r)-\frac{r^3}{a}\right)dr\right\}d\theta=\int_0^{2\pi}\left\{\left[\left(-\frac{1}{2}\frac{\left(2a^2-r^2\right)^{3/2}}{3/2}-\frac{r^4}{4a}\right)\right]_0^a\right\}d\theta$$

$$=\int_0^{2\pi}\left\{\left(-\frac{1}{3}\left(2a^2-a^2\right)^{3/2}-\frac{a^4}{4a}\right)-\left(-\frac{1}{3}\left(2a^2-0\right)^{3/2}-\frac{0}{4a}\right)\right\}d\theta$$

$$=\int_0^{2\pi}\left\{\left(-\frac{1}{3}\left(a^2\right)^{3/2}-\frac{a^3}{4}\right)-\left(-\frac{1}{3}\left(2a^2\right)^{3/2}\right)\right\}d\theta$$

$$=\int_0^{2\pi}\left\{\left(-\frac{a^3}{3}-\frac{a^3}{4}\right)-\left(-\frac{2\sqrt{2}a^3}{3}\right)\right\}d\theta=\int_0^{2\pi}\left\{-\frac{7a^3}{12}+\frac{2\sqrt{2}a^3}{3}\right\}d\theta$$

$$=\left\{-\frac{7a^3}{12}+\frac{2\sqrt{2}a^3}{3}\right\}2\pi=\left\{-\frac{7}{12}+\frac{2\sqrt{2}}{3}\right\}2\pi a^3=\left\{\frac{4\sqrt{2}}{3}-\frac{7}{6}\right\}\pi a^3.$$ Cubic units.

Q.No.14.: Find the volume bounded by $xy=z$, $z=0$ and $\left(x-1\right)^2+\left(y-1\right)^2=1$.

Sol.: Required volume $=\int\int_R\int dzdydx=\iint_{(x-1)^2+(y-1)^2\le 1}\left(\int_0^{xy}dz\right)dydx$

$$=\iint_{(x-1)^2+(y-1)^2\le 1}xydydx$$

Let $x-1=u$ and $y-1=v \Rightarrow dx=du,\ dy=dv$.

Then the required volume $=\iint_{u^2+v^2\le 1}\left(u+1\right)\left(v+1\right)dudv$

Put $u=r\cos\theta,\ \ v=r\sin\theta,\ \ J=r$, we get

the required volume $=\int_0^{2\pi}\int_0^1\left(r\cos\theta+1\right)\left(r\sin\theta+1\right)rdrd\theta$

$$= \int_0^{2\pi}\int_0^1 \left[r^3\cos\theta\sin\theta + r^2(\cos\theta+\sin\theta) + r\right]drd\theta$$

$$= \int_0^{2\pi}\left[\frac{r^4}{4}\cos\theta\sin\theta + \frac{r^3}{3}(\cos\theta+\sin\theta) + \frac{r^2}{2}\right]_0^1 d\theta$$

$$= \int_0^{2\pi}\left[\frac{1}{4}\cos\theta\sin\theta + \frac{1}{3}(\cos\theta+\sin\theta) + \frac{1}{2}\right]d\theta = \int_0^{2\pi}\left[\frac{2\cos\theta\sin\theta}{8} + \frac{1}{3}(\cos\theta+\sin\theta) + \frac{1}{2}\right]d\theta$$

$$= \int_0^{2\pi}\left[\frac{\sin 2\theta}{8} + \frac{1}{3}(\cos\theta+\sin\theta) + \frac{1}{2}\right]d\theta = \left[\frac{\cos 2\theta}{16} + \frac{1}{3}(\sin\theta-\cos\theta) + \frac{1}{2}\theta\right]_0^{2\pi}$$

$$= \left[\frac{(0-0)}{16} + \frac{1}{3}\{(0-0)-(1-1)\} + \frac{1}{2}(2\pi-0)\right] = \pi \text{ Cubic units. Ans.}$$

Q.No.15.: Compute the volume of solid bounded by planes, $2x+3y+4z=12$, xy-plane and the cylinder $x^2+y^2=1$.

Sol.: Required volume $= \int\int_R\int dzdydx = \iint_{x^2+y^2\le 1}\left(\int_0^{\frac{1}{4}(12-2x-3y)} dz\right)dydx$

$$= \iint_{x^2+y^2\le 1}\frac{1}{4}(12-2x-3y)\,dydx = \int_{-1}^{+1}\left[\int_{-\sqrt{1-x^2}}^{+\sqrt{1-x^2}}\frac{1}{4}(12-2x-3y)\,dy\right]dx$$

$$= \frac{1}{4}\int_{-1}^{+1}\left[12y - 2xy - 3\frac{y^2}{2}\right]_{-\sqrt{1-x^2}}^{+\sqrt{1-x^2}} dx$$

$$= \frac{1}{4}\int_{-1}^{+1}\left[\left(12\sqrt{1-x^2} - 2x\sqrt{1-x^2} - 3\frac{\left(\sqrt{1-x^2}\right)^2}{2}\right) - \left(-12\sqrt{1-x^2} + 2x\sqrt{1-x^2} - 3\frac{\left(-\sqrt{1-x^2}\right)^2}{2}\right)\right]dx$$

$$= \frac{1}{4}\int_{-1}^{+1}\left[24\sqrt{1-x^2} - 4x\sqrt{1-x^2}\right]dx = \int_{-1}^{+1}\left[6\sqrt{1-x^2} - x\sqrt{1-x^2}\right]dx$$

$$= \int_{-1}^{+1} 6\sqrt{1-x^2}dx - \int_{-1}^{+1} x\sqrt{1-x^2}dx = 6\left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x\right]_{-1}^{+1} + \left[\frac{1}{2}\frac{(1-x^2)^{3/2}}{3/2}\right]_{-1}^{+1}$$

$$= 6\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] + [0-0] = 6.\frac{2\pi}{4} = 3\pi \text{ Cubic units. Ans.}$$

Q.No.15.: Compute the volume in the first octant bounded by the cylinder $x = 4 - y^2$ and the planes $z = y, x = 0, z = 0$.

Sol.: Required volume $= \int\int_R\int dzdydx = \int_0^4 \int_0^{\sqrt{4-x}} \left(\int_0^y dz\right) dydx$

$$= \int_0^4 \int_0^{\sqrt{4-x}} [z]_0^y \, dydx = \int_0^4 \left(\int_0^{\sqrt{4-x}} ydy\right) dx$$

$$= \int_0^4 \left[\frac{y^2}{2}\right]_0^{\sqrt{4-x}} dx = \int_0^4 \frac{\left(\sqrt{4-x}\right)^2}{2} dx = \int_0^4 \frac{4-x}{2} dx$$

$$= \frac{1}{2}\left(4x - \frac{x^2}{2}\right)_0^4 = \frac{1}{2}\left(4.4 - \frac{4^2}{2}\right) = \frac{1}{2}(16-8)$$

$= 4$ Cubic units. Ans.

Q.No.16.: Find the volume cut from the sphere of radius b and the cone $\varphi = \alpha$. Hence deduce the volumes of the hemisphere and sphere (by triple integrals).

Sol.: Volume $= \iiint \delta x \delta y \delta z$

We can solve this problem by changing rectangular co-ordinates (x, y, z) to spherical polar co-ordinates (r, θ, ϕ).

As we know, when we change rectangular co-ordinates (x, y, z) to spherical polar co-ordinates (r, θ, ϕ), we have put $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$ and

$$J = \frac{\partial(x, y, z)}{\partial(r, \theta, \phi,)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} = r^2\sin\theta.$$

Now $V = 2\times\int_0^\alpha \int_0^\pi \left(\int_0^b r^2\right) \sin\phi\delta x\delta\theta\delta\phi = 2\left(\frac{b^3}{3}\right)\int_0^\alpha \int_0^\pi \sin\phi(\delta\theta)(\delta\phi)$

$$= \frac{2b^3}{3}\int_0^\alpha [\theta]_0^\pi \sin\phi\delta\phi = \frac{2\pi b^3}{3}\left[-(\cos\phi)_0^\alpha\right] = \frac{2b^3\pi}{3}(1-\cos\alpha) = \frac{2b^3}{3}(1-\cos\alpha)\pi$$

For volume of the hemisphere, put $\alpha = \frac{\pi}{2}$, we get $V = \frac{2b^3}{3}\pi$. Ans.

For volume of the sphere, put $\alpha = \frac{\pi}{2}$, we get $V = \frac{2b^3}{3}\pi(1-\cos\pi) = \frac{4\pi b^3}{3}$. Ans.

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Home Assignments

5^{th} Topic

Integral Calculus

Double Integrals

(Change of variables)

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NIT Hamirpur (HP)

Change of variables:

The evaluation of the double integrals is greatly simplified by a suitable change of variables. By changing the variables, a given integral can be transformed into a simpler integral involving the new variables.

In a double integrals:

Let the variables x, y in the double integral $\iint_R f(x, y)dxdy$ be changed to the new variables u. v by the transformation

$$x = \phi(u, v), \quad y = \psi(u, v),$$

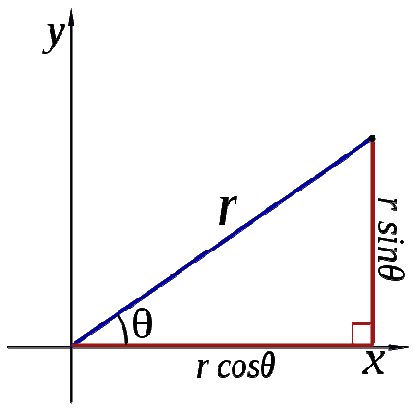
where $\phi(u, v)$ and $\psi(u, v)$ are continuous and have continuous first order derivatives in some region R'_{uv} in the uv-plane which corresponds to the region R_{xy} in the xy-plane. Then

$$\iint_{R_{xy}} f(x, y)dxdy = \iint_{R'_{uv}} f[\phi(u, v), \psi(u, v)]|J|dudv, \qquad (1)$$

where $J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} (\neq 0)$ is the **Jacobian** of transformation from (x, y) to (u, v) co-ordinates.

Particular case:

CONVERSION OF CARTESION TO POLAR SYSTEM



A diagram illustrating the relationship between polar and Cartesian coordinates.

To change Cartesian co-ordinates (x, y) to polar co-ordinates (r, θ),

we have put $x = r\cos\theta$, $y = r\sin\theta$ and

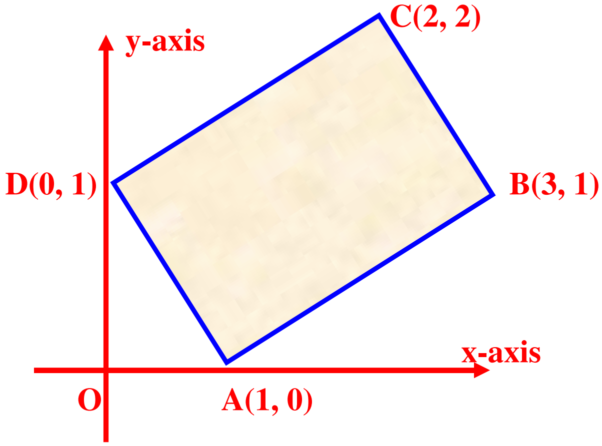
$$J = \frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\left(\cos^2\theta + \sin^2\theta\right) = r.$$

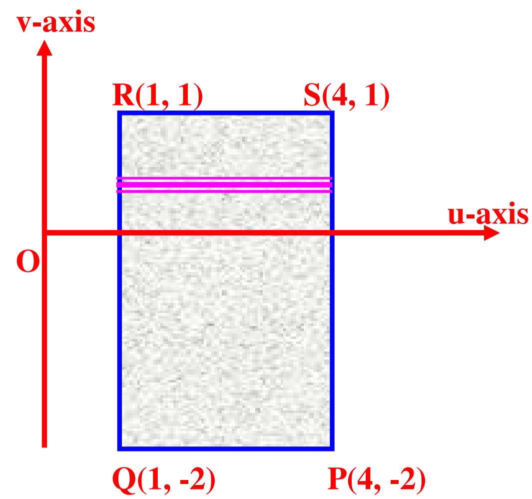
Then $\iint_{R_{xy}} f(x, y)dxdy = \iint_{R'_{r\theta}} f(r\cos\theta, r\sin\theta).r\,dr\,d\theta$.

Q.No.1.: Evaluate $\iint_R (x+y)^2 dxdy$, where R is the parallelogram in the xy-plane with vertices (1, 0), (3, 1), (2, 2), (0, 1) using the transformation $u = x + y$ and $v = x - 2y$.

Sol.: The region R, i.e. parallelogram ABCD in the xy-plane becomes the region R', i.e. rectangle RSPQ in the uv-plane, as shown in the figure, by taking

$u = x + y$ and $v = x - 2y$ (i)





From (i), we have $x = \frac{1}{3}(2u + v)$, $\quad y = \frac{1}{3}(u - v)$.

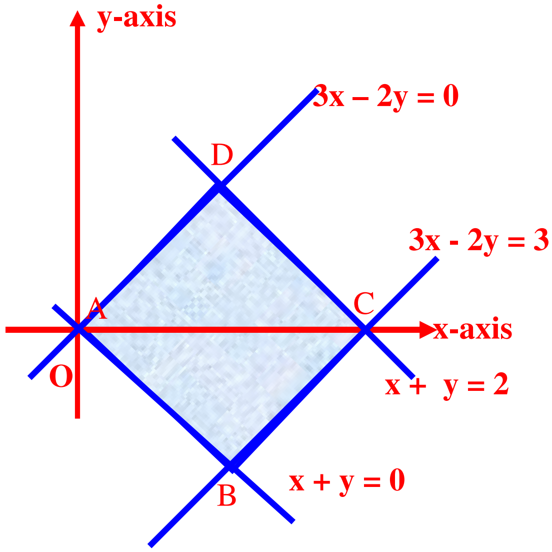
$$\therefore J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{vmatrix} = -\frac{2}{9} - \frac{1}{9} = -\frac{3}{9} = -\frac{1}{3}.$$

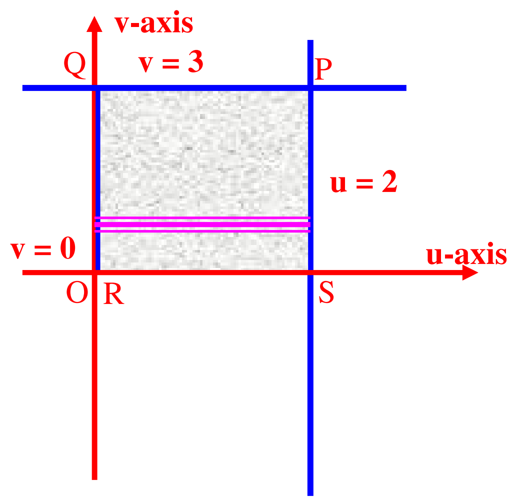
Thus $|J| = \frac{1}{3}$.

Hence, the given integral $= \iint\limits_{R'} u^2 |J| du dv = \frac{1}{3} \int_{-2}^{1} \left(\int_{1}^{4} u^2 du \right) dv = \frac{1}{3} \left| \frac{u^3}{3} \right|_1^4 . |v|_{-2}^{1} = 21$. Ans.

Q.No.2.: Evaluate $\iint_R (x+y)^2 dxdy$, where R is the region bounded by parallelogram

$$x+y=0,\ x+y=2,\ 3x-2y=0,\ 3x-2y=3.$$





Sol.: By changing the variables x, y to the new variables u, v, by the substitution (transformation) $x+y=u$, $3x-2y=v$, then the region R, i. e. parallelogram ABCD in the xy-plane becomes the region R', i. e. rectangle RSPQ in the uv-plane, as shown in the figure, by taking $x+y=u$, $3x-2y=v$. (i)

From (i), we have $x=\frac{1}{5}(2u+v)$, $y=\frac{1}{5}(3u-v)$.

$$\therefore J=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}\frac{2}{5} & \frac{1}{5}\\ \frac{3}{5} & -\frac{1}{5}\end{vmatrix}=-\frac{2}{25}-\frac{3}{25}=-\frac{5}{25}=-\frac{1}{5}.$$

Thus $|J|=\frac{1}{5}$.

Since, $u=x+y=0$ and $u=x+y=2$. Thus u varies from 0 to 2.

Also since $3x-2y=v=0$, $3x-2y=v=3$. Thus v varies from 0 to 3.

Thus the given integral in terms of new variables u, v is $\iint_R (x+y)^2 dxdy=\iint_{R'} u^2|J|dudv$

$$= \frac{1}{5}\int_0^3 \left(\int_0^2 u^2 du\right) dv = \frac{1}{5}\left|\frac{u^3}{3}\right|_0^2 . \left|v\right|_0^3 = \frac{24}{15} = \frac{8}{5}. \text{ Ans.}$$

Q.No.3.: Evaluate $\int_0^{\infty}\int_0^{\infty} e^{-\left(x^2+y^2\right)} dxdy$ by changing to polar co-ordinates.

Hence show that $\int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$.

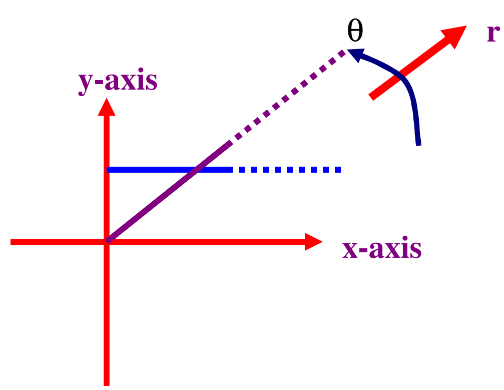
Sol.: To change cartesian co-ordinates (x, y) to polar co-ordinates (r, θ),

we have put $x = r\cos\theta$, $y = r\sin\theta$ and

$$J = \frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\left(\cos^2\theta + \sin^2\theta\right) = r .$$

Then $\iint_{R_{xy}} f(x, y) dxdy = \iint_{R'_{r\theta}} f(r\cos\theta, r\sin\theta).r\, dr\, d\theta$.

The region of integration being the first quadrant of the xy-plane, r varies from 0 to ∞ and θ varies from 0 to $\frac{\pi}{2}$. Hence,



$$I = \int_0^{\infty}\int_0^{\infty} e^{-\left(x^2+y^2\right)} dxdy = \int_{\theta=0}^{\pi/2}\int_{r=0}^{\infty} e^{-r^2} r dr d\theta = -\frac{1}{2}\int_0^{\pi/2}\left\{\int_0^{\infty} e^{-r^2}(-2r) dr\right\} d\theta$$

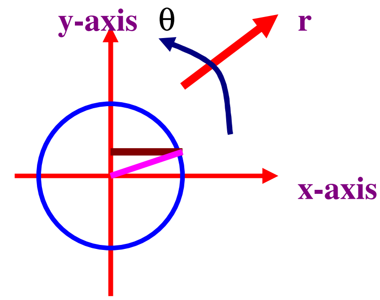
$$= -\frac{1}{2}\int_0^{\pi/2}\left|e^{-r^2}\right|_0^{\infty} d\theta = \frac{1}{2}\int_0^{\pi/2} d\theta = \frac{\pi}{4}. \text{ Ans.} \qquad \text{(i)}$$

Also $I = \int_0^\infty e^{-x^2}dx \times \int_0^\infty e^{-y^2}dy = \left\{\int_0^\infty e^{-x^2}dx\right\}^2$, when $y = x$. (ii)

Thus, from (i) and (ii), we have $\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$. Ans.

Q.No.4.: Evaluate the integral by changing to polar co-ordinates

$$\int_0^a \int_0^{\sqrt{(a^2-y^2)}} (x^2+y^2)dxdy.$$



Sol.: We have to evaluate the integral $I = \int_0^a \left(\int_0^{\sqrt{(a^2-y^2)}} (x^2+y^2)dx\right)dy$,

by changing cartesian co-ordinates to polar co-ordinates.

To change cartesian co-ordinates (x, y) to polar co-ordinates (r, θ), we have $x = r\cos\theta$,

$$y = r\sin\theta \text{ and } J = \frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\left(\cos^2\theta + \sin^2\theta\right) = r.$$

$$\therefore \iint_{R_{xy}} f(x, y)dxdy = \iint_{R'_{r\theta}} f(r\cos\theta, r\sin\theta).r\,dr\,d\theta.$$

Also when $x = 0, r = 0$; $x = \sqrt{a^2 - y^2}$, $r = a$

$y = 0, \ \theta = 0$; $y = a, \ \theta = \frac{\pi}{2}$

$$\therefore \int_0^a \left(\int_0^{\sqrt{(a^2-y^2)}} (x^2+y^2)dx\right)dy = \int_0^{\pi/2}\left(\int_0^a r^2.rdr\right)d\theta = \int_0^{\pi/2}\left[\int_0^a r^3.dr\right]d\theta = \int_0^{\pi/2}\left|\frac{r^4}{4}\right|_0^a d\theta$$

$$= \int_0^{\pi/2}\left(\frac{a^4}{4}\right)d\theta = \left(\frac{a^4}{4}\right)\int_0^{\pi/2}d\theta = \frac{a^4}{4}\times\left(\frac{\pi}{2} - 0\right) = \frac{\pi a^4}{8}. \text{ Ans.}$$

Q.No.5.: Evaluate the integral by changing to polar co-ordinates $\int_0^a \int_y^a \frac{x^2 dxdy}{\sqrt{x^2+y^2}}$.

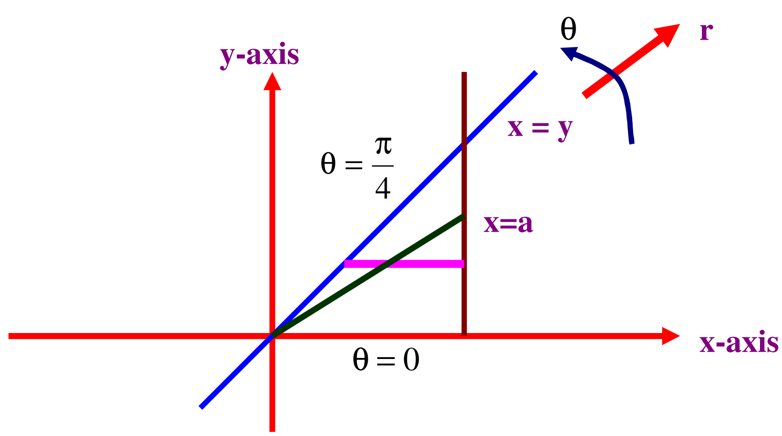
Sol.: We have to evaluate the integral $I = \int_0^a \int_y^a \frac{x^2 dxdy}{\sqrt{x^2+y^2}}$,

by changing cartesian co-ordinates to polar co-ordinates.

To change cartesian co-ordinates (x, y) to polar co-ordinates (r, θ), we have $x = r\cos\theta$,

$$y = r\sin\theta \text{ and } J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\left(\cos^2\theta + \sin^2\theta\right) = r.$$

$$\therefore \iint_{R_{xy}} f(x,y)dxdy = \iint_{R'_{r\theta}} f(r\cos\theta, r\sin\theta).r\,dr\,d\theta.$$



Also when x varies from y to a, r varies from 0 to $\frac{a}{\cos\theta}$, $[\because x = r\cos\theta]$

And as y varies from 0 to a, θ varies from 0 to $\frac{\pi}{4}$.

$$\therefore \int_0^a \left(\int_y^a \frac{x^2}{\sqrt{x^2+y^2}} dx \right) dy = \int_0^{\pi/4} \int_0^{a/\cos\theta} \frac{r^2\cos^2\theta}{r} r dr d\theta = \int_0^{\pi/4} \left[\int_0^{a/\cos\theta} r^2 dr \right] \cos^2\theta d\theta$$

$$= \int_0^{\pi/4} \left| \frac{r^3}{3} \right|_0^{a/\cos\theta} \cos^2\theta d\theta = \int_0^{\pi/4} \left(\frac{1}{3} \frac{a^3}{\cos^3\theta} - 0 \right) \cos^2\theta d\theta$$

$$= \int_0^{\pi/4} \frac{1}{3} a^3 \sec\theta d\theta = \frac{a^3}{3} \left[\log|\sec\theta + \tan\theta| \right]_0^{\pi/4}$$

$$= \frac{a^3}{3}\left[\log\left(\sqrt{2}+1\right) - \log(1+0)\right]_0^{\pi/4} = \frac{a^3}{3}\log\left(1+\sqrt{2}\right). \text{ Ans.}$$

Q.No.6.: Evaluate the integral by changing to polar co-ordinates

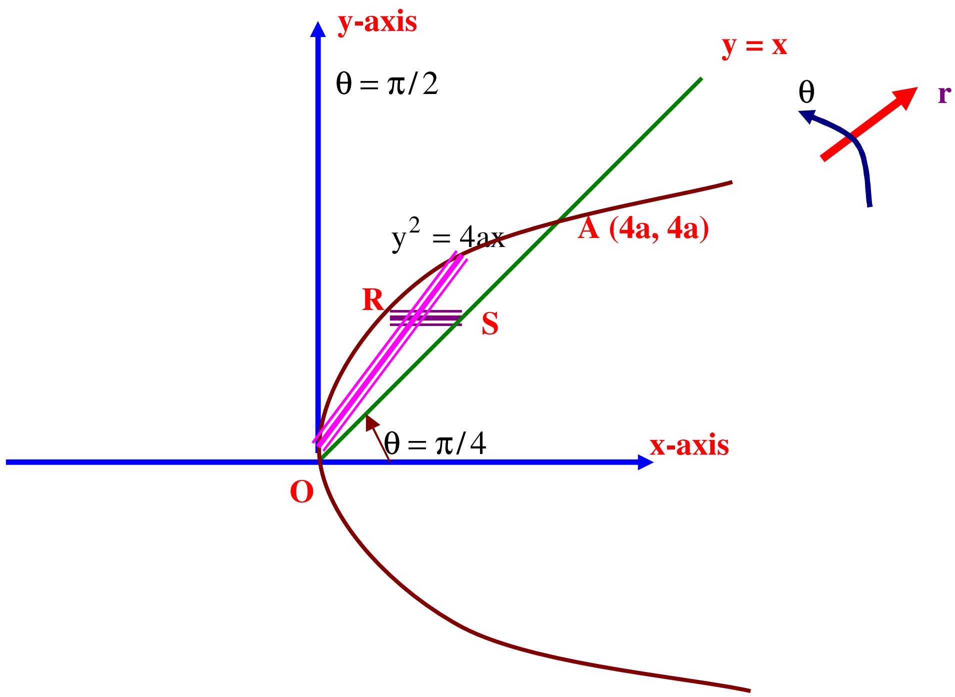
$$\int_0^{4a}\int_{y^2/4a}^{y} \frac{x^2-y^2}{x^2+y^2}\,dxdy.$$

Sol.: We have to evaluate the integral $I = \int_0^{4a}\int_{y^2/4a}^{y} \frac{x^2-y^2}{x^2+y^2}\,dxdy$, by changing Cartesian co-ordinates to Polar co-ordinates.

To change cartesian co-ordinates (x, y) to polar co-ordinates (r, θ), we have $x = r\cos\theta$,

$$y = r\sin\theta \text{ and } J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\left(\cos^2\theta + \sin^2\theta\right) = r.$$

$$\therefore \iint_{R_{xy}} f(x,y)dxdy = \iint_{R'_{r\theta}} f(r\cos\theta, r\sin\theta).r\,dr\,d\theta.$$



Now $\frac{x^2-y^2}{x^2+y^2} = \frac{r^2\cos^2\theta - r^2\sin^2\theta}{r^2\cos^2\theta + r^2\sin^2\theta} = \frac{r^2\cos 2\theta}{r^2} = \cos 2\theta.$

Since $y^2 = 4ax \Rightarrow r^2\sin^2\theta = 4ar\cos\theta \Rightarrow r\left(r\sin^2\theta - 4a\cos\theta\right) = 0$

$\Rightarrow r = 0$ and $r = \dfrac{4a\cos\theta}{\sin^2\theta}$. Thus

$$I = \int_{\pi/4}^{\pi/2}\left(\int_0^{4a\cos\theta/\sin^2\theta}\cos 2\theta r dr\right)d\theta = \int_{\pi/4}^{\pi/2}\left[\frac{r^2}{2}\right]_0^{4a\cos\theta/\sin^2\theta}\cos 2\theta d\theta$$

$$= 8a^2\int_{\pi/4}^{\pi/2}\frac{\cos^2\theta}{\sin^4\theta}\left(\cos^2\theta - \sin^2\theta\right)d\theta = 8a^2\int_{\pi/4}^{\pi/2}\left(\cot^4\theta - \cot^2\theta\right)d\theta$$

$$= 8a^2\int_{\pi/4}^{\pi/2}\left[\cot^2\theta\left(\operatorname{cosec}^2\theta - 1\right) - \left(\operatorname{cosec}^2\theta - 1\right)\right]d\theta$$

$$= 8a^2\int_{\pi/4}^{\pi/2}\left[\cot^2\theta\operatorname{cosec}^2\theta - \cot^2\theta - \left(\operatorname{cosec}^2\theta - 1\right)\right]d\theta$$

$$= 8a^2\left\{\left[-\frac{\cot^3\theta}{3}\right]_{\pi/4}^{\pi/2} - \int_{\pi/4}^{\pi/2}\left(\operatorname{cosec}^2\theta - 1\right)d\theta - \left[-\cot\theta - \theta\right]_{\pi/4}^{\pi/2}\right\}\left[\because \int [f(\theta)]^n f'(\theta)d\theta = \frac{[f(\theta)]^{n+1}}{n+1}\right]$$

$$= 8a^2\left[\frac{-\cot^3\theta}{3} + \cot\theta + \theta + \cot\theta + \theta\right]_{\pi/4}^{\pi/2}$$

$$= 8a^2\left\{\left[0 + 0 + \frac{\pi}{2} + 0 + \frac{\pi}{2}\right] - \left[\frac{-1}{3} + 1 + \frac{\pi}{4} + 1 + \frac{\pi}{4}\right]\right\}$$

$$= 8a^2\left[\pi - \left(-\frac{1}{3} + 2 + \frac{\pi}{2}\right)\right] = 8\left[\frac{\pi}{2} - \frac{5}{3}\right]a^2. \text{ Ans.}$$

Q.No.7.: Evaluate $\iint xy\left(x^2 + y^2\right)^{n/2} dxdy$ over the positive quadrant of $x^2 + y^2 = 4$ supposing $n + 3 > 0$ by changing to polar co-ordinates.

Sol.: We have to evaluate the integral $I = \iint xy\left(x^2 + y^2\right)^{n/2} dxdy$, by changing cartesian co-ordinates to polar co-ordinates.

To change cartesian co-ordinates (x, y) to polar co-ordinates (r, θ), we have $x = r\cos\theta$,

$y = r\sin\theta$ and $J = \dfrac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\left(\cos^2\theta + \sin^2\theta\right) = r$.

$$\therefore \iint_{R_{xy}} f(x, y)dxdy = \iint_{R'_{r\theta}} f(r\cos\theta, r\sin\theta).r\,dr\,d\theta.$$

Also in +ve quadrant of $x^2 + y^2 = 4$, r varies from 0 to 2 and θ varies from 0 to $\dfrac{\pi}{2}$.

$$I = \iint xy\left(x^2 + y^2\right)^{n/2} dxdy = \int_0^{\pi/2}\int_0^2 (r\cos\theta.\,r\sin\theta)r^n.r\,drd\theta$$

$$= \int_0^{\pi/2}\left[\int_0^2 r^{n+3}dr\right](\cos\theta\sin\theta)d\theta$$

$$= \int_0^{\pi/2}\left|\frac{r^{n+4}}{n+4}\right|_0^2 \cos\theta\sin\theta d\theta = \int_0^{\pi/2}\frac{2^{n+4}}{n+4}\cos\theta.\sin\theta d\theta = \frac{2^{n+4}}{n+4}\int_0^{\pi/2}\cos\theta\sin\theta d\theta$$

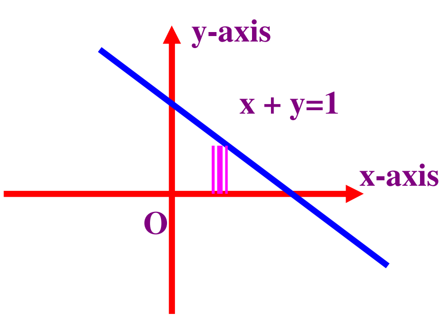
$$= \frac{2^{n+4}}{n+4}.\frac{1}{2} = \frac{2^{n+3}}{n+4}.$$ Ans.

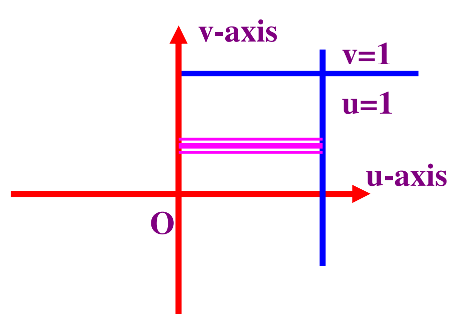
Q.No.8.: By using the transformation $x + y = u$, $y = uv$, show that

$$\int_0^1\left(\int_0^{1-x} e^{y/(x+y)}dy\right)dx = \frac{1}{2}(e-1).$$

Sol.: We have to evaluate $I = \int_0^1\left(\int_0^{1-x} e^{y/(x+y)}dy\right)dx$, by using the transformation

$x + y = u$, $y = uv$.





Since we have given $x+y=u$ and $y=uv$ $\therefore x=u(1-v)$, $y=uv$ and

$$\therefore |J| = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix} = u - uv + uv = u.$$

Since $x+y=u$ and $\frac{y}{u}=v \Rightarrow \frac{y}{x+y}=v$, then

When $y=0$, $v=0$; $y=1-x$, $u=1$,

When $x=0$, $v=1$; $x=1$, $u=\frac{y}{v}=0$. (because at $x=1$: $y=0$)

Thus u and v varies from 0 to 1.

$$\therefore I = \int_0^1 \left(\int_0^{1-x} e^{y/(x+y)} dy \right) dx = \int_0^1 \int_0^1 e^v u du dv = \int_0^1 \left[\int_0^1 u du \right] e^v dv$$

$$= \int_0^1 \left| \frac{u^2}{2} \right|_0^1 e^v dv = \frac{1}{2} \int_0^1 e^v dv = \frac{1}{2} \left| e^v \right|_0^1 = \frac{1}{2}(e-1). \text{ Ans.}$$

Q.No.9.: Show that $\iint \frac{dxdy}{4-x^2-y^2} = \pi \log 3$, over the region between the concentric circle $x^2+y^2=1$ and $x^2+y^2=3$ by changing to polar co-ordinates.

Sol.: We have to evaluate the integral $I = \iint \frac{dxdy}{4-x^2-y^2}$,

by changing cartesian co-ordinates to polar co-ordinates.

To change cartesian co-ordinates (x, y) to polar co-ordinates (r, θ), we have $x = r\cos\theta$,

$$y = r\sin\theta \text{ and } J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r.$$

$$\therefore \iint_{R_{xy}} f(x,y) dxdy = \iint_{R'_{r\theta}} f(r\cos\theta, r\sin\theta).r\, dr\, d\theta.$$

$$\text{Thus } I = \int_0^{2\pi} \left[\int_1^{\sqrt{3}} \frac{rdr}{4-r} \right] d\theta$$

Putting $4-r^2 = t \Rightarrow -2rdr = dt \Rightarrow dr = -\dfrac{dt}{2r}$.

$\therefore$ The limits changed from $\sqrt{3}$ to 1 and 1 to 3.

$$\therefore I = \int_0^{2\pi}\left[-\int_3^1 \frac{r}{t}\frac{dt}{2r}\right]d\theta = \int_0^{2\pi}\left[-\frac{1}{2}\log t\right]_3^1 dt = \frac{1}{2}\int_0^{2\pi}\log 3 d\theta = \frac{1}{2}\log 3\int_0^{2\pi} d\theta$$

$$= \frac{1}{2}\log 3[\theta]_0^{2\pi} = \frac{1}{2}\log 3.2\pi = \pi\log 3$$, which is the required proof.

Q.No.10.: Evaluate $\iint_R \left[\dfrac{1-x^2-y^2}{1+x^2+y^2}\right]^{1/2} dxdy$ by changing to polar co-ordinates, over the positive quadrant of the circle $x^2+y^2=1$ by changing to polar co-ordinates.

Sol.: The region for integration is bounded by the curves.

$x=0$, $x=1$, $y=0$, $y=1$, and $x^2+y^2=1$.

We have to evaluate the integral $\iint_R \left[\dfrac{1-x^2-y^2}{1+x^2+y^2}\right]^{1/2} dxdy$,

by changing cartesian co-ordinates to polar co-ordinates.

To change cartesian co-ordinates (x, y) to polar co-ordinates (r, θ), we have $x = r\cos\theta$,

$$y = r\sin\theta \text{ and } J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}\end{vmatrix} = \begin{vmatrix}\cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta\end{vmatrix} = r\left(\cos^2\theta+\sin^2\theta\right) = r.$$

$$\therefore \iint_{R_{xy}} f(x,y)dxdy = \iint_{R'_{r\theta}} f(r\cos\theta, r\sin\theta).r\,dr\,d\theta.$$

In the integral, we have

$$\iint_R \left[\frac{1-x^2-y^2}{1+x^2+y^2}\right]^{1/2} dxdy = \iint_R \left[\frac{1-r^2\cos^2\theta - r^2\sin^2\theta}{1+r^2\cos^2\theta+r^2\sin^2\theta}\right] rdrd\theta$$

$\therefore$ Limits required are $r=0$, $r=1$ and $\theta=0$, $\theta=\dfrac{\pi}{2}$ (positive quadrant).

Thus, we need to evaluate

$$I=\int_0^{\pi/2}\int_0^1\left[\frac{1-r^2\left(\cos^2\theta+\sin^2\theta\right)}{1+r^2\left(\cos^2\theta+\sin^2\theta\right)}\right]rdrd\theta=\int_0^{\pi/2}\left[\int_0^1\left[\frac{1-r^2}{1+r^2}\right]rdr\right]d\theta.$$

Substitute $r^2=\cos\phi\Rightarrow 2rdr=-\sin\phi d\phi$.

$\therefore$ New limits are

at $r=0\Rightarrow\cos\phi=0\Rightarrow\phi=\frac{\pi}{2}$, $r=1\Rightarrow\cos\phi=1\Rightarrow\phi=0$, .

So the value to be integrated is

$$I=\int_0^{\pi/2}\left[\frac{1}{2}\int_{\pi/2}^0\left[\frac{1-\cos\phi}{1+\cos\phi}\right]^{1/2}(-\sin\phi)d\phi\right]d\theta$$

$$=\int_0^{\pi/2}\left[\frac{1}{2}\int_{\pi/2}^0\left[\frac{2\sin^2\frac{\phi}{2}}{2\cos^2\frac{\phi}{2}}\right]^{1/2}\sin\phi d\phi\right]d\theta \qquad \left[\because -\int_a^b f(x)dx=\int_a^b f(x)dx\right]$$

$$=\int_0^{\pi/2}\left[\frac{1}{2}\int_0^{\pi/2}\left(\tan\frac{\phi}{2}.2\sin\frac{\phi}{2}.\cos\frac{\phi}{2}\right)d\phi\right]d\theta=\int_0^{\pi/2}\left[\frac{1}{2}\int_0^{\pi/2}2\sin^2\frac{\phi}{2}d\phi\right]d\theta$$

$$=\int_0^{\pi/2}\left[\frac{1}{2}\int_0^{\pi/2}(1-\cos\phi)d\phi\right]d\theta=\int_0^{\pi/2}\left[\frac{1}{2}[\phi-\sin\phi]_0^{\pi/2}\right]d\theta=\int_0^{\pi/2}\frac{1}{2}\left(\frac{\pi}{2}-1\right)d\theta$$

$$=\frac{1}{2}\left(\frac{\pi-2}{2}\right)[\theta]_0^{\pi/2}=\frac{1}{2}\left(\frac{\pi-2}{2}\right)\frac{\pi}{2}=\frac{\pi}{8}(\pi-2).\text{ Ans.}$$

Q.No.11.: Evaluate $\int_0^a\left(\int_0^{\sqrt{a^2-x^2}}e^{-\left(x^2-y^2\right)}dy\right)dx$ by changing to polar co-ordinates.

Sol.: Let $y=\sqrt{a^2-x^2}\Rightarrow y^2=a^2-x^2\Rightarrow x^2+y^2=a^2$ $\therefore$ The graph is a circle.

We have to evaluate the integral $\int_0^a\left(\int_0^{\sqrt{a^2-x^2}}e^{-\left(x^2-y^2\right)}dy\right)dx$,

by changing cartesian co-ordinates to polar co-ordinates.

To change cartesian co-ordinates (x, y) to polar co-ordinates (r, θ), we have $x = r\cos\theta$,

$y = r\sin\theta$ and $J = \dfrac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\left(\cos^2\theta + \sin^2\theta\right) = r$.

$$\therefore \iint_{R_{xy}} f(x, y)dxdy = \iint_{R'_{r\theta}} f(r\cos\theta, r\sin\theta).r\,dr\,d\theta.$$

$$\therefore I = \int_0^{\pi/2}\int_0^a e^{-r^2} r dr d\theta = \frac{1}{2}\int_0^{\pi/2}\int_0^a 2e^{-r^2} r dr d\theta = -\frac{1}{2}\int_0^{\pi/2}\left[e^{-r^2}\right]_0^a d\theta = -\frac{1}{2}\int_0^{\pi/2}\left[e^{-a^2} - 1\right]d\theta$$

$$= -\frac{1}{2}\left(e^{-a^2} - 1\right)\int_0^{\pi/2} d\theta = -\frac{1}{2}\left(e^{-a^2} - 1\right)[\theta]_0^{\pi/2} = -\frac{1}{2}\left(e^{-a^2} - 1\right)\frac{\pi}{2} = -\frac{\pi}{4}\left(e^{-a^2} - 1\right).$$

Q.No.12.: Evaluate $\displaystyle\int_0^a\int_y^a \frac{xdxdy}{x^2 + y^2}$ by changing the polar co-ordinates.

Sol.: The region of integration is the area of integration is the area bounded by the curve, $x = a$, $y = 0$, $y = a$.

We have to evaluate the integral $\displaystyle\int_0^a\left(\int_y^a \frac{xd}{x^2 + y^2}dx\right)dy$,

by changing cartesian co-ordinates to polar co-ordinates.

To change cartesian co-ordinates (x, y) to polar co-ordinates (r, θ), we have $x = r\cos\theta$,

$y = r\sin\theta$ and $J = \dfrac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\left(\cos^2\theta + \sin^2\theta\right) = r$.

$$\therefore \iint_{R_{xy}} f(x, y)dxdy = \iint_{R'_{r\theta}} f(r\cos\theta, r\sin\theta).r\,dr\,d\theta.$$

$\therefore$ In polar co-ordinates area of integration is bounded by curves

$r\cos\theta = a \Rightarrow r = a\sec\theta$

$r\sin\theta = 0 \Rightarrow \theta = 0$ and $r = 0$

$r\sin\theta = r\cos\theta \Rightarrow \theta = \dfrac{\pi}{4}$.

$$I = \int_0^a \left(\int_y^a \frac{x}{x^2 + y^2} dx \right) dy = \int_0^{\pi/4} \int_0^{a\sec\theta} \frac{r^2 \cos\theta}{r^2} .dr.d\theta = \int_0^{\pi/4} \int_0^{a\sec\theta} \cos\theta.dr.d\theta$$

$$= \int_0^{\pi/4} \left[\cos\theta \int_0^{a\sec\theta} dr \right] d\theta = \int_0^{\pi/4} \left[\cos\theta [r]_0^{a\sec\theta} \right] d\theta = \int_0^{\pi/4} (\cos\theta \times \sec\theta) d\theta$$

$$= \int_0^{\pi/4} a d\theta = a[\theta]_0^{\pi/4} = \frac{\pi}{4} a \text{ . Ans.}$$

Q.No.13 .: Evaluate $\int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{xdydx}{\sqrt{x^2+y^2}}$ by changing to polar co-ordinates.

Sol.: Let $I = \int_0^2 \left(\int_0^{\sqrt{2x-x^2}} \frac{xd}{\sqrt{x^2+y^2}} dy \right) dx$

Taking $y = \sqrt{2x - x}$ (i)

$\Rightarrow (x-1)^2 + y^2 = 1^2$ which is the equation of circle having its centre at (1,0) and a radius of 1 unit, we can draw the curve represented by the given integral as shown in Fig. 1.

Comparing the given integral I with the general form represented by $\int_{x_1}^{x_2} \left[\int_{y_1}^{y_2} f(x, y) \right] dx$,

we can conclude that the units of variable 'y' are given in terms of 'x' and hence the integration should be first carried and on dy, taking an elemental ship parallel to y-axis.

We have to evaluate the integral $\int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{xdydx}{\sqrt{x^2+y^2}}$,

by changing cartesian co-ordinates to polar co-ordinates.

To change cartesian co-ordinates (x, y) to polar co-ordinates (r, θ), we have $x = r\cos\theta$,

$$y = r\sin\theta \text{ and } J = \frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\left(\cos^2\theta + \sin^2\theta\right) = r \text{ .}$$

$$\therefore \iint_{R_{xy}} f(x, y) dxdy = \iint_{R'_{r\theta}} f(r\cos\theta, r\sin\theta).r\, dr\, d\theta \text{ .}$$

Using (i) and (ii), we write r in terms of θ,

$$r\sin\theta = \sqrt{2r\cos\theta - r^2\cos^2\theta}$$

$$\therefore r^2\left(\sin^2 + \cos^2\theta\right) = 2r\cos\theta \Rightarrow r = 2\cos\theta. \qquad \text{(iii)}$$

Thus Fig (i) can be redrawn as in Fig. (ii)

Using (ii) and (iii), the integral

$$\int_{x_1}^{x_2}\int_{y_1}^{y_2} f(x, y)dxdy \text{ can be changed to}$$

$$\int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} (r\cos\theta,\ r\sin\theta)J\left(\frac{x, y}{r, \theta}\right)drd\theta.$$

$$\therefore I = \int_0^{\pi/2}\int_0^{2\cos\theta} \frac{r\cos\theta.\, xr\, drd\theta}{\sqrt{r^2\cos^2\theta + r^2\sin^2\theta}} = \int_0^{\pi/2}\left[\int_0^{2\cos\theta} r\cos\theta dr\right]d\theta$$

$$= \int_0^{\pi/2} 2\cos^3\theta d\theta \qquad \left[\begin{array}{l}\because \cos 3A = 4\cos^3 A - 3\cos A \\ \Rightarrow \cos^3 A = \dfrac{\cos 3A + 3\cos A}{A}\end{array}\right]$$

$$= \int_0^{\pi/2} (\cos 3\theta + 3\cos\theta)d\theta = \left[\frac{\sin 3\theta}{6} + \frac{3\sin\theta}{2}\right]_0^{\pi/2} = \frac{4}{3}. \text{ Ans.}$$

Q.No.14.: Transform the following to cartesian form and hence evaluate

$$\int_0^{\pi}\int_0^{a} r^3\sin\theta\cos\theta\, drd\theta.$$

Sol.: We have evaluate the integral $I = \int_0^{\pi}\int_0^{a} r^3\sin\theta\cos\theta\, drd\theta$,

by changing polar co-ordinates to Cartesian co-ordinates.

To change polar co-ordinates (r, θ) to Cartesian co-ordinates (x, y), we have $x = r\cos\theta$,

$$y = r\sin\theta \text{ and } J = \frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\left(\cos^2\theta + \sin^2\theta\right) = r.$$

$\therefore \iint\limits_{R'_{r\theta}} f(r\cos\theta, r\sin\theta).r\,dr\,d\theta = \iint\limits_{R_{xy}} f(x, y)dxdy$.

$$\therefore I = \int_0^{\pi}\int_0^{a} r^3\sin\theta\cos\theta drd\theta = \int_0^{\pi}\int_0^{a} r\sin\theta.r\cos\theta.rdrd\theta = \int_0^{a}\int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} xydxdy$$

[Here we suppose that the strip is parallel to x-axis]

$$I = \int_0^{a}\left[\int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} xdx\right]ydy = \int_0^{a}\left[\left|\frac{x^2}{2}\right|_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}}\right]ydy$$

$$= \frac{1}{2}\int_0^{1}\left[\left(a^2-y^2\right)-\left(a^2-y^2\right)\right]ydy = \int_0^{a} 0dy = 0. \text{ Ans.}$$

Q.No.15.: Evaluate $\iint\limits_D (y-x)dxdy$, where D is the region in xy-plane bounded by the straight lines $y = x+1$, $y = x-x$, $y = -\frac{1}{3}x+\frac{7}{3}$, $y = -\frac{1}{3}x+5$ using the transformation $u = y-x$ and $v = y+\frac{x}{3}$.

Sol.: Given transformations are $u = y-x$ and $v = y+\frac{x}{3}$

$$\Rightarrow -\frac{4}{3}x = u-v,\ \frac{4}{3}y = \frac{1}{3}u+v \Rightarrow y = \frac{3}{4}\left(\frac{u}{3}+v\right)$$

Here $J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} -3/4 & 1/4 \\ 3/4 & 3/4 \end{vmatrix} = -\frac{9}{12}-\frac{3}{16} = -\frac{12}{16} = -\frac{3}{4} \Rightarrow |J| = \frac{3}{4}$

As given, $y-x=1$, $y-x=-3$

$u=-3$, $v=1$

$$-3 \le u \le 1,$$

Again, $y+\frac{1}{3}x = \frac{7}{3}$, $y+\frac{1}{3}x = 5$

$v = \frac{7}{3}$, $v = 5$

$$\frac{7}{3} \le v \le 5$$

We will now integrate,

$$\iint\limits_D (y-x)dydx = \int_{7/3}^{5}\int_{-3}^{1} u|J|dudv = \int_{7/3}^{5}\left[\frac{u^2}{2}\right]_{-3}^{1}\left(\frac{3}{4}\right)dv = \left(-\frac{8}{2}\times\frac{3}{4}\right)[v]_{7/3}^{5}$$

$-\frac{8}{2}\times\frac{3}{4}\times\left(5-\frac{7}{3}\right)=-8$. Ans.

Q.No.16.: Evaluate $\int_0^1\int_0^{1-x} e^{\frac{y}{(x+y)}}dydx$, using the transformation $x+y=u$ and $y=uv$.

Sol.: Given transformations are $x+y=u$ and $y=uv$.

$$\text{Here } J=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}1-v & -u\\ v & u\end{vmatrix}=u\Rightarrow |J|=u.$$

$$\iint e^v ududv=\int_0^1\left(\int_0^1 e^v udu\right)dv=\int_0^1\left[e^v\frac{u^2}{2}\right]_0^1 dv=\frac{1}{2}\int_d^1 e^v dv=\frac{e-1}{2}. \text{ Ans.}$$

Q.No.17.: Evaluate $\iint_D [xy(1-x-y)]^{1/2}dxdy$, where D is the region in bounded by the Δ with sides x = 0, y = 0, x + y =1 using the transformation $u=x+y$ and $uv=y$.

Sol.: Given transformations are $x+y=u$ and $y=uv$.

$x+uv=u\Rightarrow x=u(1-v)$

$$\iint_D u(1-v)uv(1-u)^{1/2}ududv=\int_0^1\int_0^1 u^2(1-v)(1-u)^{1/2}vdudv$$

$$=\int_0^1\left[\int_0^1(v-v^2)dv\right]u^2(1-u)^{1/2}du=\int_0^1\left[\frac{v^2}{2}-\frac{v^3}{3}\right]_0^1 u^3(1-u)^{1/2}du$$

$$=\int_0^1\left[\frac{1}{2}-\frac{1}{3}\right]u^3(1-u)^{1/2}du=\int_0^1 u^3(1-u)^{1/2}\frac{1}{6}du=\frac{1}{6}\int_0^1 u^3(1-u)^{1/2}du$$

Put $u=\sin^2\theta,\quad u^3=\sin^6\theta$

$du=2\sin\theta\cos\theta d\theta$

$\theta_1=0,\quad \theta_2=\frac{\pi}{2}$

$$=\frac{1}{6}\int_0^{\pi/2}\sin^6\theta(\cos\theta)2\sin\theta\cos\theta d\theta=\frac{1}{3}\int_0^{\pi/2}\sin^7\theta\cos^2\theta d\theta$$

$$=\frac{1}{3}\left[\frac{6.1}{9.7.5.3}\right]=\frac{1}{3}\left[\frac{6}{9\times105}\right]=\frac{2}{945}. \text{ Ans.}$$

Q.No.18.: Evaluate $\iint\limits_{R} (x-y)^4 e^{x+y} dxdy$, where R is the square with vertices at (1, 0), (2, 1), (1, 2), (0, 1) using the transformation $x + y = u, \quad x - y = v$.

Sol.: Given transformations are $x + y = u, \quad x - y = v$.
Solving above two, we get

$$x = \frac{u+v}{2}, \quad y = \frac{v-v}{2}.$$

Now Jacobion $\frac{J(x,y)}{(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial x}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = \left| -\frac{1}{4} \quad -\frac{1}{4} \right| = \left| -\frac{1}{2} \right| = \frac{1}{2}.$

Plotting graph taking x and y as coordinate axes

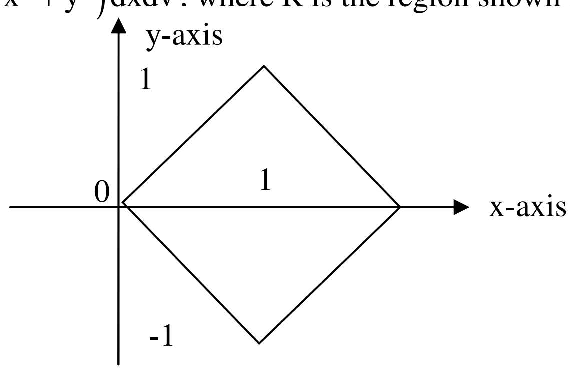
Now plotting graph taking u and v as axes.

(x,y)	(u,v)
1, 0	1, 1
2, 1	3, 1
1, 2	3, −1
0,1	1, −1

Now $\iint\limits_{R(x,y)} (x-y)^4 e^{x+y} dxdy = \iint\limits_{R(u,v)} v^4 e^u |J| dudv = \int_1^3 \left(\int_{-1}^{1} v^4 e^u \times \frac{1}{2} \times dv \right) du$

$= \int_1^3 \left| \frac{v^5}{5} \right|_{-1}^{1} \times \frac{1}{2} e^u du = \left(\frac{1}{5} + \frac{1}{5} \right) \times \frac{1}{2} \int_1^3 e^u du = \frac{2}{5} \times \frac{1}{2} \left| e^u \right|_1^3 = \left(\frac{e^3 - e}{5} \right)$. Ans.

Q.No.19.: Evaluate $\iint\limits_{R} (x^2 + y^2) dxdv$. where R is the region shown in figure



Sol.: Point in (x, y) coordinates (0, 0), (1, 1), (2, 0), $(-1,1)$

Let x + y = u and $x - y = v$

So point in $u - v$ coordinates (0, 0), (2, 0), (2, 2), (0, 2)

$$x = \frac{u+v}{2} \quad \text{and} \quad y = \frac{u-v}{2}$$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = -1/2.$$

so $|J| = 1/2$

Now $\iint_R (x^2+y^2)dxdy = \iint_R \frac{(x+y)^2+(x-y)^2}{2}dxdy = \frac{1}{2}\int_0^2 \int_0^2 (u^2+v^2)|J|dudv$

$$= \frac{1}{4}\int_0^2 \left(\int_0^2 (u^2+v^2)du \right) dv = \frac{1}{4}\int_0^2 \left[\frac{u^3}{3} + uv^2 \right]_0^2 dv = \frac{1}{4}\int_0^2 \left(\frac{8}{3} + 2v^2 \right) dv = \frac{1}{4}\left[\frac{8}{3}v + \frac{2v^3}{3} \right]_0^2$$

$$= \frac{1}{4} \times \frac{4\times 8}{3} = \frac{8}{3}. \text{ Ans.}$$

Q.No.20.: Evaluate $\int_0^e \left(\int_{\alpha x}^{\beta x} f(x,y)dy \right) dx$, using the transformation $x = u - uv$ and $y = uv$

Sol.:. Consider $x = u - uv$, $y = uv$

Since from the given integral, we have

$x = 0$, $x = e$ and $y = \alpha x$, $y = \beta x$.

Substituting the values of x, y we get the values of u, v as

$0 = u(1-v) \Rightarrow u = 0$

$e = u(1-v) \Rightarrow u = \frac{e}{1-v}$

Now $\alpha x = uv \Rightarrow \alpha(u - uv) = uv \Rightarrow v = \frac{\beta}{1+\beta}$

Finally, $\beta x = uv \Rightarrow u\alpha(1-v) = uv \Rightarrow \alpha - \alpha v = v \Rightarrow v = \frac{\alpha}{1+\alpha}$

Also Jacobian $J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1-v & -v \\ v & u \end{vmatrix} = u - uv + uv = u$

$$\therefore \int_0^c \int_{\alpha x}^{\beta x} f(x,y)\,dy\,dx = \int_{\alpha/1+\alpha}^{\beta/1+\beta} \int_0^{e/1-v} f(u-uv, uv)\,u\,du\,dv.$$

Home Assignments

Q.No.1.: Evaluate $\iint_R xy\,dx\,dy$, where R is the region in the first quadrant bounded by the hyperbola $x^2 - y^2 = a^2$, $x^2 - y^2 = b^2$ and the circle $x^2 + y^2 = c^2$, $x^2 + y^2 = d^2$, $0 < a < b < c < d$, using the transformation $x^2 - y^2 = u$ and $x^2 + y^2 = v$.

Hint: Put $x^2 - y^2 = u$, $x^2 + y^2 = v$, $J = 8xy$

R* : rectangle $a^2 \le u \le b^2$, $c^2 \le v \le d^2$

Answer: $\frac{1}{8}\left(b^2 - a^2\right)\left(d^2 - c^2\right)$

Q.No.2.: Evaluate $\iint_D e^{(x-y)/(x+y)}\,dx\,dy$, D is the triangle bounded by x = 1, x = 1, y = x, using the transformation $x = v - uv$ and $y = uv$.

Hint: Use $x = v - uv$ and $y = uv$ to transform the double integral

Answer: $\dfrac{e^2 - 1}{4e}$

Q.No.3.: Evaluate $\int_0^c \int_0^b f(x,y)\,dy\,dx$, using the transformation $x = u - uv$ and $y = uv$

Answer: $$\int_0^c \int_0^b f(x,y)\,dy\,dx = \int_0^{\frac{b}{b+c}} \int_0^{\frac{c}{1-v}} f(u-uv, uv)\,u\,du\,dv + \int_{\frac{b}{b+c}}^{b} \int_0^{\frac{b}{v}} f(u-uv, uv)\,u\,du\,dv.$$

Q.No.4.: Evaluate $\int_0^\infty \int_0^\infty \dfrac{x^2 + y^2}{1 + \left(x^2 - y^2\right)^2} e^{-2xy}\,dx\,dy$, using the transformation $u = x^2 - y^2$, and $v = 2xy$.

Answer: $\int_0^{\infty}\int_0^{\infty} \frac{x^2+y^2}{1+\left(x^2-y^2\right)^2} e^{-2xy} dxdy = \frac{\pi}{4}$.

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6^{th} Topic

Integral Calculus

Double Integrals

[Area enclosed by plane curves]

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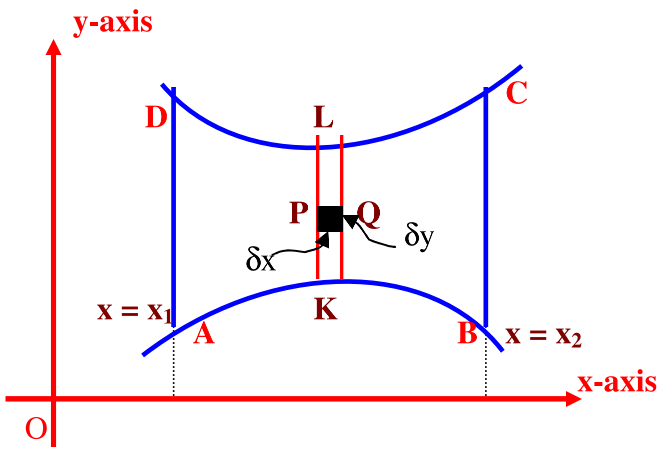
Area enclosed by plane curves:

Cartesian co-ordinates:

Case1a.:

Consider the area enclosed by the curves $y = f_1(x)$, $y = f_2(x)$ and the ordinates $x = x_1, x = x_2$. Divide this area into vertical strips of width δx. If P(x, y), $Q(x+\delta x, y+\delta y)$ be two neighbouring points, then the area of the small rectangle $PQ = \delta x \delta y$.



$\therefore$ Area of the strip $KL = \lim_{\delta y \to 0} \sum \delta x \delta y$.

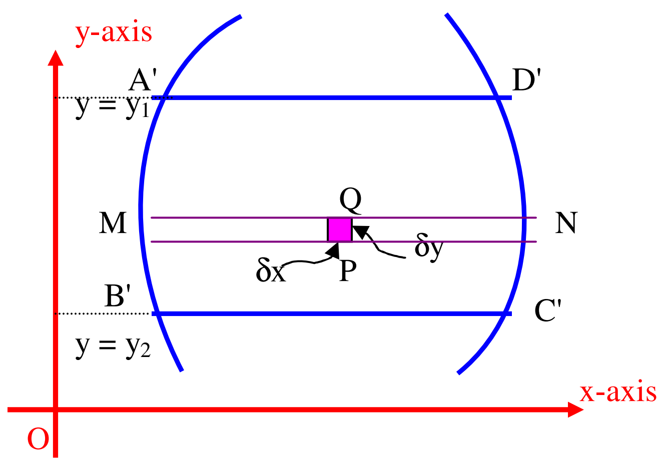
Since for all rectangles in this strip δx in the same and y varies from $y = f_1(x)$ to $y = f_2(x)$.

$\therefore$ Area of the strip $KL = \delta x \lim_{\delta y \to 0} \sum_{f_1(x)}^{f_2(x)} \delta y = \delta x \int_{f_1(x)}^{f_2(x)} dy$.

Now adding up all such strips from $x = x_1$ to $x = x_2$, we get the area ABCD

$$= \lim_{\delta x \to 0} \sum_{x_1}^{x_2} \delta x . \int_{f_1(x)}^{f_2(x)} dy = \int_{x_1}^{x_2} dx \int_{f_1(x)}^{f_2(x)} dy = \int_{x_1}^{x_2} \int_{f_1(x)}^{f_2(x)} dx dy$$

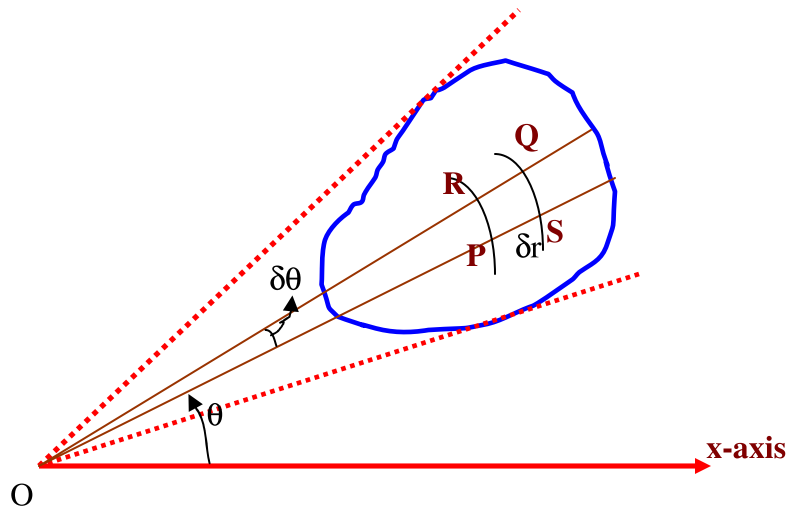
Case1b.: Similarly, dividing the area $A'B'C'D'$ as in the figure, into horizontal strips of width δy, we get the area $A'B'C'D' = \int_{y_1}^{y_2} \int_{f_1(y)}^{f_2(y)} dx dy$.



Case2.: Polar co-ordinates:

Consider an area A enclosed by a curve whose equation is in polar co-ordinates. Let $P(r, \theta)$, $Q(r + \delta r, \theta + \delta\theta)$ be two neighbouring points. Mark circular areas of radii r and $r + \delta r$ meeting OQ in R and OP (produced) in S.

Since arc $PR = r\delta\theta$ and $PS = \delta r$.



$\therefore$ Area of curvilinear rectangle PRQS is approximately $= PR\,.\,PS = r\delta\theta\,.\,\delta r$.

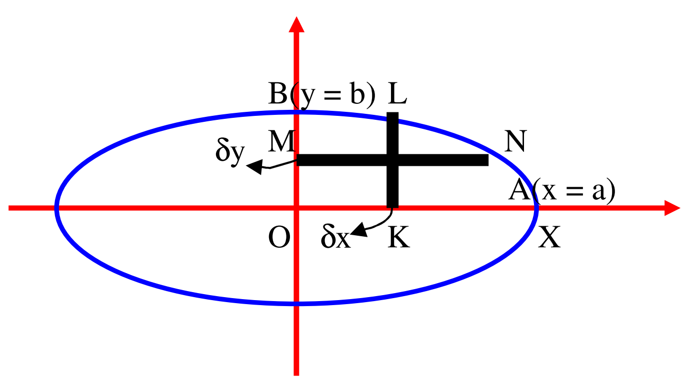
If the whole area is divided into such curvilinear rectangles, the sum $\sum\sum r\delta\theta\delta r$ taken for all these rectangles, gives in the limit the area A.

Hence, $A = \lim\limits_{\substack{\delta r\to 0\\ \delta\theta\to 0}} \sum\sum r\delta\theta\delta r = \iint r\,d\theta\,dr$,

where the limits are to be so chosen as it cover the entire area.

Q.No.1.: Find, by double integration, the area of a plate in the form of a quadrant of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Sol.: Here we suppose that the strip is parallel to the y-axis, therefore y varies from K(y = 0) to L$\left[y = b\sqrt{\left(1-\frac{x^2}{a^2}\right)}\right]$ and this strip slides from $x = 0$ to $x = a$.



$$\therefore \text{ The required area } = \int_0^a \left(\int_0^{b\sqrt{1-x^2/a^2}} dy \right) dx = \int_0^a \left([y]_0^{b\sqrt{1-x^2/a^2}} \right) dx = \frac{b}{a}\int_0^a \sqrt{\left(a^2 - x^2\right)}\, dx$$

Now put $x = a\sin t$, $dx = a\cos dt$ and when $x = 0$, $t = 0$; when $x = a$, $t = \frac{\pi}{2}$.

$$\text{Hence the required area } = \frac{b}{a}\int_0^{\pi/2} \left(a^2 - a^2\sin^2 t\right) a\cos t dt$$

$$= \frac{b}{a}\int_0^{\pi/2} a^2\cos^2 t dt = ab\left(\frac{1}{2}\times\frac{\pi}{2}\right) = \frac{\pi ab}{4}. \text{ Square units. Ans.}$$

Second Method: Here we suppose that the strip is parallel to the x-axis, therefore x varies from $M(x = 0)$ to $N\left[x = a\sqrt{\left(1 - \frac{y^2}{b^2}\right)}\right]$ and this strip slides from $y = 0$ to $y = b$.

$$\therefore \text{ The required area } = \int_0^b dy \int_0^{a\sqrt{1-y^2/b^2}} dx = \int_0^b dy [x]_0^{a\sqrt{1-y^2/b^2}} = \frac{a}{b}\int_0^b \sqrt{\left(b^2 - y^2\right)}\, dy$$

Now put $x = a\sin t$, $dx = a\cos dt$ and when $x = 0$, $t = 0$; when $x = a$, $t = \frac{\pi}{2}$.

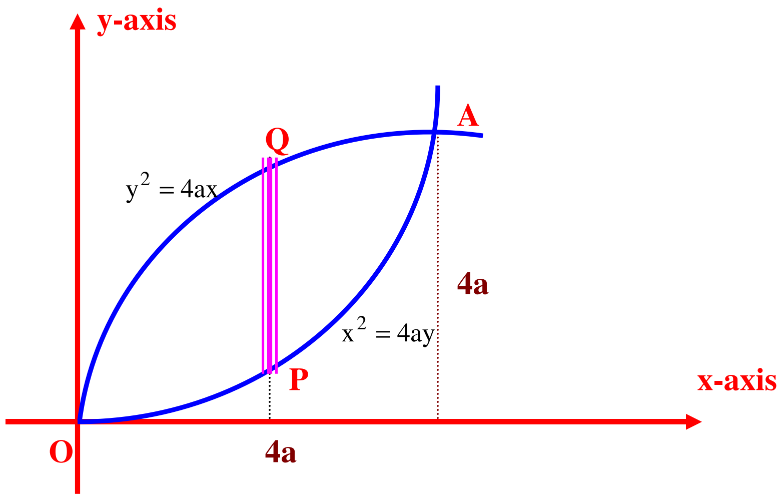
$$\text{Hence the required area } = \frac{b}{a}\int_0^{\pi/2} \left(a^2 - a^2\sin^2 t\right) a\cos t dt$$

$$= \frac{b}{a}\int_0^{\pi/2} a^2\cos^2 t dt = ab\left(\frac{1}{2}\times\frac{\pi}{2}\right) = \frac{\pi ab}{4}. \text{ Square units. Ans.}$$

Remarks: *The change of the order of integration does not in any way affect the value of the area.*

Q.No.2.: Show, by double integration, that area between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$ is $\frac{16}{3}a^2$.

Sol.: Solving the equations $y^2 = 4ax$ and $x^2 = 4ay$, it is seen that the parabolas intersect at O(0, 0) and A(4a, 4a). Here we suppose that the strip is parallel to the y-axis, therefore y varies from P to Q i. e. from $y = \frac{x^2}{4a}$ to $y = 2\sqrt{(ax)}$ and this strip slides from x = 0 to x = 4a.



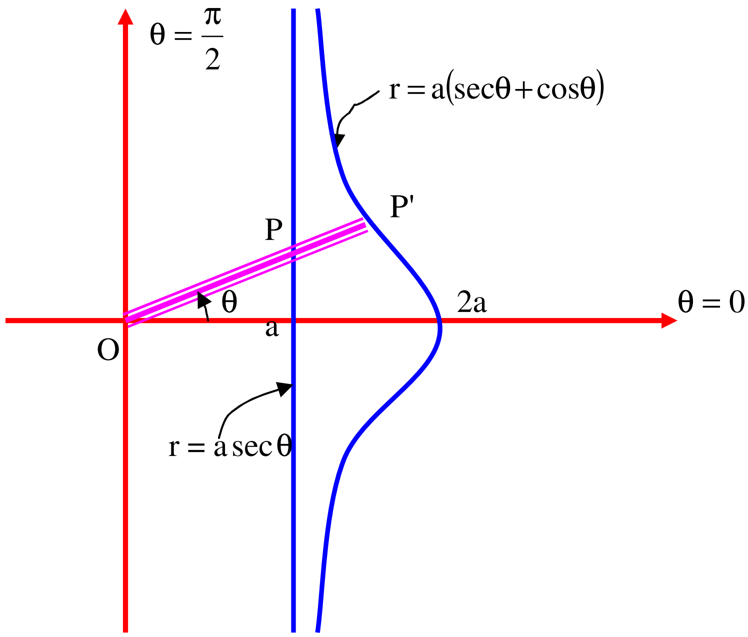
$\therefore$ The required area $= \int_0^{4a} \left(\int_{x^2/4a}^{2\sqrt{(ax)}} dy \right) dx = \int_0^{4a} \left[2\sqrt{(ax)} - \frac{x^2}{4a} \right] dx$

$$= \left| 2\sqrt{a}.\frac{2}{3}x^{3/2} - \frac{1}{4a}.\frac{x^3}{3} \right|_0^{4a} = \frac{32}{3}a^2 - \frac{16}{3}a^2 = \frac{16}{3}a^2 \text{. Square units.}$$

Q.No.3.: Calculate the area, by double integration, included between the curve

$r = a(\sec\theta + \cos\theta)$ and its asymptote.

Sol.: The curve is symmetrical about the initial line and has an asymptote $r = a\sec\theta$. Draw any line OP cutting the curve at P and its asymptote at P'. Along this line, θ is constant and r varies from $a\sec\theta$ at P' to $a(\sec\theta + \cos\theta)$ at P. Then to get the upper half of the area, θ varies from 0 to $\frac{\pi}{2}$.



$$\therefore \text{The required area } = 2\int_0^{\pi/2}\int_{a\sec\theta}^{a(\sec\theta+\text{coa}\theta)} r dr d\theta = 2\int_0^{\pi/2}\left[\frac{r^2}{2}\right]_{a\sec\theta}^{a(\sec\theta+\cos\theta)} d\theta$$

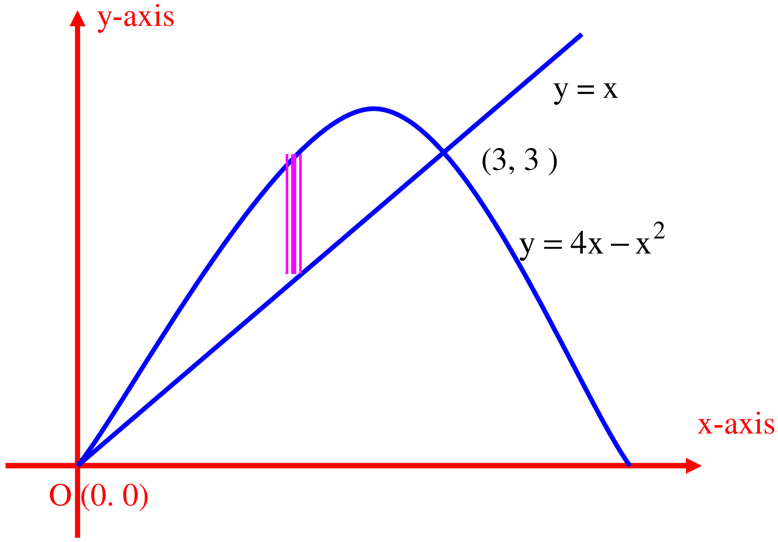
$$= a^2\int_0^{\pi/2}\left(2+\cos^2\theta\right)d\theta = a^2\left[2.\frac{\pi}{2}+\frac{1}{2}.\frac{\pi}{2}\right] = a^2\left[\pi\left(1+\frac{1}{4}\right)\right] = \frac{5\pi a^2}{4}.$$

Square units. Ans.

Q.No.4.: Find, by double integration, the area lying between the parabola $y = 4x - x^2$

and the line $y = x$.

Sol.:



The required area $= \int_0^3 \left(\int_x^{4x-x^2} dy \right) dx = \int_0^3 [y]_x^{4x-x^2} dx = \int_0^3 (4x - x^2 - x) dx$

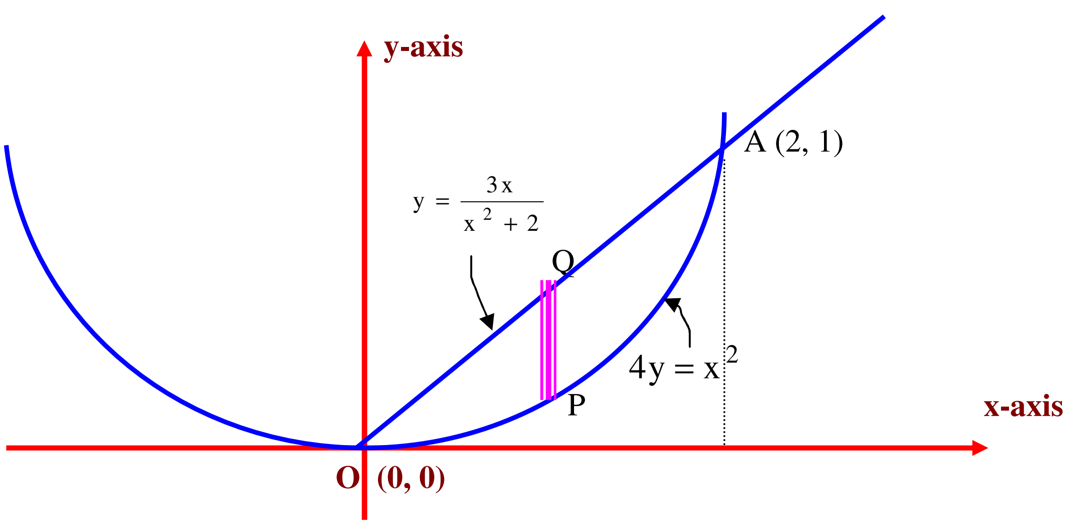
$$= \int_0^3 (3x - x^2) dx = \left[\frac{3x^2}{2} - \frac{x^3}{3} \right]_0^3$$

$$= \frac{27}{2} - \frac{27}{3} = \frac{27}{6} = \frac{9}{2} = 4.5 \text{ . Sq. units. Ans.}$$

Q.No.5.: Find, by double integration, the area enclosed by the curves $y = \frac{3x}{(x^2+2)}$ and

$4y = x^2$.

Sol.:



Let us suppose that the strip is parallel to y-axis. Then integrate w. r. t. y first and then w. r. t. x.

The required area $A = \int_0^2 \left(\int_{x^2/4}^{3x/\left(x^2+2\right)} dy \right) dx = \int_0^2 [y]_{x^2/4}^{3x/\left(x^2+2\right)} dx = \int_0^2 \left(\frac{3x}{x^2+2} - \frac{x^2}{4} \right) dx$.

Let $t = x^2 + 2 \Rightarrow dt = 2xdx \Rightarrow \frac{dt}{2} = xdx$.

$$\therefore \int \frac{3x}{x^2+2} dx = \int \frac{3\frac{dt}{2}}{t} = \frac{3}{2} \int \frac{dt}{t} = \frac{3}{2} \log t .$$

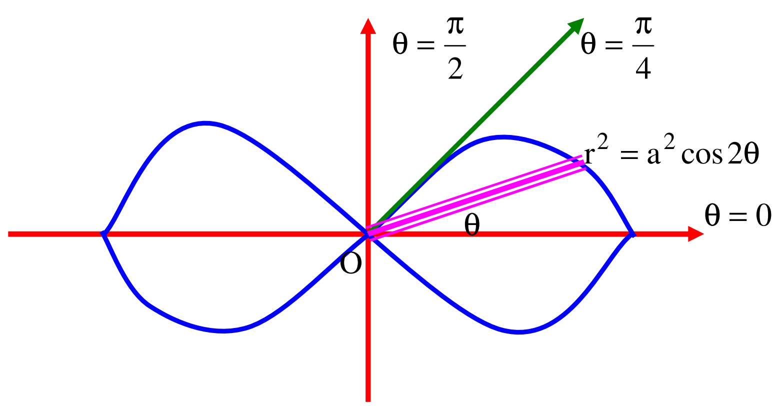
At x = 0, t = 2; x = 2, t = 6.

$$\therefore A = \int_0^2 \left(\int_{x^2/4}^{3x/\left(x^2+2\right)} dy \right) dx = \left[\frac{3}{2} \log_e t \right]_2^6 - \left[\frac{x^3}{12} \right]_0^2 = \frac{3}{2} [\log_e 6 - \log_e 2] - \frac{8}{12}$$

$$= \left(\frac{3}{2} \log_e \frac{6}{2} - \frac{2}{3} \right) = \left(\frac{3}{2} \log_e 3 - \frac{2}{3} \right) \text{. Sq. units. Ans.}$$

Q.No.6.: Find, by double integration, the area of lemniscate $r^2 = a^2 \cos 2\theta$.

Sol.:



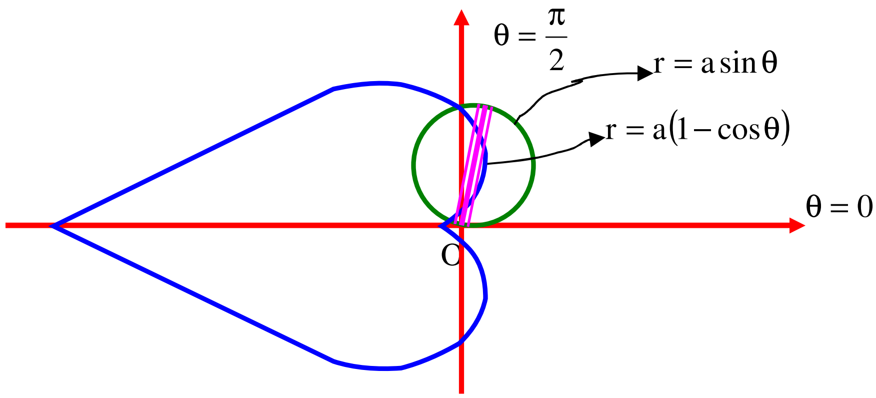
The required area $A = 4 \times$ [Area in the first quadrant]

$$= 4 \times \int_0^{\pi/4} \left(\int_0^{a\sqrt{\cos 2\theta}} r dr \right) d\theta = 4 . \int_0^{\pi/4} \left[\frac{r^2}{2} \right]_0^{a\sqrt{\cos 2\theta}} d\theta = \frac{4}{2} \int_0^{\pi/4} a^2 \cos 2\theta . d\theta$$

$$= 2a^2 \left[\frac{\sin 2\theta}{2} \right]_0^{\pi/4} = a^2 \left[\sin \frac{\pi}{2} - \sin 0 \right] = a^2 \text{. Sq. units. Ans.}$$

Q.No.7.: Find, by double integration, the area lying inside the circle $r = a\sin\theta$ and outside the cardioid $r = a(1-\cos\theta)$.

Sol.:



The required area $A = \int\limits_0^{\pi/2}\left(\int\limits_{a(1-\cos\theta)}^{a\sin\theta} r dr\right)d\theta = \int\limits_0^{\pi/2}\left[\frac{r^2}{2}\right]_{a(1-\cos\theta)}^{a\sin\theta} d\theta$

$$= \frac{1}{2}\int\limits_0^{\pi/2}\left[a^2\sin^2\theta - a^2\left(1-2\cos\theta+\cos^2\theta\right)\right]d\theta$$

$$= \frac{a^2}{2}\int\limits_0^{\pi/2}\left(\sin^2\theta - 1 + 2\cos\theta - \cos^2\theta\right)d\theta.$$

$$= \frac{a^2}{2}\left[\int\limits_0^{\pi/2}\sin^2\theta d\theta - \int\limits_0^{\pi/2}d\theta + \int\limits_0^{\pi/2}2\cos\theta d\theta - \int\limits_0^{\pi/2}\cos^2\theta d\theta\right]$$

$$= \frac{a^2}{2}\left[\left(\frac{1}{2}\times\frac{\pi}{2}\right) - \left(\frac{\pi}{2}\right) + (2\times 1) - \left(\frac{1}{2}\times\frac{\pi}{2}\right)\right]$$

$$= \frac{a^2}{2}\left[-\frac{\pi}{2}+2\right] = a^2\left[1-\frac{\pi}{4}\right].$$ Square units. Ans.

Q.No.8.: Find the area between the curve $\sqrt{x}+\sqrt{y}=\sqrt{a}$ and the axis.

Sol.: Since the x and y are under radical sign, x and y can take only positive values, therefore the curve lies in the first quadrant.

Now for $x = 0$, $y = a$ and $y - 0$, $x = a$ (here it is important that a is also positive)

Also $x = y = \frac{a}{4}$, satisfy the equation of the curve. Thus the curve can be plotted as shown in the figure.

To find the area, we have to calculate the following integral.

$$A = \int_0^a \left[\int_0^{\left(\sqrt{a}-\sqrt{x}\right)^2} dy \right] dx = \int_0^a [y]_0^{\left(\sqrt{a}-\sqrt{x}\right)^2} dx = \int_0^a \left(\sqrt{a}-\sqrt{x}\right)^2 dx$$

$$= \int_0^a \left(a + x - 2\sqrt{ax}\right) dx = \left[ax + \frac{x^2}{2} - 2\times\frac{2}{3}\sqrt{a}x^{3/2} \right]_0^a$$

$$= a^2 + \frac{a^2}{2} - \frac{4}{3}a^2 = \frac{a^2}{6}.$$ Square units. Ans.

Q.No.9.: Find, by double integration, the smaller of the areas bounded by the ellipse $4x^2 + 9y^2 = 36$ and the straight line $2x + 3y = 6$.

Sol.: Equation of the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$.

Area required $\int_0^2 \int_{\frac{6-3y}{2}}^{\sqrt{\frac{36-9y^2}{4}}} dxdy$

$$A = \frac{1}{2}\int_0^2 \left[\sqrt{6^2 - (3y)^2} - (6-3y) \right] dy = \frac{3}{2}\int_0^2 \sqrt{2^2 - y^2}\,dy - \int_0^2 \frac{6-3y}{2} dy$$

$$= \frac{3}{2}\left[\frac{y}{2}\sqrt{4-y^2} + \frac{4}{2}\sin^{-1}\frac{y}{2} \right]_0^2 - \left[3y - \frac{3}{4}y^2 \right]_0^2 = \frac{3}{2}\left[2\times\frac{\pi}{2} \right] - [6-3]$$

$$= \frac{3\pi}{2} - 3 = \frac{3}{2}(\pi - 2).$$ Square units.

Q.No.10.: Find, by double integration, the smaller of the areas bounded by the circle $x^2 + y^2 = 9$ and the line $x + y = 3$.

Sol.: Equation of the circle $x^2 + y^2 = 3^2$.

Area required $= \int_0^3 \int_{3-y}^{\sqrt{9-y^2}} dxdy = \int_0^3 \left[\sqrt{9-y^2} - (3-y) \right] dy$

$$=\left[\frac{y}{2}\sqrt{9-y}+\frac{9}{2}\sin^{-1}\frac{y}{3}\right]_0^3-\left[3y-\frac{y^2}{2}\right]_0^3=\frac{9}{2}\times\frac{\pi}{2}-\left(9-\frac{9}{2}\right)=\frac{9\pi}{4}-\frac{9}{2}=\frac{9}{4}(\pi-2).\text{ Sq.units.}$$

Q.No.11.: Find, by double integration, the area bounded by the parabola $y=x^2$ and the line $y=2x+3$.

Sol.: Required area

$$A=\int_{-1}^{3}\int_{x^2}^{2x+3}dydx=\int_{-1}^{3}\left(2x+3-x^2\right)dx=2\left[\frac{x^2}{2}\right]_{-1}^{3}+3[x]_{-1}^{3}-\left[\frac{x^3}{3}\right]_{-1}^{3}$$

$$=2\left[\frac{9}{2}-\frac{1}{2}\right]+3(3+1)-\left(\frac{27}{3}+\frac{1}{3}\right)=8+12-\frac{28}{3}=20-\frac{28}{3}=\frac{32}{3}=10\frac{2}{3}\text{ . Square units.}$$

Q.No.12.: Find, by double integration, the area bounded by the parabolas $y^2=4-x$ and $y^2=4-4x$.

Sol.: Area required $=\int\int_R dxdy$

$$A=\int_{-2}^{2}\int_{\frac{4-y^2}{4}}^{4-y^2}dxdy=\int_{-2}^{2}\left[4-y^2-\left(\frac{4-y^2}{4}\right)\right]dy=\int_{-2}^{2}\frac{16-4y^2-4+y^2}{4}dy$$

$$=\int_{-2}^{2}\frac{12-3y^2}{4}dy=3[y]_{-2}^{2}-\frac{3}{4\times3}\left[y^3\right]^2=3(2+2)-\frac{1}{4}(8+8)=12-4=8\text{ . Square units.}$$

Q.No.13.: Find, by double integration, the area bounded by the circles $r=2\sin\theta$ and $r=4\sin\theta$.

Sol.: Area required $=2\int\int_R rdrd\theta=2\int_0^{\pi/2}\int_{2\sin\theta}^{4\sin\theta}rdrd\theta=2\int_0^{\pi/2}\left[\frac{r^2}{2}\right]_{2\sin\theta}^{4\sin\theta}d\theta$

$$=2\int_0^{\pi/2}\frac{16\sin^2\theta-4\sin^2\theta}{2}d\theta=2\int_0^{\pi/2}6\sin^2\theta d\theta.$$

$$=12\int_0^{\pi/2}\sin^2\theta d\theta=12\times\frac{1}{2}\times\frac{\pi}{2}=3\pi\text{ Square units.}$$

Q.No.14.: Find, by double integration, the area outside the circles $r=a$ and inside the cardioids $r=a(1+\cos\theta)$.

Sol.: Required area $=2\int\limits_R\int r dr d\theta$

$$A=2\int_0^{\pi/2}\int_a^{a(1+\cos\theta)} r dr d\theta=2\int_0^{\pi/2}\left[\frac{r^2}{2}\right]_a^{a(1+\cos\theta)}d\theta=\frac{2}{2}\int_0^{\pi/2}\left[a^2(1+\cos\theta)^2-a^2\right]d\theta$$

$$=\frac{2a^2}{2}\int_0^{\pi/2}\left(1+\cos^2\theta+2\cos\theta-1\right)d\theta=\frac{2a^2}{2}\int_0^{\pi/2}\cos^2\theta+2a^2\int_0^{\pi/2}2\cos\theta$$

$$=\frac{2a^2}{2}\times\frac{1}{2}\times\frac{\pi}{2}+\frac{2a^2}{2}\times 2[\sin\theta]_0^{\pi/2}=\frac{\pi a^2}{4}+2a^2=\frac{a^2}{4}(\pi+8).\ \text{Square units.}$$

Q.No.15.: Find, by double integration, the area of the curvilinear quadrilateral bounded by four parabolas $y^2=ax,\ y^2=bx,\ x^2=cy,\ x^2=dy$.

Sol.: Area required $=\iint\limits_R dxdy$

Given parabolas are

$y^2=ax,\ y^2=bx,\ x^2=cy,\ x^2=dy$ (i, ii, iii, iv)

Now substituting $y^2=u^3x$ and $x^2=v^3y$

Now from (i) we know

$ax=u^3x,\quad u^3=a\quad$ and $x=0\quad u=a^{1/3}$ (A)

Also from (ii)

$bx=u^3x,\quad u=b^{1/3}$ (B)

and from (iii)

$cy=u^3y,\quad v=c^{1/3}$ (C)

From (iv)

$dy=v^3y,\quad v=d^{1/3}$ (D)

Considering $b>a$ and $d>c$

Now from A, B, C and D

$$J=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}v^2 & 2uv\\ 2uv & u^2\end{vmatrix}=u^2v^2-4u^2v^2-3u^2v^2.$$

$\Rightarrow |J|=3u^2v^2$.

$$\therefore A = \int_{c^{1/3}}^{d^{1/3}} \left(\int_{a^{1/3}}^{b^{1/3}} 3u^2v^2 du \right) dv = 3\int_{c^{1/3}}^{d^{1/3}} \left(\int_{a^{1/3}}^{b^{1/3}} u^2v^2 du \right) dv = 3\int_{c^{1/3}}^{d^{1/3}} \left(\frac{u^3}{3} \right)_{a^{1/3}}^{b^{1/3}} v^2 dv$$

$$= \frac{3}{3}\int_{c^{1/3}}^{d^{1/3}} (b-a)v^2 dv = (b-a)\left(\frac{v^3}{3} \right)_{e^{1/3}}^{d^{1/3}} = \frac{(b-a)(d-c)}{3}$$. Square units. Ans.

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7th Topic

Integral Calculus

Double Integrals

[Volume of solids by double integrals]

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Evaluation of Volume by double integrals:

Consider a surface $z = f(x, y)$. (i)

Let the orthogonal projection on xy-plane of its portion S' be the area S given by $\phi(x, y) = 0$. (ii)

Now (ii) represents a cylinder with generators parallel to z-axis and guiding curve given by (ii). Let V be the volume of this cylinder between S and S'.

Divide S into **elementary rectangles** of area $\delta x\, \delta y$ by drawing lines parallel to x and y-axes. With each of these rectangles as base, erect a prism having its length parallel to OZ.

$\therefore$ Volume of this **prism** between S and the given surface $z = f(x, y)$ is $(z\ \delta x\, \delta y)$.

Hence, the **volume of the solid cylinder** on S as base, bounded by the given surface with generators parallel to the z-axis

$$V = \underset{\substack{\delta x \to 0 \\ \delta y \to 0}}{\text{Lt}} \sum \sum z\, \delta x\, \delta y = \iint z\,dx\,dy = \iint f(x, y)\,dx\,dy,$$

where the integration is carried over the area S.

Remarks: While using polar co-ordinates, divide S into elements of area $r\,\delta\theta\,\delta r$.

$\therefore$ By replacing dxdy by $r\,d\theta\,dr$, we get the required volume $= \iint z r\,d\theta\,dr$.

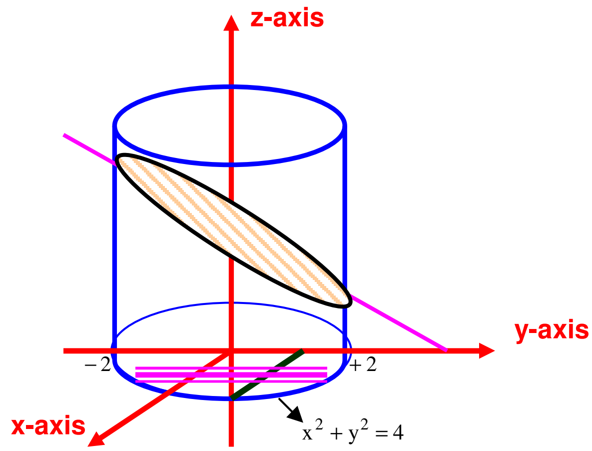
Q.No.1.: Find the volume bounded by the cylinder $x^2 + y^2 = 4$ and the planes

$y + z = 4$ and $z = 0$.

Sol.: The required volume $= \iint zdxdy = \iint (4 - y)dxdy$,

where the integration is carried over the area of circle $x^2 + y^2 = 4$.

Let us suppose strip is parallel to x-axis, then to cover the whole circle, x varies from $-\sqrt{4 - y^2}$ to $\sqrt{4 - y^2}$ and y varies from -2 to 2.



$$\therefore \text{Required volume} = \int_{-2}^{2} \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} (4 - y)dxdy = 2\int_{-2}^{2} \left(\int_{0}^{\sqrt{(4-y^2)}} (4 - y)dx \right) dy$$

$$= 2\int_{-2}^{2} (4 - y)[x]_0^{\sqrt{(4-y^2)}} dy = 2\int_{-2}^{2} (4 - y)\sqrt{(4 - y^2)}dy$$

$$= 2\int_{-2}^{2} 4(4 - y^2)dy - 2\int_{-2}^{2} y\sqrt{(4 - y^2)}dy$$

$$= 8\int_{-2}^{2} \sqrt{(4 - y^2)}dy - 0. \left[\begin{array}{l}\text{The second term vanishes as the} \\ \text{integrand is an odd function.}\end{array}\right]$$

Put $y = 2\sin\theta$ so that $dy = 2\cos\theta d\theta$.

And as y varies from -2 to 2, θ varies $-\frac{\pi}{2}$ to $\frac{\pi}{2}$.

$\therefore$ Required volume $= 8\int_{-\pi/2}^{\pi/2} 2\cos\theta . 2\cos\theta d\theta = 32\int_{-\pi/2}^{\pi/2} \cos^2\theta d\theta = 64\int_{0}^{\pi/2} \cos^2\theta d\theta$

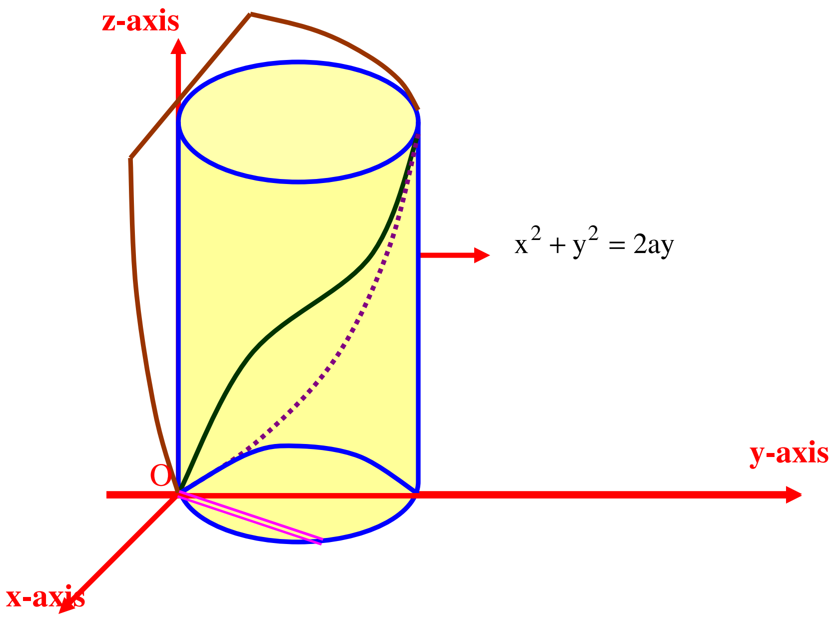
$= 64\times\frac{1}{2}\times\frac{\pi}{2} = 16\pi$. Cubic units. Ans.

Q.No.2.: Find the volume bounded by the paraboloid $x^2 + y^2 = az$, the cylinder

$x^2 + y^2 = 2ay$ and the plane z = 0.

Sol.: The required volume $V = \iint z dx dy = \iint \frac{x^2 + y^2}{a} dx dy$,

over the circle $x^2 + y^2 = 2ay$.



To change cartesian co-ordinates (x, y) to polar co-ordinates (r, θ),

we have put $x = r\cos\theta$, $y = r\sin\theta$ and

$$J = \frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\left(\cos^2\theta + \sin^2\theta\right) = r .$$

Then $\iint_{R_{xy}} f(x, y) dx dy = \iint_{R'_{r\theta}} f(r\cos\theta, r\sin\theta) . r\, dr\, d\theta$.

$\therefore$ Paraboloid $x^2 + y^2 = az \Rightarrow z = \frac{r^2}{a}$ and the polar equation of the circle is $r = 2a\sin\theta$.

To cover the circle, r varies from 0 to $2a\sin\theta$ and θ varies from 0 to π.

$$\therefore \text{ Required volume } = \iint \frac{x^2+y^2}{a}dxdy = \int_0^{\pi}\int_0^{2a\sin\theta}\frac{r^2}{a}.r\,d\theta\,dr = \frac{1}{a}\int_0^{\pi}\left(\int_0^{2a\sin\theta} r^3dr\right)d\theta$$

$$= \frac{1}{a}\int_0^{\pi}\left(\left|\frac{r^4}{4}\right|_0^{2a\sin\theta}\right)d\theta = 4a^3\int_0^{\pi}\sin^4\theta d\theta = 4a^3.2\int_0^{\pi/2}\sin^4\theta d\theta$$

$$= 8a^3\times\left(\frac{3\times1}{4\times2}\times\frac{\pi}{2}\right) = \frac{3\pi a^3}{2}. \text{ Cubic units Ans.}$$

Q.No.3.: Find the volume bounded by the xy-plane, the paraboloid $2z = x^2+y^2$ and the cylinder $x^2+y^2=4$.

Sol.: Required volume is found by integrating $z = \dfrac{x^2+y^2}{2}$ over $x^2+y^2=4$.

$$\text{i. e. } V = \iint zdxdy = \iint_{x^2+y^2\le4}\frac{x^2+y^2}{2}dxdy$$

To change cartesian co-ordinates (x, y) to polar co-ordinates (r, θ),

we have put $x = r\cos\theta$, $y = r\sin\theta$ and

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial\theta}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\theta}\end{vmatrix} = \begin{vmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{vmatrix} = r\left(\cos^2\theta+\sin^2\theta\right) = r.$$

$$\text{Then } \iint_{R_{xy}} f(x,y)dxdy = \iint_{R'_{r\theta}} f(r\cos\theta, r\sin\theta).r\,dr\,d\theta.$$

$$\text{Paraboloid } 2z = x^2+y^2 \Rightarrow z = \frac{x^2+y^2}{2} = \frac{r^2}{2} \text{ and}$$

cylinder $x^2+y^2=4 \Rightarrow r^2=4$, $\therefore r = 2, -2$ (Rejected) $\therefore r = 2$

To cover full circle, r varies from 0 to 2 and θ varies from 0 to 2π

$$V = \int_0^{2\pi}\left(\int_0^2\frac{r^2}{2}rdr\right)d\theta \Rightarrow \frac{1}{2}\int_0^{2\pi}\left(\int_0^2 r^3dr\right)d\theta = \frac{1}{2}\int_0^{2\pi}\left|\frac{r^4}{4}\right|_0^2 d\theta = \frac{1}{2}\int_0^{2\pi}4d\theta$$

$$= 2\int_0^{2\pi} d\theta = 2\times 2\pi = 4\pi \text{. Cubic units. Ans.}$$

Q.No.4.: Find the volume enclosed by the cylinders $x^2 + y^2 = 2ax$ and $z^2 = 2ax$.

Sol.: The required volume $= 2\iint zdxdy = 2\iint \sqrt{2ax}dxdy$

To change cartesian co-ordinates (x, y) to polar co-ordinates (r, θ),

we have put $x = r\cos\theta$, $y = r\sin\theta$ and

$$J = \frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\left(\cos^2\theta + \sin^2\theta\right) = r.$$

Then $\iint\limits_{R_{xy}} f(x, y)dxdy = \iint\limits_{R'_{r\theta}} f(r\cos\theta, r\sin\theta).r\, dr\, d\theta$.

Now $x^2 + y^2 = 2ax \Rightarrow r^2 = 2ar\cos\theta \Rightarrow r[r - 2a\cos\theta] = 0$.

So r varies from 0 to $2a\cos\theta$ and θ varies from 0 to π.

$\therefore$ Required volume $= 2\iint \sqrt{2ax}dxdy = 2\int_0^{\pi}\int_0^{2a\cos\theta} \sqrt{2ar\cos\theta}rdrd\theta$

$$= 4\int_0^{\pi/2}\int_0^{(2a\cos\theta)} \sqrt{2a\cos\theta}r^{3/2}drd\theta = 4\int_0^{\pi/2}\left[\int_0^{2a\cos\theta} r^{3/2}\, dr\right]\sqrt{2a\cos\theta}d\theta$$

$$= 4\int_0^{\pi/2} \sqrt{2ar\cos\theta}\left[\frac{r^{5/2}}{\frac{5}{2}}\right]_0^{2a\cos\theta} d\theta = 4\int_0^{\pi/2} \sqrt{2ar\cos\theta}\,\frac{2}{5}\left[(2a)^{5/2}\cos^{5/2}\theta\right]d\theta$$

$$= \frac{2^6}{5}a^3\left[\int_0^{\pi/2} \cos^3\theta d\theta\right] = \frac{64a^3}{5}.\frac{2}{3\times 1} = \frac{128}{15}a^3 \text{. Cubic units. Ans.}$$

Q.No.5.: Find the volume of the cylinder $x^2 + y^2 - 2ax = 0$, intercepted between the paraboloid $x^2 + y^2 = 2az$ and the xy-plane.

Sol.: The required volume $= \iint zdxdy = \iint \frac{x^2 + y^2}{2a}dxdy$

To change cartesian co-ordinates (x, y) to polar co-ordinates (r, θ), we have put $x = r\cos\theta$, $y = r\sin\theta$ and

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\left(\cos^2\theta + \sin^2\theta\right) = r .$$

Then $\iint\limits_{R_{xy}} f(x,y)dxdy = \iint\limits_{R'_{r\theta}} f(r\cos\theta, r\sin\theta).r\,dr\,d\theta$.

Since $x^2 + y^2 = 2ax \Rightarrow r^2 = 2ar\cos\theta \Rightarrow r[r - 2a\cos\theta] = 0$

To cover the circle r varies from 0 to $2a\cos\theta$ and θ varies from 0 to π.

$$\therefore \text{Required volume } = \int_0^{\pi}\int_0^{2a\cos\theta} \frac{r^2}{2a} r\,dr d\theta = \frac{1}{2a}\int_0^{\pi} \left|\frac{r^4}{4}\right|_0^{2a\cos\theta} d\theta = \frac{1}{2a}\times\frac{16a^4}{4}\int_0^{\pi} \cos^4\theta d\theta$$

$$= 2a^3 \times 2\int_0^{\pi/2} \cos^4\theta d\theta = 4a^3 \times \frac{3}{4\times 2}\times\frac{\pi}{2} = \frac{3\pi a^3}{4}. \text{ Cubic units. Ans.}$$

Q.No.6.: Find the volume of the region bounded by $z = x^2 + y^2$, $z = 0$, $x = -a$, $x = a$, and $y = -a$, $y = a$.

Sol.: Required volume $= \iint zdxdy = \int_{-a}^{a}\int_{-a}^{a} \left(x^2 + y^2\right)dxdy = \int_{-a}^{a} \left|x^2y + \frac{y^3}{3}\right|_{-a}^{a} dx$

$$= \int_{-a}^{a}\left(x^2a + \frac{a^3}{3} + x^2a + \frac{a^3}{3}\right)dx = 2\int_{-a}^{a}\left(x^2a + \frac{a^3}{3}\right)dx$$

$$= 2\left|\frac{x^3}{3}a + \frac{a^3}{3}x\right|_{-a}^{a} = \frac{2}{3}\left(a^4 + a^4 + a^4 + a^4\right) = \frac{8}{3}a^4. \text{ Cubic units. Ans.}$$

Q.No.7.: Find the volume V of a solid bounded by the spherical surface $x^2 + y^2 + z^2 = 4a^2$ and the cylinder $x^2 + y^2 - 2ay = 0$.

Sol.: $V = \iint\limits_R zdxdy$.

R is a region defined by $x^2 + y^2 - 2ay = 0$.

Putting $z = \sqrt{4a^2 - \left(x^2 + y^2\right)}$

$V = \iint_R \sqrt{4a^2 - \left(x^2 + y^2\right)}dxdy$.

Putting $x = r\cos\theta$, $y = r\sin\theta$.

$J = \dfrac{\partial(x,y)}{\partial(r,\theta)} = r$

$$V = 2\int_0^{\pi}\int_0^{2a\sin\theta} \sqrt{4a^2 - \left(r^2\cos^2\theta + r^2\sin^2\theta\right)}rdrd\theta = 4\int_0^{\pi/2}\int_0^{2a\sin\theta}\sqrt{4a^2 - r^2}rdrd\theta$$

Putting $4a^2 - r^2 = t^2 \Rightarrow -2rdr = 2tdt$

$$V = 4\int_0^{\pi/2}\int_{2a}^{2a\cos\theta} t(-tdt)d\theta = 4\int_0^{\pi/2}\left[\frac{-t^3}{3}\right]_{2a}^{2a\cos\theta} d\theta = 4\int_0^{\pi/2}\frac{(2a)^3}{3}\left[1-\cos^3\theta\right]d\theta$$

$$= 4\times\frac{8a^3}{3}\int_0^{\pi/2} d\theta - 4\times\frac{8a^3}{3}\int_0^{\pi/2}\cos^2\theta.\cos\theta d\theta = 4\left[\frac{8a^3}{3}\left(\frac{\pi}{2}\right) - \frac{8a^3}{3}\int_0^{\pi/2}\left[1-\sin^2\theta\right]\cos\theta d\theta\right]$$

$$= 4\left[\frac{8a^3}{3}.\frac{\pi}{2} - \frac{8a^3}{3}\int_0^{\pi/2}\cos\theta d\theta + \frac{8a^3}{3}\int_0^{\pi/2}\sin^2\theta\cos\theta d\theta\right]$$

$$= 4\left[\frac{8a^3}{3}.\frac{\pi}{2} - \frac{8a^3}{3} + \frac{8a^3}{3}.\left[\frac{\sin^3\theta}{3}\right]_0^{\pi/2}\right] = 4\times\frac{8a^3}{3}\left[\frac{\pi}{2} - 1 + \frac{1}{3}\right]$$

$$= 4\times\frac{8a^3}{3}\left[\frac{\pi}{2} - \frac{2}{3}\right] = \frac{16a^3}{3}\left[\pi - \frac{4}{3}\right].\ \text{Ans.}$$

Q.No.8.: Find, by double integration, the volume of the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$.

Sol.: The volume of the required ellipsoid is equal to 8 times the volume of ellipsoid in any one octant (say XOY).

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \Rightarrow z = c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}.$$

For plane XOY: $z = 0$, $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \Rightarrow y = b\sqrt{1 - \dfrac{x^2}{a^2}}$.

Required volume = $8\int_0^a ydx \int_0^{b\sqrt{1-x\frac{2}{a^2}}} zdydx = 8\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{x^2}}} c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}dydx$

Putting $\sqrt{1-\frac{x^2}{a^2}} = t$

$$V = 8\int_0^a \left(\int_0^{bt} c\sqrt{t^2-\frac{y^2}{b^2}}dy\right)dx = 8\int_0^a \left(\int_0^{bt} \frac{c}{b}\sqrt{(bt)^2-y^2}dy\right)dx$$

$$= \frac{8c}{b}\left[\frac{y}{2}\sqrt{(bt)^2-y^2}+\frac{(bt)^2}{2}\sin^{-1}\frac{y}{bt}\right]_0^{bt} dx = \frac{8c}{b}\int_0^a \left[\frac{(bt)^2}{2}\times\frac{\pi}{2}\right]dx = \frac{dc}{b}\times\frac{b^2}{2}\times\frac{\pi}{2}\int_0^a t^2dx$$

$$= \frac{8bc}{4}\times\pi\int_0^a \left(1-\frac{x^2}{a^2}\right)dx = 2\pi bc\int_0^a dx - \frac{2\pi bc}{a^2}\int_0^a x^2dx = 2\pi bc[x]_0^a - \frac{2\pi bc}{a^2}\left[\frac{x^3}{3}\right]_0^a$$

$$= 2\pi abc - \frac{2\pi abc}{3} = \frac{4}{3}\pi abc \text{ Cubic units.}$$

Q.No.9.: Find, by double integration, the volume of the tetrahedron bounded by the coordinate planes and the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

Sol.: $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ is the given equation of tetrahedron.

$$\Rightarrow z = c\left(1-\frac{x}{a}-\frac{y}{b}\right)$$

For plane XOY: $z = 0$, $\frac{x}{a}+\frac{y}{b}=1$

$$\text{Volume } = \int_0^a \int_0^{b(1-\frac{x}{a})} zdydx = \int_0^a \left(\int_0^{b\left(1-\frac{x}{a}\right)} c\left(1-\frac{x}{a}-\frac{y}{b}\right)dy\right)dx$$

$$= \int_0^a \left[c\left(b-\frac{xb}{a}\right)-\frac{cx}{a}\left(b-\frac{xb}{a}\right)-\frac{c}{2b}b^2\left(1-\frac{x}{a}\right)^2\right]dx$$

$$= bc[x]_0^b - \frac{bc}{2a}\left[x^2\right]_0^a - \frac{bc}{2a}\left[x^2\right]_0^a + \frac{bc}{3a^2}\left[x^3\right]_0^a + \left[-\left(\frac{bc}{2}[x]_0^a - \frac{bc}{2\times 3a^2}\left[x^3\right]_0^a - \frac{bc}{2a}\left[x^2\right]_0^a\right)\right]$$

$$= abc - \frac{abc}{2} - \frac{abc}{2} + \frac{abc}{3} \mp \left(\frac{abc}{2} + \frac{abc}{2\times 3} - \frac{abc}{2} \right)$$

$$= abc - \frac{abc}{2} - \frac{abc}{2} + \frac{abc}{3} - \frac{abc}{2} - \frac{abc}{2\times 3} + \frac{abc}{2}$$

$$= abc - \frac{abc}{2} - \frac{abc}{2} + \frac{abc}{3} - \frac{abc}{2} - \frac{abc}{6} + \frac{abc}{2} = \frac{abc}{6}.\quad \text{Cubic units.}$$

Q.No.10.: Find the volume common to the cylinders $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$.

Sol.: $z = \sqrt{a^2 - x^2}$, $\quad y = \sqrt{a^2 - x^2}$.

Required volume $= 8\iint zdydx = 8\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2 - x^2}\,dxdy$

$$= 8\int_0^a \sqrt{a^2 - x^2}\,[y]_a^{\sqrt{a^2-x^2}}\,dx = 8\int_0^a \left(a^2 - x^2\right)dx = 8a^2[x]_0^a - \frac{8}{3}\left[x^3\right]_0^a$$

$$= 8a^3 - \frac{8a^3}{3} = \frac{16}{3}a^3.\quad \text{Cubic units}$$

Q.No.11.: Find, by double integration, the volume common to the sphere

$$x^2 + y^2 + z^2 = a^2 \text{ and the cylinder } x^2 + y^2 = ay.$$

Sol.: The required volume is the part of the sphere $x^2 + y^2 + z^2 = a^2$ lying within the cylinder $z = a^2 - y^2 - x^2$. On the account of symmetry of the sphere, half of it lies above the plane XOY and half below it.

$\therefore$ Required volume $= 2\int\int zdydx$,

where $z = \sqrt{\left(a^2 - y^2 - x^2\right)}$, and the region of integration is the area inside the circle $x^2 + y^2 = ay$.

On the account of symmetry, the volume above the two parts of circle $x^2 + y^2 = ay$ in the first and the second quadrants are equal.

Total volume required $= 2\times 2\int\int_R \sqrt{\left(a^2 - y^2 - x^2\right)}dydx$

where R is half of the circle $x^2 + y^2 = ay$ lying in the first quadrant.

Changing to polar coordinates by putting $x = r\cos\theta$, $y = r\sin\theta$ so that $x^2 + y^2 = r^2$.

Equation of the circle $x^2 + y^2 = ay$ becomes

$r^2 = ar\sin\theta \Rightarrow r = a\sin\theta$

Thus the region of integration is bounded by $r = 0, r = a\sin\theta$ and $\theta = 0, \theta = \dfrac{\pi}{2}$.

$\therefore$ Required volume $V = 4\int_0^{\pi/2}\int_0^{a\sin\theta}\sqrt{a^2 - r^2}\; r\,dr\,d\theta$

Now put $a^2 - r^2 = t^2 \Rightarrow rdr = -t$

$$V = 4\int_0^{\pi/2}\int_a^{a\cos\theta} t^{2/2}(-t)dtd\theta = 4\int_0^{\pi/2}\int_a^{a\cos\theta} t^2 dtd\theta = -\frac{4}{3}\int_0^{\pi/2}\left[t^3\right]_a^{a\cos\theta} d\theta$$

$$= -\frac{4}{3}\int_0^{\pi/2}\left(a^3\cos^3\theta - a^3\right)d\theta = -\frac{4}{3}\left[a^3 \times \frac{2.1}{3.1} - a^3 \times \frac{\pi}{2}\right] = \frac{2}{9}a^3[3\pi - 4].$$ Cubic units

Q.No.12.: Find, by double integration, the volume common to the sphere

$x^2 + y^2 + z^2 = a^2$ and the cylinder $x^2 + y^2 = ax$.

Sol.: The required volume is the part of the sphere $x^2 + y^2 + z^2 = a^2$ lying within the cylinder $z = a^2 - y^2 - x^2$. On the account of symmetry of the sphere, half of it lies above the plane XOY and half below it.

$\therefore$ Required volume $= 2\int\int zdydx$,

where $z = \sqrt{\left(a^2 - y^2 - x^2\right)}$, and the region of integration is the area inside the circle $x^2 + y^2 = ax$.

On the account of symmetry, the volume above the two parts of circle $x^2 + y^2 = ay$ in the first and the second quadrants are equal.

Total volume required $= 2 \times 2\int\int_R \sqrt{\left(a^2 - y^2 - x^2\right)}dydx$

where R is half of the circle $x^2 + y^2 = ax$ lying in the first quadrant.

Changing to polar coordinates by putting $x = r\cos\theta$, $y = r\sin\theta$ so that $x^2 + y^2 = r^2$.

Equation of the circle $x^2 + y^2 = ax$ becomes

$r^2 = ar\cos\theta \Rightarrow r = a\cos\theta$

Thus the region of integration is bounded by $r = 0, r = a\cos\theta$ and $\theta = 0, \theta = \dfrac{\pi}{2}$.

$\therefore$ Required volume $V = 4\int\limits_0^{\pi/2}\int\limits_0^{a\cos\theta} \sqrt{a^2 - r^2}\ r\,dr d\theta$

$$V = \frac{4}{-2}\int\limits_0^{\pi/2}\int\limits_a^{a\cos\theta} \sqrt{a^2 - r^2}(-2r)dr d\theta = -2\int\limits_0^{\pi/2}\left[\frac{\left(a^2 - r^2\right)}{3/2}\right]_a^{a\cos\theta} d\theta$$

$$= -\frac{4}{3}\int\limits_0^{\pi/2}\left(a^3\sin^3\theta - a^3\right)d\theta = -\frac{4}{3}\left[a^3 \times \frac{2.1}{3.1} - a^3 \times \frac{\pi}{2}\right] = \frac{2}{9}a^3[3\pi - 4].$$ Cubic units

Q.No.13.: Find, by double integration, the volume bounded by the cylinder $x^2 + y^2 = 4$ and the hyperboloid $x^2 + y^2 - z^2 = 1$.

Sol.: $z = \sqrt{x^2 + y^2 - 1}$, $y = \sqrt{4 - x^2}$

$$\text{Volume } = 2\int\int\limits_R z dy dx = 2\int\limits_0^2\int\limits_0^{\sqrt{4-x^2}} \sqrt{x^2 + y^2 - 1}dy dx$$

Putting $r\cos\theta = x$, $r\sin\theta = y$ and $|J| = r$

$x^2 + y^2 = r^2$

Also $x^2 + y^2 = 4 \Rightarrow r^2 = 4 \Rightarrow r = 2$.

$$V = 2\int\limits_0^{2\pi}\int\limits_1^2 \sqrt{r^2 - 1}\ r dr d\theta.$$

Putting $r^2 - 1 = t^2 \Rightarrow 2rdr = 2tdt$.

$$V = 2\int\limits_0^{2\pi}\int\limits_0^{\sqrt{3}} t^2 dt d\theta \Rightarrow \int\limits_0^{\pi}\left[\frac{t^3}{3}\right]_0^{\sqrt{3}} d\theta = 2\int\limits_0^{2\pi}\sqrt{3}d\theta = 4\sqrt{3}\pi.$$ Cubic units.

Q.No.14.: Find, by double integration, the volume under the plane $z = x + y$ and above the area cut from the first quadrant by the ellipse $4x^2 + 9y^2 = 36$.

Sol.: Given $z = x + y$, $4x^2 + 9y^2 = 36$.

Required volume $= \int\int_R z\,dy\,dx = \int_0^3 \left[\int_0^{\sqrt{\frac{36-4x^2}{9}}} (x+y)\,dy \right] dx$

$$= \int_0^3 \left[x\frac{\sqrt{36-4x^2}}{3} + \frac{1}{2}\frac{(36-4x^2)}{9} \right] dx = \frac{1}{3}\int_0^3 x\sqrt{36-4x^2}\,dx + \frac{1}{18}\int_0^3 (36-4x^2)\,dx$$

I II

For I, putting $36-4x^2 = t^2$, $-8x\,dx = 2t\,dt \Rightarrow x\,dx = -\frac{t}{4}dt$

$$V = -\frac{1}{3}\int_6^0 \frac{t^2}{4}dt + 2[x]_0^3 - \frac{4}{18}\times\frac{1}{3}\left[x^3\right]_0^3 = \frac{1}{3}\int_0^6 \frac{t^2}{4}dt + 6 - \frac{4}{18}\times\frac{1}{3}[27]$$

$$= \frac{1}{12}\times\frac{1}{3}\times 6\times 6\times 6 + 6 - \frac{4}{18}\times\frac{1}{3}\times 3\times 3\times 3 = 6+6-2 = 10.$$ Cubic units

Q.No.15.: Find, by double integration, the volume bounded by the plane $z = 0$, surface $z = x^2 + y^2 + 2$ and the cylinder $x^2 + y^2 = 4$.

Sol.: Given $z = x^2 + y^2 + 2$, $x^2 + y^2 = 4$

Volume of required region $= 4\int\int_R z\,dy\,dx$

$$V = 4\int_0^2 \left(\int_0^{\sqrt{4-x^2}} \left[x^2 + y^2 + 2\right] dy \right) dx = 4\int_0^2 \left[x^2[y]_0^{\sqrt{4-x^2}} + \frac{1}{3}\left[y^3\right]_0^{\sqrt{4-x^2}} + 2[y]_0^{\sqrt{4-x^2}} \right] dx$$

$$= 4\int_0^2 \left[x^2\sqrt{4-x^2} + \frac{1}{3}\left(4-x^2\right)^{3/2} + 2\sqrt{4-x^2} \right] dx$$

I II III

For I and II

Putting $x = 2\sin\theta$, $dx = 2\cos\theta\,d\theta$

$$= 4\int_0^{\pi/2} 4\sin^2\theta\sqrt{4-4\sin^2\theta}\,2\cos d\theta + \frac{4}{3}\int_0^{\pi/2} \left(4-4\sin^2\theta\right)^{3/2} 2\cos\theta + 2\times 4\int_0^2 \sqrt{4-x^2}$$

$$= 4\int_0^{\pi/2} 16\sin^2\theta\cos^2\theta\,d\theta + \frac{4}{3}\int_0^{\pi/2} 8\cos^3\theta.2\cos\theta\,d\theta + 8\left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]$$

$$= 64\int_0^{\pi/2} \sin^2\theta\cos^2\theta d\theta + \frac{32}{3}\times 2\int_0^{\pi/2} \cos^4\theta + 8\left[2\times\frac{\pi}{2}\right]$$

$$= 64\times\frac{1.1}{4.2}\times\frac{\pi}{2} + \frac{64}{3}\times\frac{3.1}{4.2}\times\frac{\pi}{2} + 8\pi = 4\pi + 4\pi + 8\pi = 16\pi.\ \text{Cubic units}$$

Q.No.16.: Find, by double integration, the volume bounded by the cylinder $x^2 + y^2 = 1$ and the plane $x + y + z = 3$.

Sol.: Given $x + y + z = 3 \Rightarrow z = 3 - x - y$

$$x^2 - y^2 = 1 \Rightarrow y = \sqrt{1 - x^2}$$

$$\text{Volume} = 4\iint_R zdydx = 4\int_0^1 \int_0^{R/\sqrt{1-x^2}} 3 - x - ydydx$$

$$= 4\int_0^1 3[y]_0^{\sqrt{1-x^2}} - x[y]_0^{\sqrt{1-x^2}} - \frac{1}{2}\left[y^2\right]_0^{\sqrt{1-x^2}} dx = 4\int_0^1 \underset{\text{I}}{3\sqrt{1-x^2}} - \underset{\text{II}}{x\sqrt{1-x^2}} - \underset{\text{III}}{\frac{1}{2}\left(1-x^2\right)}dx$$

For II, $1 - x^2 = t^2 \Rightarrow -2xdx = 2tdt \Rightarrow xdx = -tdt$.

$$V = 4.3\left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x\right]_0^1 - 4\int_1^0 -t^2dt - \frac{4}{2}\left[x - \frac{1}{3}x^3\right]_0^1$$

$$= 4.3\left[\frac{1}{2}\times\frac{\pi}{2}\right] + 4\left[\frac{t^3}{3}\right]_1^0 - \frac{4}{2}\left[1 - \frac{1}{3}\right] = 4.3\times\frac{\pi}{4} - \frac{4}{3} - \frac{4}{3} = 4.\frac{3\pi}{4} - \frac{2.4}{3} = 3\pi - \frac{8}{3}\ \ \text{Cubic units}$$

Q.No.17.: A rectangular prism is formed by the planes whose equations are ay = bx, y = 0 and $x = a$. Find, by double integration, the volume of this prism between the plane z = 0 and the surface $z = c + xy$.

Sol.: Volume $= 4\iint_R zdydx$

$$V = \int_0^a \int_0^{\frac{b}{a}x} (c + xy)dydx = \int_0^a \left(c[y]_0^{\frac{bx}{a}} + \frac{x}{2}\left[y^2\right]_0^{\frac{bx}{a}}\right)dx = \int_0^a \left(\frac{bcx}{a} + \frac{b^2x^3}{2a^2}\right)dx$$

$$= \frac{bc}{a}\times\frac{1}{2}\left[x^2\right]_0^a + \frac{b^2}{2a^2}\times\frac{1}{4}\left[x^4\right]_0^a = \frac{bc}{2a}\times a^2 + \frac{b^2}{8a^2}\times a^4$$

$$= \frac{abc}{2} + \frac{a^2b^2}{8} = \frac{ab}{8}(4c + ab).$$ Cubic units

Q.No.18.: Find, by double integration, the volume of the sphere $x^2 + y^2 + z^2 = 9$.

Sol.: Required volume will be equal to 8times the volume of XOY, $z = 0$

$$z = \sqrt{9 - x^2 - y^2}$$

$$\text{Volume } = 8\int\int_R z\,dydx = 8\int_0^3 \int_0^{\sqrt{9-x^2}} \sqrt{9 - x^2 - y^2}\,dydx$$

Put $x = r\cos\theta$, $y = r\sin\theta$ and $|J| = r$

$$x^2 + y^2 = r^2$$

$$\text{Volume } = 8\int_0^{\pi/2}\int_0^3 \sqrt{9 - r^2}\, rdrd\theta$$

Put $9 - r^2 = t^2 \Rightarrow -2rdr = 2tdt \Rightarrow rdr = -tdt$

$$V = 8\int_0^{\pi/2}\int_3^0 -t^2dtd\theta = 8\int_0^{\pi/2}\int_0^3 t^2dtd\theta = \frac{8}{3}\int_0^{\pi/2}\left[t^3\right]_0^3 d\theta = 8\int_0^{\pi/2}\int_0^3 t^2dtd\theta$$

$$= \frac{8}{3}\int_0^{\pi/2}\left[t^3\right]_0^3 d\theta = \frac{8}{3} \times 3^3 \times \frac{\pi}{2} = 36\pi.$$ Cubic units

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8^{th} Topic

Integral Calculus

Double Integrals

[Volumes of solids of revolution]

(Cartesian and Polar co-ordinates)

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Volumes of solids of revolution:

Cartesian co-ordinates:

Consider an elementary area $\delta x\, \delta y$ at the point P(x, y) of a plane area A.

As this elementary area revolves about x-axis, we get a ring of volume

$= \pi\left[(y + \delta y)^2 - y^2\right]\delta x = 2\pi y\, \delta x\, \delta y$,

nearly to the first powers of δy.

Hence, the total volume of the solid formed by the revolution of the area A about x-axis

$= \iint_A 2\pi y dx dy$.

Similarly, the volume of the solid formed by the revolution of the area A about y-axis

$= \iint_A 2\pi x dx dy$.

Polar co-ordinates:

In polar co-ordinates, the above formula for the volume becomes

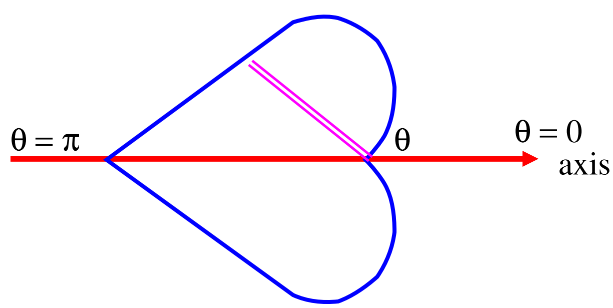
$$\iint_A 2\pi r\sin\theta . r d\theta dr = \iint_A 2\pi r^2 \sin\theta . d\theta dr .$$

Q.No.1.: Calculate by double integration, the volume generated by the revolution of the cardioid $r = a(1-\cos\theta)$ about its axis.

Sol.: In polar co-ordinates, the formula for evaluating the volume of revolution is

$$\iint_A 2\pi r\sin\theta . r d\theta dr = \iint_A 2\pi r^2 \sin\theta . d\theta dr .$$

Here $r = a(1-\cos\theta)$.



$$\therefore \text{Required volume } = \int_0^{\pi}\int_0^{a(1-\cos\theta)} 2\pi r^2 \sin\theta dr d\theta = 2\pi\int_0^{\pi}\left|\frac{r^3}{3}\right|_0^{a(1-\cos\theta)} \sin\theta d\theta$$

$$= \frac{2\pi a^3}{3}\int_0^{\pi} (1-\cos\theta)^3 . \sin\theta d\theta .$$

Put $1-\cos\theta = t$, so that $\sin\theta d\theta = dt$.

And when $\theta = 0$, $t = 0$, and when $\theta = \pi$, $t = 2$.

$$\therefore \text{Required volume of revolution } = \frac{2\pi a^3}{3}\int_0^{2} t^3 dt$$

$$= \frac{2\pi a^3}{3}\left[\frac{t^4}{4}\right]_0^2 dt = \frac{8\pi a^3}{3}. \text{ Cubic units. Ans.}$$

Q.No.2.: Prove, by using a double integral that the volume generated by the revolution of the cardioid $r = a(1+\cos\theta)$ about its axis is $\frac{8\pi a^3}{3}$.

Sol.: In polar co-ordinates, the formula for evaluating the volume of revolution is

$$\iint_A 2\pi r\sin\theta . rd\theta dr = \iint_A 2\pi r^2 \sin\theta . d\theta dr .$$

Here $r = a(1+\cos\theta)$.

$\therefore$ Required volume of revolution $= \int_0^{\pi}\left(\int_0^{a(1+\cos\theta)} 2\pi r^2 dr\right)\sin\theta d\theta$

$$= 2\pi\int_0^{\pi}\left[\frac{r^3}{3}\right]_0^{a(1+\cos\theta)} \sin\theta d\theta = 2\pi\int_0^{\pi}\left[\frac{a^3(1+\cos\theta)^3}{3} - 0\right]\sin\theta d\theta$$

$$= \frac{2\pi a^3}{3}\int_0^{\pi}(1+\cos\theta)^3 \sin\theta d\theta .$$

Put $1+\cos\theta = t$, so that $-\sin\theta d\theta = dt$.

And when $\theta = 0$, $t = 2$, and when $\theta = \pi$, $t = 0$.

$\therefore$ Required volume of revolution $= -\frac{2\pi a^3}{3}\int_2^0 t^3 dt = \frac{2\pi a^3}{3}\int_0^2 t^3 dt$

$$= \frac{2\pi a^3}{3}\left[\frac{t^4}{4}\right]_0^2 dt = \frac{8\pi a^3}{3}.$$ Cubic units. Ans.

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Home Assignments

Q.No.1.: Find, by double integration, the volume of the solid generated by revolving the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ about the y-axis.

Ans.: $\frac{4}{3}\pi a^2 b$. Cubic units.

Q.No.2.: Find, by double integration, the volume of the solid generated by revolving the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ about the x-axis.

Ans.: $\frac{4}{3}\pi ab^2$. Cubic units.

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